New Insights on the Stabilization of Generalized Serially Connected Piezoelectric and Elastic Beams

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Outline

1. Introduction
2. Generalized PEP Systems
3. Main Results
4. Conclusion

Physical Interpretation

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- **Serial connection:** vibrations travel across junctions → complex dynamics.
- **Without control:** vibrations may persist or amplify, leading to instability.
- **Stabilization goal:**
  - Use piezoelectric actuation and sensing,
  - Introduce damping to dissipate vibration energy,
  - Ensure the structure returns to equilibrium.

Physical Applications

- **Aerospace structures (airplanes)**
    - Wings modeled as elastic beams with piezoelectric patches.
    - Piezoelectric layers used for *vibration suppression* and *flutter control*.
    - Enhance stability and fuel efficiency by reducing structural oscillations.

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  - Piezoelectric elements allow for precise sensing and active damping.

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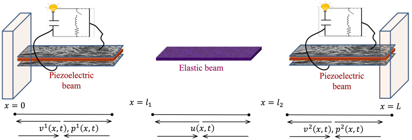
  - Piezoelectric elements allow for precise sensing and active damping.
- **Civil and mechanical engineering**
  - Smart bridges and tall buildings using piezoelectric-elastic beam models.
  - Energy harvesting from vibrations.

Serially Connected Piezoelectric-Elastic Beams



- $v(x,t)$: longitudinal vibrations of the center line beam.
- $p(x,t)$: total charges accumulated at the electrodes.

$$\begin{cases} \rho_1 v^1_{tt} - \alpha_1 v^1_{xx} + \gamma_1\beta_1 p^1_{xx} + r_1 v^1_t = 0, & (x,t) \in (0,l_1)\times(0,\infty), \\ \mu_1 p^1_{tt} - \beta_1 p^1_{xx} + \gamma_1\beta_1 v^1_{xx} = 0, & (x,t) \in (0,l_1)\times(0,\infty), \\ u_{tt} - c_1 u_{xx} + r_2 u_t = 0, & (x,t) \in (l_1,l_2)\times(0,\infty), \\ \rho_2 v^3_{tt} - \alpha_3 v^3_{xx} + \gamma_3\beta_3 p^3_{xx} + r_3 v^3_t = 0, & (x,t) \in (l_2,L)\times(0,\infty), \\ \mu_3 p^3_{tt} - \beta_2 p^3_{xx} + \gamma_3\beta_3 v^3_{xx} = 0, & (x,t) \in (l_2,L)\times(0,\infty), \\ v^1(0,t) = p^1(0,t) = v^3(L,t) = p^3(L,t) = 0, & \\ v^1(l_1,t) = u(l_1,t), & \\ v^3(l_2,t) = u(l_2,t), & \\ \alpha_1 v^1_x(l_1,t) - \gamma_1\beta_1 p^1_x(l_1,t) = c_1 u_x(l_1,t), & \\ \alpha_3 v^3_x(l_2,t) - \gamma_3\beta_3 p^3_x(l_2,t) = c_1 u_x(l_2,t), & \\ \beta_j p^j_x(l_j,t) - \gamma_j\beta_j v^j_x(l_j,t) = (j-2) r_{j+2} p^j_t(l_j,t), & t \in (0,\infty),\ j = 1,3, \\ (v^1, p^1, u, v^2, p^2)(.,0) = (v^1_0, p^1_0, u_0, v^2_0, p^2_0)(.), & \\ (v^1_t, p^1_t, u_t, v^2_t, p^2_t)(.,0) = (v^1_1, p^1_1, u_1, v^2_1, p^2_1)(.), & \end{cases} \tag{1}$$

Physical Parameters and Conditions

Parameter	Definition
ρ	Mass density
α	Elastic stiffness
β	Impermeability
γ	Piezoelectric constant
μ	Magnetic permeability

