A model of rigid vortex filament in Euler flows

Olivier Glass

Ceremade, Université Paris-Dauphine PSL

Joint work with David Meyer (Madrid) and Franck Sueur (Luxembourg),

with references to works with C. Lacave (Grenoble), A. Munnier (Nancy) and F. Sueur

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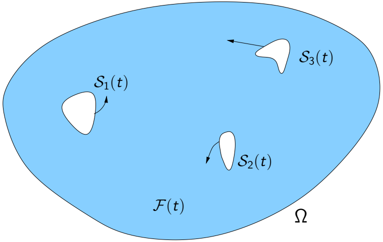
Presentation of the model: a rigid body immersed in an incompressible perfect fluid

- We consider the motion of a rigid body (or N rigid bodies) immersed in an incompressible perfect fluid filling a regular domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$.



- Each solid occupies at each instant $t \geqslant 0$ a closed subset $\mathcal{S}_i(t) \subset \Omega$, $i = 1, \ldots, N$ and the fluid occupies $\mathcal{F}(t) := \Omega \setminus \cup_{i=1}^{N} \mathcal{S}_i(t)$.
- We will work in the absence of collisions:

$$\forall i = 1, \ldots, N, \quad d(\mathcal{S}_i(t), \partial\Omega) > 0 \text{ and } \forall j \neq i, \quad d(\mathcal{S}_i(t), \mathcal{S}_j(t)) > 0.$$

Fluid equation

- In $\mathcal{F}(t)$, the fluid satisfies the Euler equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = 0, \quad t \in [0, T],\ x \in \mathcal{F}(t), \\ \operatorname{div} u = 0 \qquad t \in [0, T],\ x \in \mathcal{F}(t), \end{cases}$$

where

  - $u = u(t, x) : x \in \mathcal{F}(t) \to \mathbb{R}^2$ or $\mathbb{R}^3$ is the fluid velocity,
  - $p = p(t, x) : x \in \mathcal{F}(t) \to \mathbb{R}$ denotes the pressure.

Boundary conditions

- At the boundaries, the fluid satisfies the no-penetration/slip condition :

$$
\begin{gathered}
u \cdot n = 0 \text{ for } x \in \partial\Omega, \\
u \cdot n = V_{\mathcal{S},i} \cdot n \text{ for } x \in \partial\mathcal{S}_i(t),\ \ i = 1, \dots, N,
\end{gathered}
$$

where n is the normal to the boundaries $\partial\Omega$ and $\partial\mathcal{S}_i(t)$, and

$$
\begin{gathered}
V_{\mathcal{S},i}(t,x) = h_i'(t) + \vartheta_i'(t)(x - h_i(t))^{\perp} \ \ (2D), \\
V_{\mathcal{S},i}(t,x) = h_i'(t) + \Omega_i(t) \wedge (x - h_i(t)) \ \ (3D).
\end{gathered}
$$

is the i-th body's velocity, where:

  - $h_i(t)$ is the position of center of mass associated to $\mathcal{S}_i(t)$,
  - $\vartheta_i'(t)$ (resp. $\Omega_i(t)$) is the angular velocity of $\mathcal{S}_i(t)$.

Denoting $R_i(t)$ the linear rotation of the solid with respect to its initial position,

$$
\mathcal{S}_i(t) = h_i(t) + R_i(t)\left(\mathcal{S}_i(0) - h_i(0)\right).
$$

Dynamics of the solids

- The dynamics of the solids are driven by the action of the pressure on their surface:

$$m_i\, h_i''(t) = \int_{\partial \mathcal{S}_i(t)} p\, n\; ds,$$

$$\mathcal{J}_i\, \vartheta_i''(t) = \int_{\partial \mathcal{S}_i(t)} p\, (x - h_i(t))^{\perp} \cdot n\; ds \quad (2D),$$

$$(\mathcal{J}_i(t)\, R_i)'(t) = \int_{\partial \mathcal{S}_i(t)} p\, (x - h_i(t)) \wedge n\; ds \quad (3D).$$

  where
  - $m_i > 0$ is the mass of the i-th body,
  - $\mathcal{J}_i > 0$ (2D) / $\mathcal{J}_i(t) = Q(t)\mathcal{J}_0 Q^*(t) \gg 0$ (3D) denotes its moment of inertia.

