On the reachable set of perturbed heat equations

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Joint works with

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I. Introduction

Abstract linear system

$$\begin{cases} y' = Ay + Bu, & t \geq 0. \\ y(0) = y_0. \end{cases}$$

- A generates a C^0 semigroup $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ on a Hilbert space H.
- $B \in \mathcal{L}(U, H)$ (or admissible) is the control operator.
- H, U Hilbert spaces.
  - → H : state space $y(t, \cdot)$ = state
  - → U : control space $u \in L^2(0, T; U)$: control

Goal: Understand how the control can act on the system

$$\begin{cases} y' = Ay + Bu \\ y|_{t=0} = y_0 \end{cases} \rightsquigarrow y(t) = \mathbb{T}_t y_0 + \int_0^t \mathbb{T}_{t-s} Bu(s)\, ds.$$

$\rightsquigarrow$ For $T > 0$, we want to describe the reachable space at time T starting from $y_0 = 0$

$$\mathcal{R}(T) = \left\{ \int_0^T \mathbb{T}_{T-s} Bu(s)\, ds, \quad u \in L^2(0,T;U) \right\}.$$

$$= \left\{ y(T), \text{ for } y \text{ solving } \begin{cases} y' = Ay + Bu \\ y(0) = 0 \end{cases}, \ u \in L^2(0,T;U) \right\}.$$

Goal: Describe $\mathcal{R}(T)$

Describing $\mathcal{R}(T)$

$\rightsquigarrow$ Completely characterized in finite dimension $(H = \mathbb{R}^n)$.

$$\mathcal{R}(T) = \operatorname{Ran}(B, AB, \ldots A^{n-1}B)$$

[Kalman]

$\rightsquigarrow$ For exactly controllable systems, $\mathcal{R}(T) = H$

Ex: Waves, Schrödinger under suitable geometric conditions

[Bardos Lebeau Rauch, ...].

$\rightsquigarrow$ For systems which are null controllable in any positive time, $\mathcal{R}(T)$ does not depend on T

$$\mathcal{R} = \mathcal{R}(T).$$

[Seidman]

Typical example: the heat equation.

II An abstract result

$$\begin{cases} y' = Ay + Bu \\ y|_{t=0} = 0. \end{cases}$$

Assumpt.: this system is null-controllable in any positive time.

Thm [S.E., Le Balc'h, Tucsnak '23]

The semi-group $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ defined on H is such that $\mathbb{T}_{\mathcal{R}} = \left(\mathbb{T}_t |_{\mathcal{R}} \right)_{t \geq 0}$ is a C_0 semi-group on $\mathcal{R}$. $\rightsquigarrow$ Exactly controllable system on $\mathcal{R}$.

Rk. Similar statement in [van Neerven '05].

A word on the proof.

- $\mathcal{R}$ is endowed with the norm (indexed by $T>0$)

$$\|z\|_{\mathcal{R}(T)} = \inf\left\{ \|u\|_{L^2(0,T;U)} ,\ u \in L^2(0,T;U) ,\ \left| \begin{array}{l} y' = Ay + Bu \\ y(0) = 0 \end{array} \right. \text{ and } y(T) = z \right\}.$$

Note that if $T_1 < T_2$, $\|z\|_{\mathcal{R}(T_2)} \leqslant \|z\|_{\mathcal{R}(T_1)}$.

and $\exists\, C(T_1,T_2)$, $\|z\|_{\mathcal{R}(T_1)} \leqslant C(T_1,T_2)\, \|z\|_{\mathcal{R}(T_2)}$.

- Let $\tau > 0$. For $t \in [0,\tau]$, and $z \in \mathcal{R}$

$$\|\mathbb{T}_t z\|_{\mathcal{R}(\tau)} \leqslant C(\tau)\, \|\mathbb{T}_t z\|_{\mathcal{R}(2\tau)} \leqslant C(\tau)\, \|z\|_{\mathcal{R}(2\tau - t)} \leqslant C(\tau)\, \|z\|_{\mathcal{R}(\tau)}.$$

$\Rightarrow$ Perturbative arguments can be applied. cf [S.E, Le Balc'h, Tucsnak].

E.g. $\forall T > 0$, $\exists \varepsilon(T) > 0$, if $Q \in \mathcal{L}(\mathcal{R})$ satisfies

$$\|Q\|_{\mathcal{L}(\mathcal{R}(T))} \leq \varepsilon$$

then the reachable space at time T for

$$\begin{cases} y' = (A+Q)y + Bu \\ y|_{t=0} = 0 \end{cases} \quad \text{and} \quad \begin{cases} y' = Ay + Bu \\ y|_{t=0} = 0 \end{cases}$$

are identical.

$$\mathcal{R}_{A+Q}(T) = \mathcal{R}_A(T) = \mathcal{R}$$

Problems and open questions

- When $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ is an analytic semi-group in H

is $\mathbb{T}_R = (\mathbb{T}_t|_R)$ only a C^0 semi-group on R?

