Controllability of the heat equation from very small sets

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Definition

The equation (E) is said to be **null-controllable** in time $T > 0$ if for all $y_0 \in L^2(\Omega)$, there exists a control $u \in L^2((0,T) \times \omega)$ such that $y(T, \cdot) = 0$.

Observability

Consider

$$(E^*) : \begin{cases} (\partial_t - \Delta) z(t,x) = 0 & t \geqslant 0,\ x \in \Omega, \\ z(t,x) = 0 & t \geqslant 0,\ x \in \partial\Omega, \\ z(0,x) = z_0(x) & x \in \Omega. \end{cases}$$

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Proposition (Hilbert Uniqueness Method)

Equation (E) is null-controllable in time $T > 0$ iff the solution of (E^*) satisfies

$$\|z(T)\|_{L^2(\Omega)}^2 \leqslant C_T \int_0^T \int_\omega |z(t,x)|^2 dx dt. \textbf{ (observability inequality)}.$$

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$$\|\pi_k g\|_{L^2(\Omega)} \lesssim e^{Ck^a} \|\pi_k g\|_{L^2(\omega)}, \forall g \in L^2(\Omega) \qquad (\text{spec}(a, \omega, \Omega))$$

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$$\|z(T)\|^2_{L^2(\Omega)} \leqslant C_T \int_0^T \int_\omega |z(t,x)|^2 dxdt$$

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(Equivalently, (E) is null-controllable in any time $T > 0$).

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Let $d = 1$. Then $\pi_k = \mathbb{1}_{-\Delta \leqslant k}$ is the orthogonal projection

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Theorem: Kovrijkine (2001)

If ω is **relatively dense (or thick)** i.e. there exist $r > 0$ and $\gamma > 0$ such that

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If ω is relatively dense, then the heat equation (E) is null-controllable on $\mathbb{R}^d$ in any time. The converse is also true.

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Question: Can we go beyond?

Observability inequality

$$\|z(T)\|^2_{L^2(\Omega)} \leqslant C_T \int_0^T \int_\omega |z(t,x)|^2 dxdt.$$ **(observability inequality)**.

where

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The spectral estimate to prove becomes

$$\forall f \in E_\lambda, \ \|f\|_{L^2(\Omega)} \lesssim e^{C\lambda^a} \sup_{x\in\omega} |f(x)|.$$

for some $a < 1$.

Set of zero Lebesgue measure

Theorem: Burq-Moyano 23'

There exists $0 < \delta < 1$ such that if $\dim_{\mathcal{H}}(\omega) > d - 1 + \delta$ the heat equation is observable in any time $T > 0$ from ω.

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Theorem: Green, Le Balc'h, Martin, O.

If $\dim_{\mathcal{H}}(\omega) > d - 1$ then the heat equation is observable in any time $T > 0$ from ω. This is sharp.

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Lemma (Malinnikova, 2004)

Let $\mathcal{K} \subset\subset B_1 \subset \mathbb{R}^{d+1}$. There exists $\alpha \in (0,1)$ such that for any $\mathcal{E} \subset\subset B_1$ contained in a hyperplane and satisfies $\dim_{\mathcal{H}}(\mathcal{E}) > d-1$, we have for every $u : B_1 \to \mathbb{R}$ satisfying $\Delta u = 0$ in B_1,

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We apply it on $\mathcal{E} = \omega \times \{0\}$ for $\tilde{f}$.

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Then $\dim_{\mathcal{H}}(\omega) = d - 1$.
Moreover $z(t,x) = e^{-\lambda t}\varphi_\lambda(x)$ is the solution of (E^*) with $z_0 = \varphi_\lambda$. But

$$\|z(T)\|^2_{L^2(\Omega)} \leqslant C_T \int_0^T \sup_{x\in\omega} |z(t,x)| dt$$

fails.

Approximate observability

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Proposition

Approximate observability holds from ω iff $\omega \not\subset \mathcal{N}_\lambda$ for any λ.

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Let $x_0 \notin \mathbb{Q}$. Let $T_{x_0} = -\liminf \frac{\log(|\sin(n\pi x_0)|)}{\lambda_n}$.

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Remark

However, the spectral estimate obviously never holds on $\omega = \{x_0\}$. So the Lebeau-Robbiano method is suboptimal.

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Since $|\sin(n\pi x_0)| \approx n \inf_{p\in\mathbb{N}} |x_0 - \frac{p}{n}|$, T_{x_0} is related to the diophantine approximation of x_0.

Examples

- If $x_0 \notin \mathbb{Q}$ and if there exist two sequences of integers (p_k) and (q_k) such that $\left|x_0 - \frac{p_k}{q_k}\right| \leqslant e^{-Cq_k^2}$ then $T_{x_0} = +\infty$.

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- If x_0 is an irrational algebraic number of degree $m \geqslant 2$, then $T_{x_0} = 0$.

In 1-D

Since $|\sin(n\pi x_0)| \approx n \inf_{p\in\mathbb{N}} |x_0 - \frac{p}{n}|$, T_{x_0} is related to the diophantine approximation of x_0.

Examples

- If $x_0 \notin \mathbb{Q}$ and if there exist two sequences of integers (p_k) and (q_k) such that $\left|x_0 - \frac{p_k}{q_k}\right| \leqslant e^{-Cq_k^2}$ then $T_{x_0} = +\infty$.
- If x_0 is an irrational algebraic number of degree $m \geqslant 2$, then $T_{x_0} = 0$. Moreover the observability constant is of the form $e^{C/T}$.

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Due to Müntz theorem!

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For any set $E \subset [0,1]$, define

$$\gamma_k(E) = \sup_{\{x_i\}_{i=1}^k \subset E} \min_i \prod_{j \neq i} |x_i - x_j|. \tag{1}$$

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Example: $\omega_\alpha = \{i^{-\alpha} : \ i \in \mathbb{N}\}$, $\frac{1}{\gamma_k(\omega_\alpha)} \lesssim e^{Ck\log(k)}$. Moreover ω_α is countable so $\dim_{\mathrm{H}}(\omega_\alpha) = 0$.

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- If $\text{spec}(a, \omega_1, \Omega_1)$ holds and the observability estimate holds for (ω_2, Ω_2) with a "good" constant then the observability inequality holds for $(\omega_1 \times \omega_2, \Omega_1 \times \Omega_2)$.

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Remark

Cartesian products of observability estimates fail: $\omega = \{x_0\} \times \{x_0\}$ is not an observation subset of $[0, 1]^2$.

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Corollary:

- $\omega_\alpha \times \omega_\alpha \times \cdots \times \omega_\alpha$ is an observation set on $[0, 1]^d$.
- If x_0 is algebraic then $\{x_0\} \times \omega_\alpha$ is an observation set on $[0, 1]^2$.

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- On the square $[0,1]^2$, can we control from certain non-vertical lines or other curves?
- Geometric condition related to nodal sets for observability?

Thank you!