- Remark. D'Alembert's paradox does not apply here, because it concerns fluids which are potential in $\mathbb{R}^2$, stationary and constant at infinity. In that case (only), D'Alembert's paradox states that the dynamics of a single solid are not influenced by the fluid.

Vorticity formulation

The system can also be written as

$$\partial_t \omega + (u \cdot \nabla)\omega = 0 \ (2D) \ / \ \partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla) u \ (3D) \ \text{ in } \mathcal{F}(t),$$

and

$$\begin{gathered} \operatorname{curl} u = \omega \ \text{ in } \ \mathcal{F}(t), \quad \operatorname{div} u = 0 \ \text{ in } \ \mathcal{F}(t), \\ \oint_{\Gamma_i(t)} u \cdot \tau \, ds = \oint_{\Gamma_i(0)} u \cdot \tau \, ds = \gamma_i \ \text{ (Kelvin's law)}, \\ u \cdot n = V_{i,\mathcal{S}} \cdot n \ \text{ on } \ \partial \mathcal{S}_i \ \text{ for } \ i = 1, \dots, N, \\ u \cdot n = 0 \ \text{ on } \ \partial \Omega, \end{gathered}$$

where $\Gamma_1, \dots, \Gamma_k$ is a family of curves attached to the solids and generating the fondamental group of $\mathcal{F}(t)$.

This is again coupled with the same solid equations.

References for the Cauchy problem (single body)

- Classical solutions (say at least C^1) solutions with finite energy:
  - Ortega-Rosier-Takahashi (2006) in the full plane.
  - Rosier-Rosier (2009) in the full space.
  - Houot-San Martin-Tucsnak (2010) in a bounded domain in Sobolev spaces.
- Weaker solutions in 2D (Yudovich or DiPerna-Majda type solutions):
  - G.-Sueur (2012)
  - Wang-Xin (2013)

Remark. As for the Euler equation alone, the complete system can be viewed as an equation of geodesics on an infinite dimensional Riemannian manifold, in the spirit of Arnold's work, see also Ebin-Marsden. (G.-Sueur, 2012)

Cauchy problem (2D, Yudovich-type solutions)

Theorem (*Adaptation of* G.-Sueur)

Let $\mathcal{S}_{i,0}$ *be smooth, bounded domains in* Ω*, at positive distance from* $\partial\Omega$ *and one from another, with center of mass* $h_{i,0}$*. For any* $u_0 \in C^0(\overline{\mathcal{F}}_0; \mathbb{R}^2)$, $(h'_{i,0}, \vartheta'_{i,0}) \in \mathbb{R}^{3N}$ *such that*

$$\mathit{div}\, u_0 = 0, \quad \mathit{curl}\ u_0 = \omega_0 \in L^\infty(\mathcal{F}_0),$$
$$u_0 \cdot n = (h'_{i,0} + \vartheta'_{i,0}(x - h_{i,0})^\perp) \cdot n \text{ on } \partial\mathcal{S}_{i,0}, \quad u \cdot n = 0 \text{ on } \partial\Omega,$$

there exists a unique maximal solution of the system

$$(h_i, \vartheta_i, u) \in C^2([0, T^*))^{3N} \times L^\infty([0, T^*); \mathcal{LL}(\mathcal{F}(t))),$$

where $T^* \in (0, +\infty]$ *is the first meeting time between two solids or with the outer boundary* $\partial\Omega$*.*

Here $\mathcal{LL}(U) := \Big\{ f \in C^0(U) \,/\, \exists C > 0,\ \forall x, y \in U, |f(x) - f(y)| \leqslant C|x-y|(1 + \ln^- |x-y|) \Big\}.$

Cauchy problem (3D, Hölder regularity, one solid, $\Omega = \mathbb{R}^3$)

