Optimal stabilization rate for the wave equation with hyperbolic boundary condition

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Control of Partial Differential Equations in Hauts-de-France (2nd edition).

Ongoing work with N. Vanspranghe (Tampere University).

September 16, 2025

Starting point

Beale, 1976, **Littman & Liu, 1998** studied spectral properties and stabilization of acoustic flows:

Wave equation $+$ Kinetic (second-order) BC.

Key observation by L&L: Adding boundary dynamics loss of uniform stability!

Unif. sta. $\Longleftrightarrow$ Exp. sta. for mild sol. Non-uniform (polynomial/logarithm, etc ....) for classical sol.

Dynamic BC

- 1D case. Tip mass, drilling systems: Auriol, Roman, Chitour, Nguyen...
- MultiD. Acoustic BC, dynamic Ventcel BC: Graber, Lasiecka, Vitillaro, Buffe...

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Wave with damped dynamic Ventcel BC

Let Ω be a smooth bounded domain of $\mathbb{R}^d$ with $\partial\Omega = \Gamma$ made of two disjoint parts.

$$\begin{cases} (\partial_t^2 - \Delta)u = 0 & \text{in } \Omega \times (0,+\infty), \\ (\partial_t^2 + \partial_t - \Delta_\Gamma)u = -\partial_\nu u & \text{on } \Gamma_0 \times (0,+\infty), \\ u = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{in } \Omega. \end{cases}$$

Γ_0 satisfies GCC,
Γ_1 possibly empty.

$$E(t) = \underbrace{\frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 \, \mathrm{d}x}_{\text{Interior energy}} + \underbrace{\frac{1}{2}\int_{\Gamma_0} |\nabla_\Gamma u(x,t)|^2 + |\partial_t u(x,t)|^2 \, \mathrm{d}\sigma}_{\text{Boundary energy}}.$$

Theorem (Non-uniform stabilization rate)

For initial data $(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega)$ *with* $(u_0, u_1)|_{\Gamma_0} \in H^2(\Gamma_0) \times H^1(\Gamma_0)$ *and* $(u_0, u_1)|_{\Gamma_1} = 0$,

$$E(t) \lesssim \frac{1}{1+t}\|(u_0, u_1)\|^2_{D(A)}, \quad t \to +\infty.$$

- For finite-energy data: asymptotic stability $E(t) \to 0$ (no rate).
- **Buffe 2017**, *a priori* logarithmic decay is established without geometric condition by means of Carleman estimates.

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Evolution structure

$$U' + \mathcal{A}U = 0, \quad \mathcal{A} \triangleq \begin{pmatrix} 0 & -1 \\ A & BB^* \end{pmatrix}.$$

Damped second-order structure

$$u'' + BB^* u' + Au = 0,$$

- $Au = (-\Delta u, -\Delta_\Gamma u + \partial_\nu u)$,
- $\mathrm{dom}(A) = \{u \in H^2(\Omega) : u|_{\Gamma_0} \in H^2(\Gamma_0),\ u|_{\Gamma_1} = 0\}$.
- $Be = (0, e)$, for all $e \in L^2(\Gamma_0)$.
- $\mathrm{dom}(\mathcal{A}) \triangleq \mathrm{dom}(A) \times \{u \in H^1(\Omega) : u|_{\Gamma_0} \in H^1(\Gamma_0)\}$.

- A is a nonnegative self-adjoint operator on $H = L^2(\Omega) \times L^2(\Gamma_0)$
- $B \in \mathcal{L}(L^2(\Gamma_0), H)$.

$P(z) = z^2 + zBB^* + A, \quad z \in \mathbb{C}$, Quadratic operator pencil (Laplace transform.)

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- $\lambda \in \mathbb{R} \setminus \{0\}$, $i\lambda \in \sigma(\mathcal{A}) \Leftrightarrow \exists u \neq 0$, s.t. $P(i\lambda)u = 0$. **Anantharaman-Léautaud 2014.**

$$0 = \langle P(i\lambda)u, u\rangle_H = \langle Au, u\rangle_H + i\lambda\|u\|^2_{L^2(\Gamma_0)} - \lambda^2\|u\|^2_H, \qquad \Longrightarrow u = 0 \text{ on } \Gamma_0.$$

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Unique continuation $\Longrightarrow u = 0$.

Semigroup theory (Borichev & Tomilov...) + special **wave structure** (Lebeau...):

Non-uniform decay of $E(t)$ $\Longleftrightarrow$ High-frequency asymptotics of $P(i\lambda)^{-1}$.

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Analysis of quasimodes

Semiclassical scaling

In what follows, $h > 0$ is a small parameter (here, $h = 1/\lambda$, $\lambda \to +\infty$).