- $\alpha = \alpha^1 + \gamma^2 \beta$ with $\alpha^1 > 0,.$

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Consider for $N \geq 1$:

$$\frac{\sigma_{2k+1}^{+}}{\sigma_{2k+1}^{-}} \neq \frac{n_+}{n_-}, \quad \forall n_+, n_- \in \mathbb{N}, \quad \text{where} \quad 0 < k < N, \qquad (\mathrm{SC}_{2k+1})$$

and

$$\frac{\sigma_{2k+1}^{+}}{\sigma_{2k+1}^{-}} \neq \frac{2n_+ + 1}{2n_- - 1}, \quad \forall n_+, n_- \in \mathbb{N}, \quad \text{where} \quad k = 0 \text{ or } N, \qquad (\mathrm{NC}_{2k+1})$$

where the two positive real numbers σ_j^{+} and σ_j^{-} are defined by

$$\sigma_j^{\pm} := \sqrt{\frac{(\rho_j\beta_j + \mu_j\alpha_j) \pm \sqrt{(\rho_j\beta_j - \mu_j\alpha_j)^2 + 4\gamma_j^2\beta_j^2\mu_j\rho_j}}{2\beta_j\alpha_j^1}} \quad \text{for} \quad j = 2k+1, \ k \leq N. \qquad (2)$$

Hypothesis

xp$)_{\mathrm{2k+1}}$ Suppose $\frac{\sigma_+^{2k+1}}{\sigma_-^{2k+1}} \in \mathbb{Q}$ such that $\frac{\sigma_+^{2k+1}}{\sigma_-^{2k+1}} = \frac{\xi_+^{2k+1}}{\xi_-^{2k+1}}$, where $\gcd(\xi_+^{2k+1}, \xi_-^{2k+1}) = 1$, and ξ_+^{2k+1} and ξ_-^{2k+1} are, respectively, even and odd integers or vice versa for $k = 0$ or N.

ol$)_{\mathrm{2k+1}}$ Suppose that $\frac{\sigma_+^{2k+1}}{\sigma_-^{2k+1}}$ is an irrational number. Then, consider the existence of $\varpi\left(\frac{\sigma_+^{2k+1}}{\sigma_-^{2k+1}}\right) \geq 2$, dependent on $\frac{\sigma_+^{2k+1}}{\sigma_-^{2k+1}}$, such that for all sequences $\Lambda^{2k+1} = (\xi_{1,n}^{2k+1}, \xi_{2,n}^{2k+1})_{n\in\mathbb{N}} \in (\mathbb{N} \times \mathbb{N}^*)^{\mathbb{N}}$ with $\xi_{1,n}^{2k+1} \sim \xi_{2,n}^{2k+1}$ for sufficiently large n, there exists a positive constant $c\left(\frac{\sigma_+^{2k+1}}{\sigma_-^{2k+1}}, \Lambda^{2k+1}\right)$ and a positive integer $N\left(\frac{\sigma_+^{2k+1}}{\sigma_-^{2k+1}}, \Lambda^{2k+1}\right)$, both depending on $\frac{\sigma_+^{2k+1}}{\sigma_-^{2k+1}}$ and the sequence Λ^{2k+1} such that

$$\left|\frac{\sigma_+^{2k+1}}{\sigma_-^{2k+1}} - \frac{\xi_{1,n}^{2k+1}}{\xi_{2,n}^{2k+1}}\right| > \frac{c\left(\frac{\sigma_+^{2k+1}}{\sigma_-^{2k+1}}, \Lambda^{2k+1}\right)}{\left(\xi_{2,n}^{2k+1}\right)^{\varpi\left(\frac{\sigma_+^{2k+1}}{\sigma_-^{2k+1}}\right)}}, \quad \forall n \geq N\left(\frac{\sigma_+^{2k+1}}{\sigma_-^{2k+1}}, \Lambda^{2k+1}\right) \quad \text{for} \quad 0 \leq k \leq N.$$

Cases and Results

	Piezo		Elastic		Piezo	
Cases	$(0, l_1)$	$x = l_1$	(l_1, l_2)	$x = l_2$	(l_2, L)	Type of Stability
1	VD				VD	Exponential
2	VD			BD		Exponential
3			VD			Exponential if $(\mathbf{H_{Exp}})_{\mathbf{2k+1}}$ holds for $k = 0$ and 1. Polynomial if $(\mathbf{H_{Pol}})_{\mathbf{2k+1}}$ holds for $k = 0$ and 1.
4			VD	BD		Exponential if $(\mathbf{H_{Exp}})_{\mathbf{2k+1}}$ holds for $k = 0$. Polynomial if $(\mathbf{H_{Pol}})_{\mathbf{2k+1}}$ holds for $k = 0$.
5		BD	VD	BD		Exponential

[1]

Table: Outlined above is a summary of the results, with “VD” and “BD” denoting viscous damping and boundary damping, respectively.