Theorem

Let $\mathcal{S}_0$ *be a smooth bounded domain in* $\mathbb{R}^3$ *and* $\mathcal{F}_0 = \mathbb{R}^3 \backslash \overline{\mathcal{F}_0}$*. Let* $u_0 \in C^{1,\lambda}(\mathcal{F}_0; \mathbb{R}^2)$*,* $(h_0', \Omega_0) \in \mathbb{R}^6$ *such that*

$$\begin{gathered} div\, u_0 = 0, \quad curl\ u_0 = \omega_0 \in C_c^{\lambda}(\overline{\overline{\mathcal{F}_0}}), \\ u_0 \cdot n = (h_0' + \Omega_0 \wedge (x - h_0)) \cdot n \text{ on } \partial\mathcal{S}_0, \quad u \cdot n = 0 \text{ on } \partial\Omega, \end{gathered}$$

there exists a unique maximal solution of the system

$$(h_i', \Omega_i, u) \in C^1([0, T^*))^6 \times C_w([0, T^*); C^{1,\lambda}(\mathcal{F}(t))),$$

where $T^* \in (0, +\infty]$ *is the usual possible blow-up time for the vorticity.*

General question

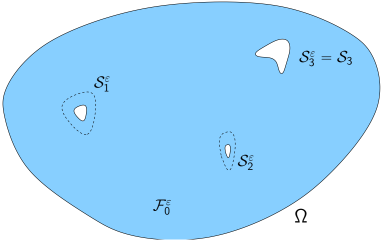
- Can we characterize whan happens when the solid(s) becomes small?
- Motivations:
  - Obtain simplified fluid-solid models. This is particularly interesting in the case of many bodies. . .
  - Give another justification of classical vortex models and obtain new ones.
  - One would like to study control problems associated to a fluid-rigid body system (cf. for instance G.-Kolumban-Sueur (2019), Dorsz-G. (2023))
- The problem is originally connected to a paper by Iftimie-Lopes-Nussnzweig Lopes (2006) on a fixed obstacle in 2D.

2D: The "bodies converging to particles" problem

- Let us be given $\mathcal{S}_{i,0}$ smooth, etc., with center of mass $h_{i,0}$, and let $h'_{i,0}$, $\vartheta'_{i,0}$, γ_i, $\omega_0 \in L^\infty(\Omega)$ fixed as above.
- Question. What can be said for when a part of the solids become small, that is, how the solution behaves when the initial position of the solids are:

$$\mathcal{S}^{\varepsilon}_{i,0} := h_{i,0} + \varepsilon_i(\mathcal{S}_{i,0} - h_{i,0}),$$

as $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_K, \varepsilon_{K+1}, \ldots, \varepsilon_N)$ goes to $(1, \ldots, 1, 0, \ldots, 0)$?



Three regimes for the solids

- We will be interested in the following three regimes for the solids:
  - Fixed size solids:
  $$m_i^\varepsilon = m_i^1 \text{ and } \mathcal{J}_i^\varepsilon = \mathcal{J}_i^1,$$
  say for $i \in \mathcal{P}_f$.
  - Massive shrinking solids:
  $$m_i^\varepsilon = m_i^1 \text{ and } \mathcal{J}_i^\varepsilon = \varepsilon_i^2 \mathcal{J}_i^1,$$
  say for $i \in \mathcal{P}_m$.
  - Massless ("light") shrinking solids (e.g. fixed density):
  $$m_i^\varepsilon = \varepsilon_i^{\alpha_i} m_i^1 \text{ and } \mathcal{J}_i^\varepsilon = \varepsilon_i^{2+\alpha_i} \mathcal{J}_i^1,$$
  for $\alpha_i > 0$, say for $i \in \mathcal{P}_\ell$. Typically, a fixed density means $\alpha_i = 2$.

A question and three assumptions

- Question: given all the data above, we can define for all $\varepsilon \in (0,1]^N$ a solution $(h_i^\varepsilon, \vartheta_i^\varepsilon, u^\varepsilon)$ to the system.
  Does it have a limit as $\varepsilon \to (1,\dots,1,0,\dots,0)$, and can one characterize it?