- Let $\delta > 0$. Elements $u \in \mathrm{dom}(A)$ such that

$$\begin{cases} \|u\|_H = 1, \\ \|(h^2 A + ihBB^* - 1)u\|_H = o(h^{1+\delta}), \quad h \to 0, \end{cases}$$

define sequences of (normalized) quasimodes of order $1 + \delta$.

- **In PDE terms:**

$$\begin{cases} (-h^2\Delta - 1)u = f & \text{in } \Omega, \\ (-h^2\Delta_\Gamma + ih - 1)u = -h^2\partial_\nu u + e & \text{on } \Gamma_0, \\ u = 0 & \text{on } \Gamma_1, \end{cases}$$

with, as $h \to 0$,

$$\|u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Gamma_0)} = 1, \quad \|f\|_{L^2(\Omega)} = o(h^{1+\delta}), \quad \|e\|_{L^2(\Gamma_0)} = o(h^{1+\delta}).$$

Standard contradiction argument

$$\text{No } (1+\delta)\text{-quasimodes} \implies \|P(i\lambda)^{-1}\| = O(|\lambda|^{-1+\delta}) \implies E(t) = o(t^{-2/\delta}).$$

(The decay rate is for classical solutions. The limit case $\delta = 0$ corresponds to uniform stability!)

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- **Starting point:** Recall the energy identity in the time domain

$$E(t)\Big|_0^T = -\underbrace{\iint_{\Gamma_0\times(0,T)} |\partial_t u|^2 \,\mathrm{d}\sigma\,\mathrm{d}t}_{\text{Dissipation}}, \quad T \geq 0.$$

For $o(h^{1+\delta})$-quasimodes, its frequency-domain counterpart yields

$$\|u\|_{L^2(\Gamma_0)} = o(h^{\delta/2}), \quad h \to 0. \tag{a}$$

- **Hidden regularity:** A standard differential multiplier argument gives

$$h\|\partial_\nu u\|_{L^2(\Gamma_0)} = O(1), \quad h \to 0. \tag{b}$$

- (a) + (b) + **boundary PDE** lead to

$$\|u\|_{L^2(\Gamma_0)} + \underbrace{h\|\nabla_\Gamma u\|_{L^2(\Gamma_0)}}_{\text{Tangential derivatives}} = o(h^{\delta/2}), \quad h \to 0.$$

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- To arrive at a contradiction, deduce that $\|u\|_{L^2(\Omega)} \to 0$ (interior estimate).
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Heuristics for the decay rate and semiclassical scale

Consider a hypothetical strip setting: $\Omega = \mathbb{R}^{d-1} \times (0, l)$, $\Gamma_{0,1} \simeq \mathbb{R}^{d-1}$.
Let
$$\xi = (\xi_\tau, \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R}.$$
(ξ_τ is the tangential dual variable.)

Quasimode PDEs:

$$\begin{cases} (-h^2\Delta - 1)u = f & \text{in } \Omega, \\ (-h^2\Delta_\Gamma + ih - 1)u = -h^2\partial_\nu u + e & \text{on } \Gamma_0, \\ u = 0 & \text{on } \Gamma_1. \end{cases}$$

- Regime $|\xi_\tau| \gg 1$. Elliptic behavior for the interior PDE. Indeed, in **symbols**:
$$|\xi_d|^2 + |\xi_\tau|^2 - 1 \gtrsim |\xi|^2.$$
Consequence: We can locally "solve" the interior PDE in terms of $u|_{\Gamma_0}$ and f.
- Regime $|\xi_\tau| \sim 1$. *Not clear yet.*
- Regime $|\xi_\tau| \ll 1$. Tangential derivatives are negligeable! Reduction to 1D case:
$$\text{(BVP)} \sim \begin{cases} (\partial_t^2 - \partial_x^2)u = 0 & \text{in } (0, l), \\ \partial_t^2 u(0,t) + \partial_t u(0,t) = \partial_x u(0,t) & \text{for } t \in (0, +\infty), \\ u(l,t) = 0 & \text{for } t \in (0, +\infty). \end{cases}$$
Heuristics: The 1D case has known non-uniform energy decay rate $o(t^{-1})$.

Conclusion

Establish energy decay rate $o(t^{-1})$ by analyzing $o(h^3)$-quasimodes.

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First step: boundary estimate

Improved normal derivative estimate

For $o(h^3)$-quasimodes, we can now prove that

$$h\|\partial_\nu u\|_{L^2(\Gamma_0)} = o(1), \quad h \to 0.$$

(o-improvement over the *a priori* estimate.)

Technical ingredients: normal geodesic coordinates near Γ_0 and microlocalization via semiclassical tangential PDOs.