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[1] M.Akil, S. Nicaise, O. Ozer, H. Saleh, *Advancing insights into the stabilization of serially-connected magnetizable piezoelectric and elastic beams*, SIAM journal on control and optimization, 2025

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Generalized PEP Design

Consider the following design for $N \geq 1$:

$$\begin{cases} \rho_{2k+1} v_{tt}^{2k+1}(x,t) - \alpha_{2k+1} v_{xx}^{2k+1}(x,t) + \gamma_{2k+1}\beta_{2k+1} p_{xx}^{2k+1}(x,t) + d_{2k+1} v_t^{2k+1}(x,t) = 0, \\ \mu_{2k+1} p_{tt}^{2k+1}(x,t) - \beta_{2k+1} p_{xx}^{2k+1}(x,t) + \gamma_{2k+1}\beta_{2k+1} v_{xx}^{2k+1}(x,t) = 0,, \end{cases} \tag{3}$$

where $(x,t) \in I_k = (l_{2k}, l_{2k+1}) \times (0,\infty), k = 0, 1, 2, \cdots, N$,

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where $(x,t) \in I_k = (l_{2k}, l_{2k+1}) \times (0,\infty), k = 0,1,2,\cdots,N$, and

$$u_{tt}^{2k+2}(x,t) - c_{2k+2} u_{xx}^{2k+2}(x,t) + d_{2k+2} u_t^{2k+2}(x,t) = 0, \qquad (x,t) \in \tilde{I}_k, \tag{4}$$

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Moreover,

$$\{d_k\}_{k=1}^{2N+1} \cup \{r_k\}_{k=1}^{2N} \subset [0,\infty)^{4N+1}, \tag{6}$$

and the stiffness constant satisfies

$$\alpha_{2k+1} = \alpha_{2k+1}^1 + \gamma_{2k+1}^2 \beta_{2k+1}, \text{ with } \alpha_{2k+1}^1 > 0, \forall 0 \leq k \leq N. \tag{7}$$

Transmission Conditions

The transmission conditions are given by

$$J_1 := \begin{cases} u^{2k}(l_{2k},t) = v^{2k+1}(l_{2k},t), & 1 \leq k \leq N, \\ c_{2k} u_x^{2k}(l_{2k},t) = (\alpha_{2k+1} v_x^{2k+1} - \gamma_{2k+1}\beta_{2k+1} p_x^{2k+1})(l_{2k},t), & 1 \leq k \leq N, \\ (\beta_{2k+1} p_x^{2k+1} - \gamma_{2k+1}\beta_{2k+1} v_x^{2k+1})(l_{2k},t) = r_{2k} p_t^{2k+1}(l_{2k},t), & 1 \leq k \leq N, \end{cases} \tag{8}$$

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and

$$J_2 := \begin{cases} u^{2k+2}(l_{2k+1},t) = v^{2k+1}(l_{2k+1},t), & 0 \le k \le N-1, \\ c_{2k+2} u_x^{2k+2}(l_{2k+1},t) = (\alpha_{2k+1} v_x^{2k+1} - \gamma_{2k+1}\beta_{2k+1} p_x^{2k+1})(l_{2k+1},t), & 0 \le k \le N-1, \\ (\beta_{2k+1} p_x^{2k+1} - \gamma_{2k+1}\beta_{2k+1} v_x^{2k+1})(l_{2k+1},t) = -r_{2k+1} p_t^{2k+1}(l_{2k+1},t), & 0 \le k \le N-1, \end{cases} \tag{9}$$

where $t \in (0,\infty)$.