We give an answer on this question with the following assumptions

- Assumption 1: $\omega_0 \in L_c^\infty(\Omega \backslash \{h_{K+1,0},\dots,h_{N,0}\})$. Hence for small ε and $i \geqslant K+1$,
  $$\operatorname{dist}\big(\operatorname{Supp}(\omega_0), \mathcal{S}_{i,0}^\varepsilon\big) > 0.$$
- Assumption 2: for $i \in \mathcal{P}_\ell$ (solids with mass tending to 0), $\gamma_i \neq 0$.
- Assumption 3: the solids are not balls.

Convergence result (2D)

Theorem (G.-Sueur, following works with Lacave, Munnier & Sueur)

Under the above assumptions consider for each scale factor ε sufficiently small the corresponding maximal solution $(h_i^\varepsilon, \vartheta_i^\varepsilon, u^\varepsilon)$ on $[0, T^\varepsilon)$.
Then the existence times T^ε satisfy

$$T^\varepsilon \geqslant T,$$

for some $T > 0$ and as $\varepsilon \to 0^+$, extending all fields by 0 in the solids, one has up to a subsequence,

$$\begin{aligned}
u^\varepsilon &\longrightarrow u^\star \text{ in } C^0([0,T]; L^q(\Omega)) \text{ for } q \in [1,2), \\
\omega^\varepsilon &\longrightarrow \omega^\star \text{ in } C^0([0,T]; L^\infty(\Omega) - w\star), \\
h_i^\varepsilon &\longrightarrow h_i^\star \text{ in } \begin{cases} W^{2,\infty}(0,T) \text{ weak} - \star \text{ for } i \in \mathcal{P}_f \cup \mathcal{P}_m, \\ W^{1,\infty}(0,T) \text{ weak} - \star \text{ for } i \in \mathcal{P}_\ell, \end{cases} \\
\vartheta_i^\varepsilon &\longrightarrow \vartheta_i^\star \text{ in } W^{2,\infty}(0,T) \text{ weak} - \star \text{ for } i \in \mathcal{P}_f,
\end{aligned}$$

Limit system for the fluid

The limit fluid domain is

$$\mathcal{F}^\star(t) = \Omega \setminus \bigcup_{i=1}^{K} \mathcal{S}_i^\star(t),$$

and the limit velocity field $u^\star$ satisfies

$$\begin{cases} \operatorname{div} u^\star = 0 \quad \text{in} \quad \mathcal{F}^\star(t), \\ \operatorname{curl}\, u^\star = \omega^\star + \displaystyle\sum_{i \in \mathcal{P}_\ell \cup \mathcal{P}_m} \gamma_i \delta_{h_i^\star} \quad \text{in} \quad \mathcal{F}^\star(t), \\ u^\star \cdot n = 0 \quad \text{on} \quad \partial\Omega, \\ u^\star \cdot n = ({h^\star}'_i + {\vartheta^\star}'_i (x - h_i^\star)^\perp) \cdot n \quad \text{on} \quad \partial\mathcal{S}_i^\star(t) \quad \text{for} \quad i = 1, \dots, K, \\ \oint_{\partial\mathcal{S}_i^\star(0)} u^\star \cdot \tau \, ds = \gamma_i \quad \text{for} \quad i = 1, \dots, K, \end{cases}$$

with the transport equation for the vorticity:

$$\partial_t \omega^\star + \operatorname{div}(u^\star \omega^\star) = 0 \quad \text{in} \quad [0, T] \times \Omega.$$

Limit system for large solids

The solids in the limit satisfy for $i \in \mathcal{P}_f$:

$$m_i\, h_i^{\star\prime\prime}(t) = \int_{\partial \mathcal{S}_i^\star(t)} p^\star\, n\, ds,$$

$$\mathcal{J}_i\, \vartheta_i^{\star\prime\prime}(t) = \int_{\partial \mathcal{S}_i^\star(t)} p^\star\, (x - h_i^\star(t))^\perp \cdot n\, ds,$$

where the pressure $p^\star$ can be defined in the fluid domain away from the point vortices by

$$\nabla p^\star = -(\partial_t u^\star - (u^\star \cdot \nabla) u^\star) \;\text{ in }\; \mathcal{F}^\star(t) \setminus \bigcup_{i \in \mathcal{P}_m \cup \mathcal{P}_\ell} \{h_i^\star(t)\}.$$

