Uniqueness and stability in bottom detection through surface measurements of water waves

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Outline

Problem setting (2d)

Consider the motion of an ideal, inviscid, incompressible, and irrotational fluid, bounded below by a solid bottom b and above by a free surface ζ:

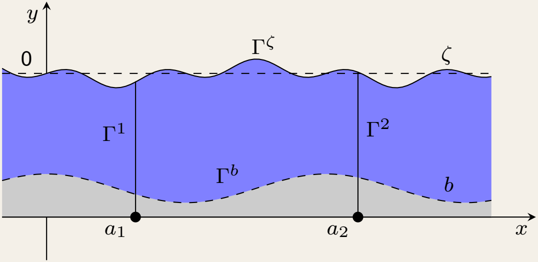

Figure: Scheme for the inverse problem

We aim to establish the uniqueness and stability of detecting any fixed portion (a_1, a_2) of the bottom b from measurements taken on the surface ζ:

► Uniqueness and upper-bound stability without imposing conditions on ζ and b. Furthermore, we avoid assumptions about the velocity of the fluid.

$\ni$ **Joint work with Prof. Lionel Rosier.**

introduction

Water waves system

We define the spatial region confined between the bottom and the free surface as

$$\Omega_t = \{(X,y) \in \mathbb{R}^d \times \mathbb{R},\ b(X) < y < \zeta(t,X)\}, \quad d = 1,2.$$

Then, the motion of the fluid can be modeled as follows:

$$\left\{ \begin{array}{rcr} \partial_t \zeta - G(\zeta,b)\psi & = & 0, \\ \partial_t \psi + g\zeta + \frac{1}{2}\left|\nabla_X \psi\right|^2 + \dfrac{\left(G(\zeta,b)\psi + \nabla_X \zeta \cdot \nabla_X \psi\right)^2}{2\left(1 + |\nabla_X \zeta|^2\right)} & = & 0, \\ \zeta(0,X) = \zeta_0(X),\ \psi(0,X) & = & \psi_0(X), \end{array} \right. \tag{1}$$

where $\psi(t,X) = \phi(t,X,\zeta(t,X))$ is the trace of the velocity potential at the free surface, and G is the Dirichlet-Neumann operator defined by

$$\psi \longrightarrow G(\zeta,b)\psi = \sqrt{1 + |\nabla_X \zeta|^2}\, \partial_n \phi_{|_{y=\zeta}},$$

such that, ϕ is the solution to the elliptic problem

$$\left\{ \begin{array}{rcll} \Delta_{X,y}\phi & = & 0, & \Omega_t, \\ \phi_{|_{y=\zeta}} = \psi,\ \partial_n \phi_{|_{y=b}} & = & 0. & \end{array} \right. \tag{2}$$

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Uniqueness and Stability

Surface measurements

Let $I = (a_1, a_2)$, based on the water wave equations (1)-(2), we assume access to the surface measurements $(\zeta, \partial_n \phi_{|_{\Gamma^\zeta}}, \psi)|_{\{t_0\}\times I}$ along Γ^ζ at time t_0. Additionally, if the walls Γ^1 and Γ^2 are virtual walls, meaning the domain is not physically limited at Γ^1 and Γ^2 we also assume we have the values of the bottom at a_1 and a_2, namely $(b(a_1), b(a_2))$.

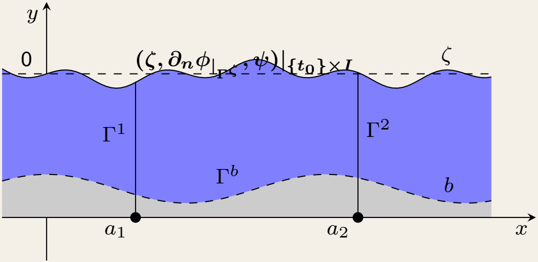

Figure: Surface measurements

- based on the first equation in (1), one can replace the measurements $\left(\zeta, \partial_n \phi_{|_{\Gamma^\zeta}}, \psi\right)|_{\{t_0\}\times I}$ by $\left(\zeta|_{[0,t^*[\times I}, \psi|_{\{t_0\}\times I}\right)$, where $t_0 \in [0, t^*)$.