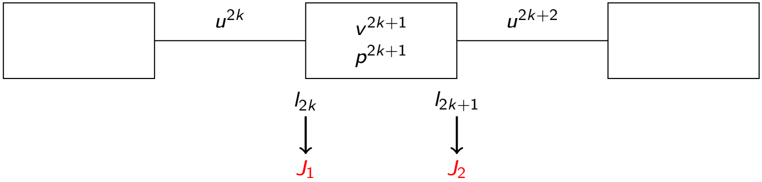
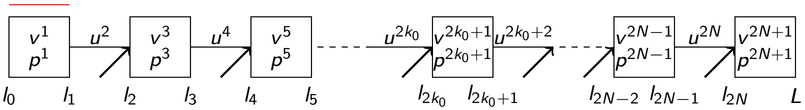


Figure: Representation of systems J_1 and J_2 around the transmission points l_{2k} and l_{2k+1}, respectively, where the straight line represents the elastic part and the rectangle represents the piezoelectric part.

Main Cases under Consideration

Case 1:

$$\begin{cases} d_{2k+1} \neq 0, \text{ for } k = 0, \text{ and } d_{2k+1} = 0 \text{ else}, \\ d_{2k+2} = 0, \text{ for } k \leq N-1, \\ r_{2k} \neq 0, \text{ for } 0 < k \leq N, \text{ and } r_{2k+1} = 0, \text{ for } k \leq N-1. \end{cases} \tag{C1}$$



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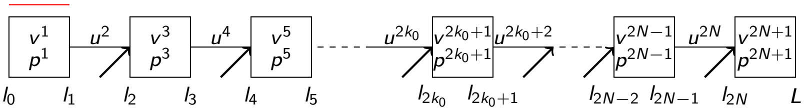


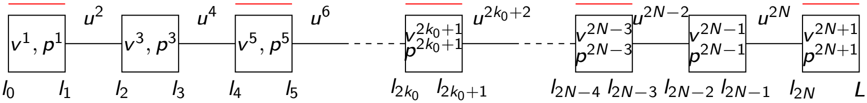
Figure: Configuration of **Case 1**.

Case 2:

$$\begin{cases} d_{2k+1} \neq 0, \text{ for } k \leq N, k \text{ even and } d_{2k+1} = 0, \text{ for } k \leq N, \text{ k odd}, \\ d_{2k+2} = 0, \text{ for } k \leq N-1, \\ r_{2k} = 0, \text{ for } k \leq N, \\ r_{2k+1} = 0, \text{ for } k \leq N-1. \end{cases} \tag{C2}$$

We now distinguish between the parity of N:

- **Case 2.a**: If N is even:



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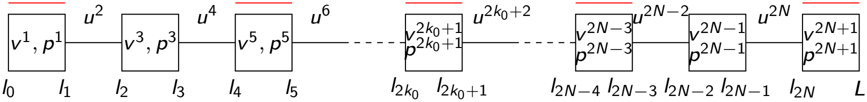

Figure: Configuration of **Case 2.a**.

- **Case 2.b**: If N is odd:

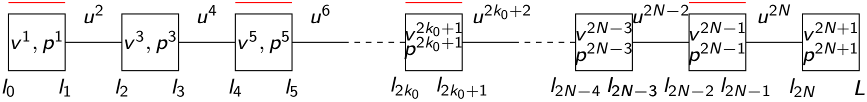

Figure: Configuration of **Case 2.b**.

Case 3:

$$\begin{cases} d_{2k+1} = 0, \ \text{for } k \leq N, \\ d_{2k+2} \neq 0, \ \text{for } k \leq N-1, \\ r_{2k} = 0, \ \text{for } k \leq N, \ \text{and } r_{2k+1} = 0, \ \text{for } k \leq N-1. \end{cases} \tag{C3}$$

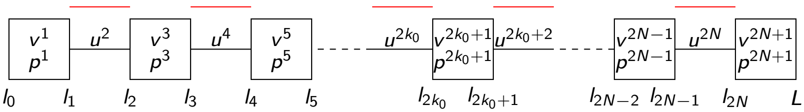

Figure: Configuration of **Case 3**.