Limit system for shrinking solids

- Desingularized velocity. For $i \in \mathcal{P}_\ell \cup \mathcal{P}_m$, we introduce the "desingularized version" $\widetilde{u}_i^\star$ of $u^\star$ at $h_i^\star$ defined by

$$\widetilde{u}_i^\star(x) = u^\star(x) - \frac{\gamma_i}{2\pi} \frac{(x - h_i)^\perp}{|x - h_i|^2}.$$

- Light small solids. For $i \in \mathcal{P}_\ell$, we obtain in the limit

$$h_i^{\star\prime} = \widetilde{u}_i^\star(h_i^\star) \quad \text{in} \quad [0, T] \quad \text{for} \quad i \in \mathcal{P}_\ell,$$

- Massive small solids. while for $i \in \mathcal{P}_m$, we obtain in the limit

$$m_i h_i^{\star\prime\prime} = \gamma_i \big[h_i^{\star\prime} - \widetilde{u}_i^\star(h_i^\star)\big]^\perp \quad \text{in} \quad [0, T] \quad \text{for} \quad i \in \mathcal{P}_m,$$

Related models 1. The wave/vortex system

- When $\mathcal{P}_m \cup \mathcal{P}_f = \varnothing$, this is a version of Marchioro and Pulvirenti's wave/vortex system, written as follows in $\mathbb{R}^2$:

$$\frac{\partial \omega}{\partial t} + \operatorname{div}\left(K[\omega + \gamma \delta_{h(t)}]\omega\right) = 0 \ \text{ in } \ [0,T] \times \mathbb{R}^2,$$
$$h'(t) = K[\omega(t,\cdot)](h(t)),$$

  where K is the usual Biot-Savart operator.

- It can be obtained as limits of regular solutions of the Euler equation (as some vorticity concentrates in h_0)

- If $h_0 \notin \operatorname{Supp} \omega_0$, then one has global existence and uniqueness, and $h(t) \notin \operatorname{Supp} \omega(t)$ for all t.

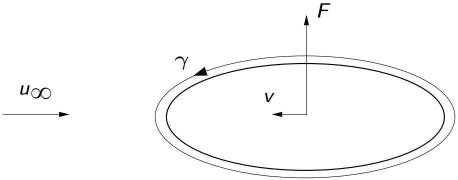
- References: Marchioro-Pulvirenti, Lacave-Miot, Bjorland, etc.

Related models 2. Kutta-Joukowski lift force

- The force appearing in the equation of the point in the limit

$$mh''(t) = \gamma\Big(h'(t) - \widetilde{u}^{\star}(h(t))\Big)^{\perp},$$

  is similar to the lift force similar to the Kutta-Joukowski force of the irrotational theory:



  the force applied to the body at speed v, with fluid velocity u_∞ at infinity and circulation γ around the body is

$$F = \gamma(v - u_\infty)^{\perp}.$$

Related models 3. Vortex points system

- The following classical system of ordinary differential equations:

$$\frac{dx_i}{dt}(t) = \sum_{j=1, j\neq i}^{N} \frac{\gamma_j}{2\pi} \frac{(x_i(t) - x_j(t))^{\perp}}{|x_i(t) - x_j(t)|^2}, \quad i = 1, \ldots, N.$$

  is called the *vortex points system*.
- It was originally introduced as a simplified model for the Euler equation (Helmholtz, Kirchhoff, Kelvin and Poincaré), where the vorticity is concentrated in a finite number of points.
- There are rigourous proofs of the limit of the solutions of Euler equations to the point vortex system (see e.g. Marchioro & Pulvirenti's book)

Controlled point vortex system

- We consider the point vortex system:

$$\frac{dx_i}{dt}(t) = \sum_{j=1, j\neq i}^{N} \frac{\gamma_j}{2\pi} \frac{(x_i(t) - x_j(t))^{\perp}}{|x_i(t) - x_j(t)|^2}, \quad i = 1, \ldots, N,$$

controlled by one of them.