Can we get better regularity results? Analytic
Gevrey
Differentiable ...

- Such result is useful if we know R:

Do we know R?

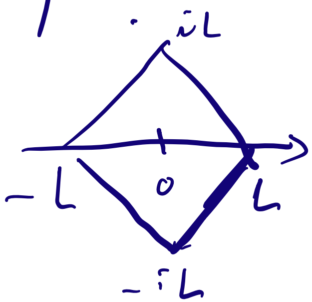
III. The heat eq.

$$\begin{cases} \partial_t y - \partial_{xx} y = 0 & \text{in } (0,T)\times(-L,L) \\ y(t,-L) = u_-(t), & y(t,L) = u_+(t). \end{cases}$$

- [Fattorini - Russell '71] on the Fourier basis
- [Martin - Rosier - Rouchon '16] in terms of holomorphy
- [Dardé - Ervedoza '18]
- [Kellay - Hartmann - Tucsnak '20], [Hartmann Orsoni]...

→ the reachable space is a space of holomorphic functions.

$\mathcal{R} = \{ z : (-L,L) \to \mathbb{C}$ which admits an holomorphic extension z_e in the square

$S =$ 

with $z_e \in L^2(S)$ $\}$

$$\mathcal{R} = L^2 \cap Hol(S).$$

Can we consider lower order perturbations?

- [Orsoni '23] specifically on $\partial_t y - \partial_{xx} y + x^2 y = 0$.

- [S.E. Le Balc'h Tucsnak '23] aplies on

$$\partial_t y - \partial_{xx} y + q y = 0 \quad \text{with } q \in L^\infty \cap Hol(S)$$

Perturbative arguments $\qquad \|q\|_{L^\infty(S)}$ small.

- [Laurent Rosier '21]

$$\partial_t y - \partial_{xx} y + f(t, x, y, \partial_x y) = 0$$

which asks strong holomorphic conditions.

Flatness / Solving an ill-posed pb in x.

The heat eq in multi-d

Only one precise result [Strohmaier – Waters '22].

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } (0,T) \times \Omega. \\ y\big|_{(0,T)\times\partial\Omega} = v(t,x) & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

When Ω is the unit ball and the control acts everywhere

on $\partial\Omega$: $\quad Hol\left(\overline{\mathcal{E}(\Omega)}\right) \subset \mathcal{R} \subset Hol\left(\mathcal{E}(\Omega)\right).$

with $\mathcal{E}(\Omega) = \{ a + ib, \ a \in \mathbb{R}^d, b \in \mathbb{R}^d, \ |a| + |b| < 1 \}$

Key point. [Strohmaier - Waters '22]

Lem. $\Omega = B(1)$. If $y_0 \in L^2(\mathbb{R}^d)$ with

$$y_0|_{B(1)} \in Hol(\mathcal{E}(B(1)))$$

Then the solution y of $\begin{cases} \partial_t y - \Delta y = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ y|_{t=0} = y_0 \end{cases}$

satisfies $\forall K$ compact of $\mathcal{E}(B(1))$,

$$\lim_{t\to 0^+} \| y(t,\cdot) - y_0 \|_{L^\infty(K)} = 0.$$

Our result (with A. Tendani-Soler).

- $\Omega = B_{\mathbb{R}^d}(1)$.

$$\begin{cases} \partial_t y - \Delta y + q y + w \cdot \nabla y = 0 & \text{in } (0,T)\times\Omega. \\ y|_{(0,T)\times\partial\Omega} = u & \text{on } (0,T)\times\partial\Omega. \\ y|_{t=0} = y_0 \end{cases}$$

Goal. Determine the reachable space in this setting in presence of lower order terms.

For $\alpha > 0$, $\Omega_\alpha = \{ a + ib,\ |a| + \alpha |b| < 1,\ a, b \in \mathbb{R}^d \}$

$$\mathcal{R}_\alpha = \{ f \in L^\infty(\Omega),\ f \text{ admits a } C^0 \text{ extension on } \overline{\Omega}_\alpha \text{ and } f_e \in Hol(\Omega_\alpha) \}$$

Thm [S.E. Tendani-Soler, ongoing].

Assume that $\exists \alpha_0 \in (0,1)$, $q, w \in L^\infty_{loc}(0, \infty; \mathcal{R}_{\alpha_0})$.