- Square: denotes the piezoelectric part,
- ——— : denotes the elastic part,
- ——— : denotes the viscous damping,
- ↗ : denotes the boundary damping at the accumulated charges.

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3) $(\mathrm{C3})$ *holds. Then, our system is strongly stable if and only if (SC_{2k+1}) holds for $0 < k < N$, and (NC_{2k+1}) holds for $k = 0$ and N.*

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Exponential and Polynomial Stabilities

- If (C1) holds $\Longrightarrow$ Exponential stability.
- If (C2) holds with N even and $(\mathbf{H_{Pol}})_{\mathbf{2k+1}}$ holds for $k \leq N$ where k odd
  $\Longrightarrow$ Polynomial stability with a decay rate of type $t^{-\frac{2}{\ell_1}}$, where

$$\ell_1 = 4\max\left(\varpi\left(\frac{\sigma_3^+}{\sigma_3^-}\right), \varpi\left(\frac{\sigma_7^+}{\sigma_7^-}\right), \cdots, \varpi\left(\frac{\sigma_{2N-5}^+}{\sigma_{2N-5}^-}\right), \varpi\left(\frac{\sigma_{2N-1}^+}{\sigma_{2N-1}^-}\right)\right) - 4.$$

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  $\Longrightarrow$ Polynomial stability with a decay rate of type $\mathrm{t}^{-\frac{2}{\ell_2}}$, where

$$\ell_2 = 4\max\left(\varpi\left(\frac{\sigma_3^+}{\sigma_3^-}\right), \varpi\left(\frac{\sigma_7^+}{\sigma_7^-}\right), \cdots, \varpi\left(\frac{\sigma_{2N-3}^+}{\sigma_{2N-3}^-}\right), \varpi\left(\frac{\sigma_{2N+1}^+}{\sigma_{2N+1}^-}\right)\right) - 4.$$

Exponential and Polynomial Stabilities

- If (C1) holds $\Longrightarrow$ Exponential stability.
- If (C2) holds with N even and $(\mathbf{H_{Pol}})_{\mathbf{2k+1}}$ holds for $k \leq N$ where k odd

  $\Longrightarrow$ Polynomial stability with a decay rate of type $\mathrm{t}^{-\frac{2}{\ell_1}}$, where

$$\ell_1 = 4\max\left(\varpi\left(\frac{\sigma_3^+}{\sigma_3^-}\right), \varpi\left(\frac{\sigma_7^+}{\sigma_7^-}\right), \cdots, \varpi\left(\frac{\sigma_{2N-5}^+}{\sigma_{2N-5}^-}\right), \varpi\left(\frac{\sigma_{2N-1}^+}{\sigma_{2N-1}^-}\right)\right) - 4.$$

- If (C2) holds with N even and $(\mathbf{H_{Pol}})_{\mathbf{2k+1}}$ holds for $k \leq N$ where k odd

  $\Longrightarrow$ Polynomial stability with a decay rate of type $\mathrm{t}^{-\frac{2}{\ell_2}}$, where

$$\ell_2 = 4\max\left(\varpi\left(\frac{\sigma_3^+}{\sigma_3^-}\right), \varpi\left(\frac{\sigma_7^+}{\sigma_7^-}\right), \cdots, \varpi\left(\frac{\sigma_{2N-3}^+}{\sigma_{2N-3}^-}\right), \varpi\left(\frac{\sigma_{2N+1}^+}{\sigma_{2N+1}^-}\right)\right) - 4.$$

- If (C3) holds and $(\mathbf{H_{Pol}})_{\mathbf{2k+1}}$ holds for $k \leq N$

  $\Longrightarrow$ Polynomial stability with a decay rate of type $\mathrm{t}^{-\frac{2}{\ell_3}}$, where

$$\ell_3 = 2\max\left(\varpi\left(\frac{\sigma_1^+}{\sigma_1^-}\right), \varpi\left(\frac{\sigma_3^+}{\sigma_3^-}\right), \cdots, \varpi\left(\frac{\sigma_{2N-1}^+}{\sigma_{2N-1}^-}\right), \varpi\left(\frac{\sigma_{2N+1}^+}{\sigma_{2N+1}^-}\right)\right) - 2.$$