- We are led to consider:

$$\frac{dx_i}{dt}(t) = \sum_{j=1, j\neq i}^{N} \frac{\gamma_j}{2\pi} \frac{(x_i(t) - x_j(t))^{\perp}}{|x_i(t) - x_j(t)|^2} + \frac{\gamma^c}{2\pi} \frac{(x_i(t) - y(t))^{\perp}}{|x_i(t) - y(t)|^2}, \quad i = 1, \ldots, N,$$

where we have N point vortices $x_1, \ldots, x_N$ and a control vortex y, whose trajectory we can choose in order to influence the others.

Main control result

Our result on the control of the point vortex system is the following.

Theorem (Dorsz-G.)

Suppose $\gamma^c \neq 0$. The point vortex system is exactly controllable in arbitrary time by means of a single control vortex.

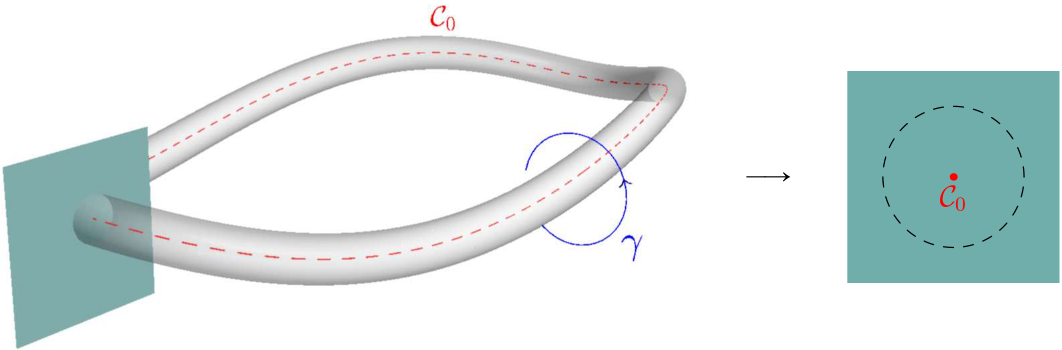
More precisely, given $T > 0$, $(x_{1,0}, \dots, x_{N,0}, y_0) \in (\mathbb{R}^2)^{N+1}$, $(x_{1f}, \dots, x_{Nf}, y_f) \in (\mathbb{R}^2)^{N+1}$, there exists $y \in C^\infty([0,T];\mathbb{R}^2)$ satisfying $y(0) = y_0$, $y(T) = y_f$, and such that the corresponding solution of the system is defined in $[0,T]$ and satisfies

$$(x_1(T), \dots, x_N(T)) = (x_{1,f}, \dots, x_{N,f}).$$

3D: the solid vortex problem

- We consider a small tube around a regular simple curve $\mathcal{C}_0$ in $\mathbb{R}^3$.
- Given $\mathcal{C}_0$ and $\varepsilon > 0$ we consider indeed

$$\mathcal{S}_0^\varepsilon := \left\{ x \in \mathbb{R}^3 \;\middle/\; d(x, \mathcal{C}_0) \leqslant \varepsilon \right\}.$$



- We let γ a curve turning around $\mathcal{C}_0$ (once), so that their linking number is 1.