Then $\forall y_0 \in H^{-1}(\Omega)$, $\forall T > 0$,

$$\bigcup_{\alpha \in (0,1)} \mathcal{R}_\alpha \subset \mathcal{R}(y_0, T).$$

Strategy Determine a functional space which looks like the expected reachable space and on which the heat semi-group is analytic.

For $\alpha > 0$ $X_\alpha(\mathbb{R}^d) = \{ f \in BUC(\mathbb{R}^d),\ f|_{B_{\mathbb{R}^d}}$ admits a continuous extension f_e on $\overline{\Omega_\alpha}$, holomorphic in $\Omega_\alpha \}$.

$$\|f\|_{X_\alpha(\mathbb{R}^d)} = \|f\|_{L^\infty(\mathbb{R}^d)} + \|f_e\|_{L^\infty(\Omega_\alpha)}.$$

Thm For $\alpha > 1$, $\mathbb{T}_\alpha = \left(\mathbb{T}_t|_{X_\alpha}\right)_{t \geq 0}$ is analytic on $X_\alpha(\mathbb{R}^d)$

where $(\mathbb{T}_t)_{t \geq 0}$ is the heat semigroup on $\mathbb{R}^d$.

Ideas of the proof in 1d.

- $\Pi_t y_0(x) = G(t) * y_0(x) = \frac{1}{\sqrt{4\pi t}} \int_{x_0 \in \mathbb{R}} e^{-\frac{(x-x_0)^2}{4t}} y_0(x_0)\, dx_0$

$\rightsquigarrow$ Natural holomorphic expansion for $t > 0$.

- Separate $\int_{x_0 \in \mathbb{R}}$ as $\int_{x_0 \in \mathbb{R} \setminus (-1,1)} + \int_{x \in (-1,1)}$

- Use Cauchy's formula to estimate the second integral.

$\rightsquigarrow$ Allows to show the well-posedness of the perturbed heat equation in $X_\alpha(\mathbb{R}^d)$.

E.g.

Thm: let $\alpha > 1$, $q \in L^\infty(0,T; X_\alpha(\mathbb{R}^d))$, $w \in L^\infty(0,T; X_\alpha(\mathbb{R}^d))$,

then $\forall y_0 \in X_\alpha(\mathbb{R}^d)$, $\forall f \in L^1(0,T; X_\alpha(\mathbb{R}^d))$, with $\sqrt{t} f \in L^\infty(0,T; X_\alpha(\mathbb{R}^d))$, the solution y of

$$\begin{cases} \partial_t y - \Delta y + q y + w \cdot \nabla y = f & \text{in } (0,T) \times \mathbb{R}^d \\ y|_{t=0} = y_0 \end{cases}$$

satisfies $y \in C^0([0,T]; X_\alpha(\mathbb{R}^d))$ and

$$\|y\|_{L^\infty(0,T; X_\alpha(\mathbb{R}^d))} + \|\sqrt{t} \nabla y\|_{L^\infty(0,T; X_\alpha(\mathbb{R}^d))} \leq C \left(\|y_0\|_{X_\alpha} + \|\sqrt{t} f\|_{L^\infty(0,T; X_\alpha)} \right).$$

Rk. Here, we use that the heat semigroup

$\mathbb{T}_\alpha = \left(\mathbb{T}_t |_{X_\alpha} \right)_{t \geq 0}$ is an analytic semigroup in X_α

$\exists C,$
$\forall t > 0$
$\forall y_0 \in X_\alpha,$

$$\| \mathbb{T}_t y_0 \|_{X_\alpha} + \sqrt{t} \, \| \nabla_x \mathbb{T}_t y_0 \|_{X_\alpha} \leq C \, \| y_0 \|_{X_\alpha}$$

but we do not know if maximal regularity results hold in the class X_α.

(or similar classes)

. Sketch of the proof stating $\bigcup_{\alpha \in (0,1)} R_\alpha \subset R(T, y_0)$

ⓐ . Null-controllability in the usual L^2 spaces

$\Rightarrow$ Null-controllability in $X_\alpha(\mathbb{R}^d)$.

ⓑ "Wick's" formula $x \longrightarrow ix$.

[Strohmaier - Waters]

IV Further comments.

- Semilinear terms can be considered in a similar fashion, thus improving the results in [Laurent - Rosier 21].

Open problems

- Can we take $\alpha = 1$ in our proofs?
- Reachable spaces in more general geometric settings?

Thank you for your attention!

Based on

- S.E., K. Le Balc'h, M. Tucsnak, 2022, JFA.
- S.E & A. Tendani-Soler, 2025, hal.