Uniqueness and Stability

Perturbations

We fix an instant of measurement t_0 and we consider ζ and ψ represent the perturbations resulting from measuring the physical correct values ζ_0 and ψ_0, and let b be the resulting bottom that replaces the exact bottom b_0, such that b, b_0, ζ and ζ_0 be $H^2(\mathbb{R})$, and ψ, ψ_0 in $\dot{H}^{3/2}(\mathbb{R})$, where

$$\dot{H}^{3/2}(\mathbb{R}) = \left\{ \psi \in L^2_{loc}(\mathbb{R}), \quad \partial_x \psi \in H^{1/2}(\mathbb{R}) \right\}.$$

Moreover, we assume that there exist two positive constants $h_0 > 0$ and $h > 0$, such that

$$\zeta_0(x) - b_0(x) \geq h_0 \text{ and } \zeta(x) - b(x) \geq h, \quad \forall x \in \mathbb{R}. \tag{3}$$

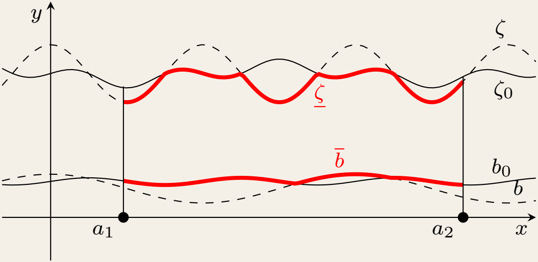

Figure: Scheme for the domains

Uniqueness and Stability

Perturbations

The following two systems are well-posed, and each admits a unique solution with H^2 regularity.

$$\left\{\begin{array}{rcll} \Delta\phi & = & 0, & \text{on } \Omega_b^\zeta, \\ \phi & = & \psi, & \text{in } \Gamma^\zeta, \\ \partial_n\phi & = & 0, & \text{in } \Gamma^b, \end{array}\right. \tag{4}$$

$$\left\{\begin{array}{rcll} \Delta\phi_0 & = & 0, & \text{on } \Omega_{b_0}^{\zeta_0}, \\ \phi_0 & = & \psi_0, & \text{in } \Gamma^{\zeta_0}, \\ \partial_n\phi_0 & = & 0, & \text{in } \Gamma^{b_0}, \end{array}\right. \tag{5}$$

To handle both domains, we introduce the following notations:

$$S_1 = (\zeta - \zeta_0)^{-1}(\mathbb{R}_*^+), \quad S_2 = (\zeta - \zeta_0)^{-1}(\mathbb{R}^-), \tag{6}$$

and we define the following Lipschitz continuous functions:

$$\underline{\zeta}(x) = \min(\zeta(x), \zeta_0(x)), \quad \overline{b}(x) = \max(b(x), b_0(x)), \ \forall x \in I. \tag{7}$$

Uniqueness and Stability

Preliminaries

Consider the following Neumann Laplace problem

$$\begin{cases} \Delta u &= 0\,, \qquad \text{on } \Omega, \\ \partial_n u &= v\,, \qquad \text{in } \partial\Omega. \end{cases} \tag{8}$$

Theorem - Lipschitz propagation of smallness

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ with constants r_0 and M_0, and let $u \in H^1(\Omega)$ be a solution of (8), such that $v \in L^2(\partial\Omega)$. Then, for every $\rho > 0$ and for every $(x, y) \in \Omega_{4\rho}$, where

$$\Omega_{4\rho} = \{(x, y) \in \Omega : \ \ d((x, y), \partial\Omega) > 4\rho\},$$

we have

$$\int_{B_\rho((x,y))} |\nabla u|^2 \geq C_\rho \int_\Omega |\nabla u|^2. \tag{9}$$

Here, C_ρ only depends on $\mu_2(\Omega)$, r_0, M_0, ρ and $\frac{||v||_{L^2(\partial\Omega)}}{||v||_{H^{-1/2}(\partial\Omega)}}$.

Uniqueness and Stability

Preliminaries

Theorem : log-log stability

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ with constants r_0 and M_0. Consider Γ^0 a nonempty open subset of $\partial\Omega$ and $s \in \left(0, \frac{1}{2}\right)$. Then there exists two constants $c > \exp(1)$ and $C > 0$, depends only on Γ^0, Ω and s, such that for any $u \in H^2(\Omega)$, $u \neq 0$ with $\Delta u = 0$ and

$$\|u\|_{H^2(\Omega)} \leq M \text{ and } \|u\|_{L^2(\Gamma^0)} + \|\nabla u\|_{L^2(\Gamma^0)} \leq \frac{M}{2c},$$

where M is a positive constant, we have

$$||u||_{H^1(\Omega)} \leq CM \left[\log\log\left(\frac{M}{||u||_{L^2(\Gamma^0)} + ||\nabla u||_{L^2(\Gamma^0)}}\right)\right]^{-s/2}.$$

Uniqueness and Stability

Preliminaries: Size Estimation Method

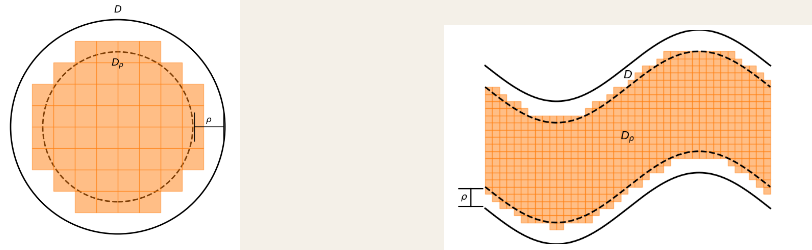

Figure: A cover of D_ρ using internally nonoverlapping closed squares $(q_n)_n$ of size less than $\rho/2$.