3D: the solid vortex problem

- The problem is a follows. We fix an initial vorticity ω_0 (compactly supported away from $\mathcal{C}_0$), a circulation μ along γ and an initial velocity $V_{\mathcal{S},0}$ for the solid.
- Then for each $\varepsilon > 0$ we have a corresponding initial velocity field for the fluid (determined as to fulfil the boundary conditions).
- This initial condition generates a (local in time) solution $(V_{\mathcal{S}}^{\varepsilon}, u^{\varepsilon})$ in $[0, T_{\varepsilon}^{\star})$.
- The question is: can we characterize the system in the limit?
- Our main result is obtained in the case of a massive filament:

$$m^{\varepsilon} = m \quad \text{and} \quad \mathcal{J}_0^{\varepsilon} = \mathcal{J}_0.$$

Standard vortex filaments

- The study of evolution of vortex filaments is a classical (but quite difficult!) topic, that goes back to Helmoltz (1858).
- It was formally shown by da Rios (1906) that if one considers a solution whose vorticity concentrates in a tube along a Jordan curve (in the direction of the tangent), then, up to a time-rescaling $t \mapsto t/|\log \varepsilon|$ where ε is the radius of the cross-section, then one obtains in the limit a curve which obeys the binormal flow equation:

$$\partial_t \chi + \chi_x \wedge \chi_{xx} = 0.$$

- Recent rigourous results in the field are Jerrard-Seis (2017), Dávila-del Pino-Musso-Wei (2024).

Change of reference frame

- It is a bit easier to state the problem in the solid's reference frame.

- Setting

$$\ell(t) := R(t)^T \, h'(t),$$

$$u_{\mathcal{S}}(t,x) := \ell(t) + \Omega(t) \wedge x,$$

$$u(t,x) := R(t)^T \, v(t, R(t)x + h(t)) \;\text{ and }\; \pi(t,x) := q(t, R(t)x + h(t)),$$

the system becomes

$$\partial_t u + (u - u_{\mathcal{S}}) \cdot \nabla u + \Omega \wedge u + \nabla \pi = 0 \;\text{ and }\; \operatorname{div} u = 0 \;\text{ for }\; x \in \mathcal{F}_0,$$

$$u \cdot n = u_{\mathcal{S}} \cdot n \;\text{ for }\; x \in \partial\mathcal{S}_0,$$

$$m\ell' = \int_{\partial\mathcal{S}_0} \pi n \, d\sigma + m\ell \wedge \Omega,$$

$$\mathcal{J}_0 \Omega' = \int_{\partial\mathcal{S}_0} \pi (x \wedge n) \, d\sigma + (\mathcal{J}_0 \Omega) \wedge \Omega.$$

Notations

- We let

$$p = \begin{pmatrix} \ell \\ \Omega \end{pmatrix},$$

$$\mathcal{M}_g := \begin{pmatrix} m\,\mathrm{Id}_3 & 0 \\ 0 & \mathcal{J}_0 \end{pmatrix} \quad \text{and} \quad \langle \Gamma_g, p, p \rangle := -\begin{pmatrix} m\ell \wedge \Omega \\ (\mathcal{J}_0 \Omega) \wedge \Omega \end{pmatrix}.$$

- We introduce the Biot-Savart kernel

$$K_{\mathbb{R}^3}[\omega](x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \wedge \omega(y)\, dy.$$

and the vortex filament velocity

$$H^* := K_{\mathbb{R}^3}\left[\tau\, \mathcal{H}^1 \llcorner \mathcal{C}_0\right],$$

where τ is the unit tangent along $\mathcal{C}_0$.

Notations, 2

- We introduce the (classical) area vector and the (more exotic) volume vector associated to $\mathcal{C}_0$ as follows:
$$\mathcal{A}_0 := \frac{1}{2} \int_{\mathcal{C}_0} x \wedge \tau d\sigma(x) \in \mathbb{R}^3 \quad \text{and} \quad \mathcal{V}_0 := \int_{\mathcal{C}_0} |x|^2 \tau d\sigma(x) \in \mathbb{R}^3.$$
- We then define the 6×6 skew-symmetric matrix B^* as follows:
$$B^* := \begin{pmatrix} 0 & -\mathcal{A}_0 \wedge \cdot \\ \mathcal{A}_0 \wedge \cdot & \mathcal{V}_0 \wedge \cdot \end{pmatrix}.$$
- Finally, we introduce the following term:
$$\mathcal{D}^*[u, \omega] := \left(\int_{\mathbb{R}^3} [\zeta_i, \omega, u] \, dx \right)_{1 \leqslant i \leqslant 6},$$
where
$$\zeta_i(x) := \begin{cases} e_i & \text{if } i = 1, 2, 3, \\ e_{i-3} \wedge x & \text{if } i = 4, 5, 6. \end{cases}$$