Other Studied Cases

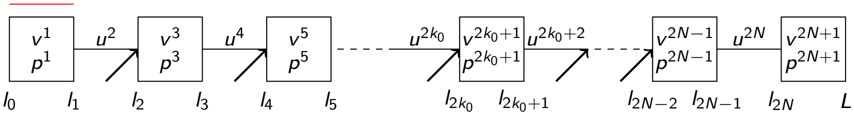

Figure: 4

Other Studied Cases

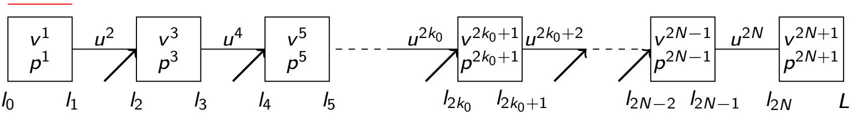

Figure: 4

- Strongly stable if and only of (NC_{2k+1}) for $k = N$.

Other Studied Cases

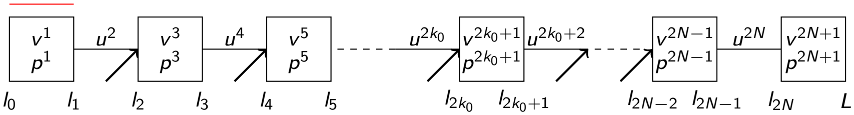

Figure: 4

- Strongly stable if and only of (NC_{2k+1}) for $k = N$.
- Exponentially stable, if $(\mathbf{H}_{\mathbf{Exp}})_{\mathrm{2N+1}}$ holds.

Other Studied Cases

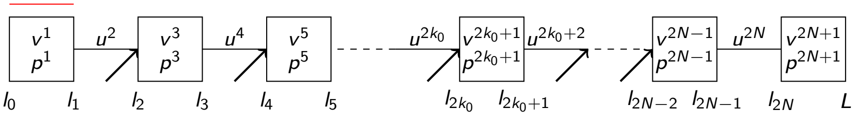

Figure: 4

- Strongly stable if and only of (NC_{2k+1}) for $k = N$.
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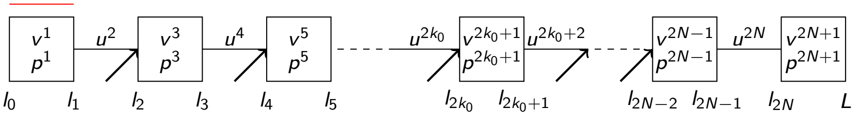
- Polynomially stable with a decay rate of type $t^{-\frac{2}{\ell}}$ and $\ell = 4\varpi\left(\frac{\sigma^{+}_{2N+1}}{\sigma^{-}_{2N+1}}\right) - 4.$, if $(\mathbf{H_{Pol}})_{2N+1}$ holds.

Other Studied Cases



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• If we consider the same assumptions as that for (C1), but with the assumption that there exist k', such that $r_{2k'} = 0$, for $0 < k' < N$, and $r_{2k} \neq 0$, for $0 < k < N$ and $k \neq k'$, then we conjecture that we lose the stability of our system.

Other Studied Cases

$$\begin{cases} d_{2k+1} \neq 0, \text{ for } k = k_1, k_2, \text{ where } k_2 - k_1 \geq 2, \text{ and } d_{2k+1} = 0 \text{ else}, \\ d_{2k+2} = 0, \text{ for } k \leq N-1, \\ r_{2k+1} \neq 0, \text{ for } k < k_1, \text{ and } r_{2k} \neq 0, \text{ for } k_2 < k \leq N, \\ \text{for } k_1 < k < k_2, \text{ either } r_{2k} \neq 0, \text{ or } r_{2k+1} \neq 0. \end{cases} \tag{10}$$