- Regime $|\xi_\tau| \gg 1$.
  1. Elliptic estimates via IPP in the normal direction and microlocal tangential Gårding inequality (cf. Le Rousseau, Lebeau & Robbiano, 2022).
  2. Normal derivate estimate with differential multipliers.
  3. General domains, compute l.o.t and commutators.
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Summary

Control on Γ_0 in tangential and normal directions at suitable orders:

$$\|u\|_{L^2(\Gamma_0)} + h\|\nabla_\Gamma u\|_{L^2(\Gamma_0)} = o(h), \quad h\|\partial_\nu u\|_{L^2(\Gamma_0)} = o(1), \quad h \to 0,$$

for $o(h^3)$-quasimodes.
Final goal: deduce that $\|u\|_{L^2(\Omega)} = o(1)$ as $h \to 0$ and arrive at a contradiction.

- Take advantage of GCC.
  1. Propagation argument for waves with second-order BC?
  2. **Lazy approach:** decoupling argument!

Idea: Find a splitting $u = u_1 + u_2$ s.t. we may:

1. Estimate u_1 (and $\partial_\nu u_1$) by propagating regularity from $u|_{\Gamma_0}$. (**Well-posedness.**)
2. Estimate u_2 by observing $\partial_\nu u_2 = \partial_\nu (u - u_1)$. (**Dynamics.**)

Summary

Control on Γ_0 in tangential and normal directions at suitable orders:

$$\|u\|_{L^2(\Gamma_0)} + h\|\nabla_\Gamma u\|_{L^2(\Gamma_0)} = o(h), \quad h\|\partial_\nu u\|_{L^2(\Gamma_0)} = o(1), \quad h \to 0,$$

for $o(h^3)$-quasimodes.
Final goal: deduce that $\|u\|_{L^2(\Omega)} = o(1)$ as $h \to 0$ and arrive at a contradiction.

- Take advantage of GCC.
  1. Propagation argument for waves with second-order BC?
  2. **Lazy approach:** decoupling argument!

Idea: Find a splitting $u = u_1 + u_2$ s.t. we may:

1. Estimate u_1 (and $\partial_\nu u_1$) by propagating regularity from $u|_{\Gamma_0}$. (**Well-posedness.**)
2. Estimate u_2 by observing $\partial_\nu u_2 = \partial_\nu(u - u_1)$. (**Dynamics.**)

Well-posedness. We would like to set

$$u_1 = (-h^2\Delta_D - 1)^{-1} Du|_{\Gamma_0},$$

with Δ_D Dirichlet Laplacian on Ω and D "Dirichlet control operator".

- **Issue:** Not well-defined for arbitrary h due to resonance!
- **Workaround:** Add a lower-order term to stay away from the spectrum by letting

$$u_1 = (-h^2\Delta_D + ih - 1)^{-1} Du|_{\Gamma_0}.$$

Then, obtain estimates for u_1 and $\partial_\nu u_1$ via differential multipliers.

Procedure related to the well-posedness of the waves under Dirichlet control.

The decoupling argument (I)

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The decoupling argument (II)

Dynamics. We let $u_2 = u - u_1$, so that $u_2|_\Gamma = 0$.

Hautus test for waves with homogeneous Dirichlet BC

Boundary observability under GCC (Bardos, Lebeau & Rauch, 1992) in resolvent form:

$$\|w\|_{L^2(\Omega)} + h\|\nabla w\|_{L^2(\Omega)} \lesssim h^{-1}\|(-h^2\Delta - 1)w\|_{L^2(\Omega)} + h\|\partial_\nu w\|_{L^2(\Gamma_0)},$$

for all $h > 0$ and $w \in \mathrm{dom}(\Delta_D) = H^2(\Omega) \cap H_0^1(\Omega)$.

- Allows us to estimate u_2 in terms of $\partial_\nu u|_{\Gamma_0}$, u_1, $\partial_\nu u_1|_{\Gamma_0}$ and f.
- **Final conclusion and contradiction:** $\|u\|_{L^2(\Omega)} = o(1)$ as $h \to 0$.

Proposition (Optimality)

For all $\varepsilon > 0$, *there exists a solution* u *with initial data* (u_0, u_1) *in* $H^2(\Omega) \times H^1(\Omega)$ *with* $(u_0, u_1)|_{\Gamma_0} \in H^2(\Gamma_0) \times H^1(\Gamma_0)$ *and* $(u_0, u_1)|_{\Gamma_1} = 0$ *(i.e., a classical solution) s.t.*

$$\sup_{t \geq 0} t^{1+\varepsilon} E(t) = +\infty. \tag{1}$$

Idea of the proof: Construct localized quasimodes near Γ_0 based on the 1D problem

$$\begin{cases} (\partial_t^2 - \partial_x^2) u = 0 & \text{in } (0, l), \\ \partial_t^2 u(0,t) + \partial_t u(0,t) = \partial_x u(0,t) & \text{for } t \in (0, +\infty), \\ u(l,t) = 0 & \text{for } t \in (0, +\infty). \end{cases}$$

(See the analysis in the microlocal regime $|\xi_\tau| \ll 1$.)

Thank you for your attention!