$$\int_D |\nabla u|^2 \geq \sum_{q_n} \int_{q_n} |\nabla u|^2 \geq \frac{\displaystyle\int_{q_{n,0}} |\nabla u|^2}{\mu_2(q_{n,0})} \sum_{q_n} \mu_2(q_n) \geq \frac{\displaystyle\int_{q_{n,0}} |\nabla u|^2}{\mu_2(q_{n,0})} \mu_2(D_\rho),$$

where $q_{n,0}$, is such that

$$\min_{q_n \subset Q_n} \int_{q_n} |\nabla u|^2 = \int_{q_{n,0}} |\nabla u|^2 \geq C_\rho \int_\Omega |\nabla u|^2.$$

Uniqueness and Stability

Conditions

We assume that there exist two positive constants $h_0 > 0$ and $h > 0$, such that

$$\zeta_0(x) - b_0(x) \geq h_0 \quad \text{and} \quad \zeta(x) - b(x) \geq h, \quad \forall x \in I. \tag{10}$$

Moreover, we assume

$$|\zeta(x) - \zeta_0(x)| \leq \min\left(\frac{h}{2}, \frac{h_0}{2}\right), \quad \forall x \in I. \tag{11}$$

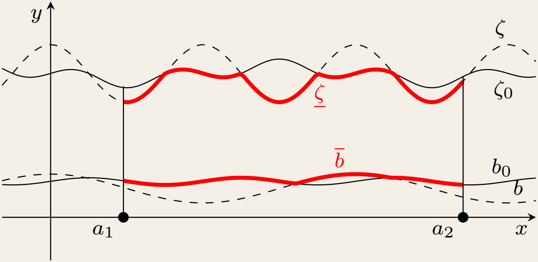

Figure: Scheme for the domains

Uniqueness and Stability

Uniqueness result

Theorem :Uniqueness

Let b, b_0, ζ and ζ_0 to be $H^2\left(\overline{I}\right)$, such that (10) and (11) holds, and consider $\phi \in H^2(\Omega_b^{\zeta})$, $\phi_0 \in H^2(\Omega_{b_0}^{\zeta_0})$ solutions of the systems (4) and (5), respectively. Then, if,

$$\zeta(x) = \zeta_0(x) \neq 0, \;\; \psi(x) = \psi_0(x) \neq 0, \;\; \partial_n \phi_{|_{\Gamma^{\zeta}}}(x) = \partial_n \phi_{0|_{\Gamma^{\zeta_0}}}(x), \quad \forall x \in I, \tag{12}$$

and

$$b(a_1) = b_0(a_1), \;\; b(a_2) = b_0(a_2), \tag{13}$$

we have,

$$b(x) = b_0(x), \quad \forall x \in I.$$

Uniqueness and Stability

Stability upper bound result

Theorem :Stability

Let fix s in the interval $]0,1/2[$. Let b, b_0, ζ and ζ_0 to be $H^2\left(\overline{I}\right)$, such that (10) and (11) holds, and consider $\phi \in H^2(\Omega_b^\zeta)$, $\phi_0 \in H^2(\Omega_{b_0}^{\zeta_0})$ solutions of the systems (4) and (5), respectively. Then, there exists two constant $c > \exp(1)$ and $C > 0$, such that, if,

$$||\phi||_{H^2(\Omega_b^\zeta)} \leq \frac{M}{2}, \ ||\phi_0||_{H^2(\Omega_{b_0}^{\zeta_0})} \leq \frac{M}{2} \text{ and } \sqrt{G_4} + \sqrt{G_5} \leq \frac{M}{2c},$$

where M is a positive constant, we have

$$\|b - b_0\|_{L^1(I)} \leq \frac{1}{\min\left\{\int_{\Omega_b^\zeta} |\nabla\phi|^2, \int_{\Omega_{b_0}^{\zeta_0}} |\nabla\phi_0|^2\right\}} \left(G_2 + G_3 + Tlog_1 + Tbot\right), \tag{14}$$