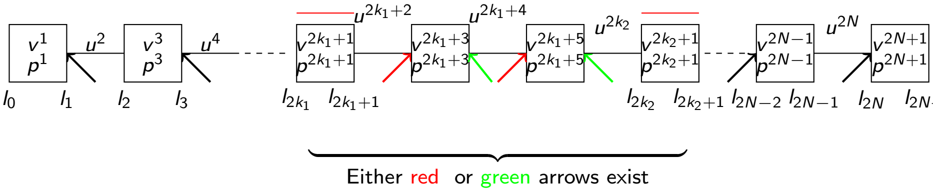
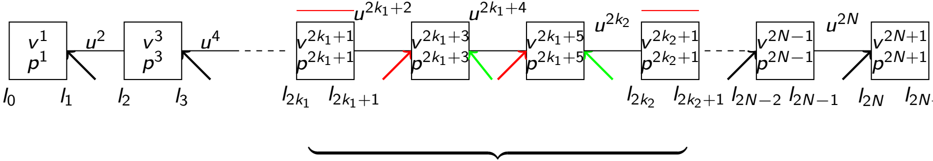


Figure: 5

Other Studied Cases

$$\begin{cases} d_{2k+1} \neq 0, \ \text{for } k = k_1, k_2, \ \text{where } k_2 - k_1 \geq 2, \ \text{and } d_{2k+1} = 0 \text{ else,} \\ d_{2k+2} = 0, \ \text{for } k \leq N-1, \\ r_{2k+1} \neq 0, \ \text{for } k < k_1, \ \text{and } r_{2k} \neq 0, \ \text{for } k_2 < k \leq N, \\ \text{for } k_1 < k < k_2, \ \text{either } r_{2k} \neq 0, \ \text{or } r_{2k+1} \neq 0. \end{cases} \tag{10}$$



Either red or green arrows exist

Figure: 5

- Strongly stable without any arithmetic conditions.
- Exponentially stable without any arithmetic conditions.

Other Studied Cases

$$\begin{cases} d_{2k+1} = 0, \text{ for } k \leq N, \\ d_{2k+2} \neq 0, \text{ for } k \leq N-1, \\ r_{2k+1} = 0, \text{ for } 0 < k \leq N-1, \text{ and } r_1 \neq 0, \\ r_{2k} \neq 0, \text{ for } 0 < k \leq N. \end{cases} \tag{11}$$

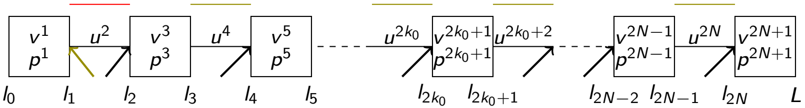


Figure: 6

Other Studied Cases

$$\begin{cases} d_{2k+1} = 0, \ \text{for } k \leq N, \\ d_{2k+2} \neq 0, \ \text{for } k \leq N-1, \\ r_{2k+1} = 0, \ \text{for } 0 < k \leq N-1, \ \text{and } r_1 \neq 0, \\ r_{2k} \neq 0, \ \text{for } 0 < k \leq N. \end{cases} \tag{11}$$

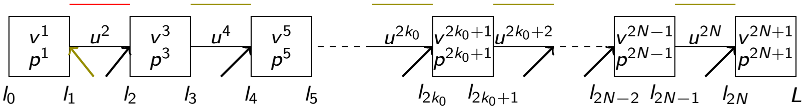


Figure: 6

- Strongly stable without any arithmetic conditions.
- Exponentially stable without any conditions.

Outline

1. Introduction

2. Generalized PEP Systems

3. Main Results

4. Conclusion

What We Conclude?

- The existence of at least one viscous damping acting on either the piezoelectric or elastic part is important to ensure the stability of our system. Indeed, if we consider only the case where electrical boundary damping is applied, we conjecture that the system is not stable.

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- The way of distribution of electrical boundary dampings is also necessary to ensure the stability of our system; otherwise, we conjecture that we lose the stability of our system.
- We conjecture that the system is not stable if two consecutive piezoelectric parts are left undamped and no other damping mechanisms are applied.

Thank You!

Questions or Comments?

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