Stability estimate

Measurements taken over Γ^{ζ} and Γ^{ζ_0}

where G_1, G_2, G_3, G_4 and G_5 are given by:

$$
\begin{aligned}
G_1(\zeta-\zeta_0,\psi-\psi_0,\partial_n\phi|_{\Gamma^{\zeta}}-\partial_n\phi_0|_{\Gamma^{\zeta_0}}) = {} & \|\partial_x\psi-\partial_x\psi_0\|^2_{L^2(I)} + C_{1,1}\left\|\partial_n\phi|_{\Gamma^{\zeta}}-\partial_n\phi_0|_{\Gamma^{\zeta_0}}\right\|^2_{L^2(I)} \\
& +C_{1,1}\|\partial_x\zeta-\partial_x\zeta_0\|^2_{L^\infty(I)} \\
& +C_{1,2}\|\partial_x\psi-\partial_x\psi_0\|_{L^2(I)}\left\|\partial_n\phi|_{\Gamma^{\zeta}}-\partial_n\phi_0|_{\Gamma^{\zeta_0}}\right\|_{L^2(I)} \\
& +C_{1,2}\|\partial_x\zeta-\partial_x\zeta_0\|_{L^\infty(I)}\|\partial_x\psi-\partial_x\psi_0\|_{L^2(I)} \\
& +2C_{1,1}\|\partial_x\zeta-\partial_x\zeta_0\|_{L^\infty(I)}\left\|\partial_n\phi|_{\Gamma^{\zeta}}-\partial_n\phi_0|_{\Gamma^{\zeta_0}}\right\|_{L^2(I)} \\
G_2(\zeta-\zeta_0,\psi-\psi_0) = {} & C_{2,1}\|\zeta-\zeta_0\|^{\frac{1}{2}}_{L^\infty(I)} + C_{2,2}\|\psi-\psi_0\|_{L^2(I)}, \\
G_3\left(\zeta-\zeta_0,\partial_n\phi|_{\Gamma^{\zeta}}-\partial_n\phi_0|_{\Gamma^{\zeta_0}}\right) = {} & C_{3,1}\left\|\partial_n\phi|_{\Gamma^{\zeta}}-\partial_n\phi_0|_{\Gamma^{\zeta_0}}\right\|_{L^2(I)} + C_{3,2}\|\zeta-\zeta_0\|^{\frac{1}{2}}_{L^\infty(I)} \\
& +C_{3,3}\|\partial_x\zeta-\partial_x\zeta_0\|_{L^\infty(I)}, \\
G_4(\zeta-\zeta_0,\psi-\psi_0) = {} & C_{4,1}\|\psi-\psi_0\|^2_{L^2(I)} + C_{4,2}\|\zeta-\zeta_0\|_{L^\infty(I)} \\
& +C_{4,3}\|\zeta-\zeta_0\|^{\frac{1}{2}}_{L^\infty(I)}\|\psi-\psi_0\|_{L^2(I)}, \\
G_5\left(\zeta-\zeta_0,\psi-\psi_0,\partial_n\phi|_{\Gamma^{\zeta}}-\partial_n\phi_0|_{\Gamma^{\zeta_0}}\right) = {} & C_{5,1}\|\zeta-\zeta_0\|_{L^\infty(I)} \\
& +C_{5,2}\|\zeta-\zeta_0\|^{\frac{1}{2}}_{L^\infty(I)}G_1^{\frac{1}{2}} \\
& +C_{5,3}G_1.
\end{aligned}
\tag{15}
$$

Stability estimate

Stability upper bound: $Tlog1$ and $Tbot$

where $Tbot$ and $Tlog_1$ are given by

$$Tbot = ||\partial_n \phi||_{L^2(\Gamma^{\bar{b}}_{b})} ||\phi||_{L^\infty(\Gamma^{\zeta}_{\bar{b}})} \left(\bar{b}(a_1) - b(a_1) + \bar{b}(a_2) - b(a_2)\right)^{\frac{1}{2}}$$
$$+ ||\partial_n \phi_0||_{L^2(\Gamma^{\bar{b}}_{b_0})} ||\phi_0||_{L^\infty(\Gamma^{\zeta}_{\bar{b}_0})} \left(\bar{b}(a_1) - b_0(a_1) + \bar{b}(a_2) - b_0(a_2)\right)^{\frac{1}{2}} . \tag{16}$$

$$Tlog_1 = C \, \|\partial_n \phi\|_{L^2(\Gamma^{\zeta}_{\bar{b}})} M \left[\log\log\left(\frac{M}{\sqrt{G_4} + \sqrt{G_5}}\right)\right]^{-s/2}$$
$$+ 3C||\phi_0||_{L^2(\partial\Omega^{\zeta}_{\bar{b}})} M \left[\log\log\left(\frac{M}{\sqrt{G_4} + \sqrt{G_5}}\right)\right]^{-s^2/2} , \tag{17}$$

- We derived the above theorem using the size estimate method without assuming the fatness condition on the region between the two bottoms, b_0 and b. Moreover, no regularity condition is imposed on the domain between b_0 and b.

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Questions & Answers

Thank you for your attention!
Any questions?