Limit system

- We obtain in the limit the following system:

$$\partial_t \omega^* + \big((u^* - u^*_{\mathcal{S}}) \cdot \nabla\big)\omega^* = \big(\omega^* \cdot \nabla\big)(u^* - u^*_{\mathcal{S}}) \quad \text{in} \quad \mathbb{R}^3,$$

$$u^* = \mu H^* + K_{\mathbb{R}^3}[\omega^*],$$

$$u^*_{\mathcal{S}}(t, x) := \ell^*(t) + \Omega^*(t) \wedge x,$$

$$\mathcal{M}_g (p^*)' + \langle \Gamma_g, p^*, p^* \rangle = \mu B^* p^* + \mathcal{D}^*[u^*, \omega^*],$$

where $p^* := (\ell^*, \Omega^*) \in \mathbb{R}^3 \times \mathbb{R}^3$.

Cauchy problem for the limit system

Proposition (G.-Meyer-Sueur)

Let ω_0 *in* $C^{k,\alpha}(\mathbb{R}^3;\mathbb{R}^3)$*, divergence-free and with compact support in* $\mathbb{R}^3\backslash\mathcal{C}_0$ *and consider other data as before.*
Then there exists $T^* > 0$ *and a unique maximal solution* (p^*,ω^*) *of the system with* p^* *in* $C^1([0,T^*);\mathbb{R}^6)$ *and* ω^* *in* $C^0([0,T^*);C^{k,\alpha}(\mathbb{R}^3;\mathbb{R}^3)-w*)$ *with for any* $t\in[0,T^*)$*,* $dist(\mathrm{Supp}(\omega^*(t)),\mathcal{C}_0) > 0$*.*

Convergence result (3D)

Theorem (G.-Meyer-Sueur)

Consider $\mathcal{C}_0$, $m > 0$, $\mathcal{J}_0$ *symmetric positive definite,* $\mu \in \mathbb{R}$, $(h'_0, \Omega_0) \in \mathbb{R}^6$ *and* ω_0 *in* $C^{0,\alpha}(\mathbb{R}^3; \mathbb{R}^3)$ *compactly supported away from* $\mathcal{C}_0$ *fixed. Consider* B^* *as introduced before. Let* T^* *the maximal existence time for the limit system and* (p^*, ω^*) *its solution.*

Let, for each $\varepsilon > 0$, $T^\varepsilon > 0$ *and* $(p^\varepsilon, u^\varepsilon, \omega^\varepsilon)$ *the unique smooth solution of the system.*

Then $\liminf_{\varepsilon \to 0} T^\varepsilon > T^*$ *and for all* $T \in (0, T^*)$ *and as* $\varepsilon \to 0^+$,

$$p^\varepsilon \longrightarrow p^* \quad in\ W^{1,\infty}([0, T]; \mathbb{R}^6) - w\star,$$

$$\omega^\varepsilon \longrightarrow \omega^* \quad in\ L^\infty([0, T]; C^{1,\alpha}(\mathbb{R}^3 \backslash \mathcal{C}_0) - w^*).$$

Non convergence in the non-massive case

- If we suppose that the solid's density is constant as $\varepsilon \to 0^+$:

  $$m^\varepsilon = \varepsilon^2 m \quad \text{and} \quad \mathcal{J}_0^\varepsilon = \varepsilon^2 \mathcal{J}_0,$$

  then the convergence fails.

Theorem (G.-Meyer-Sueur)

Let $\mathcal{C}_0$ *a horizontal circle. There exists a smooth, compactly supported* ω_0 *with* $\operatorname{dist}(Supp\, \omega_0, \mathcal{C}_0) > 0$*, an initital* $p_0 = (\ell_0, r_0)$ *and a* $T_0 > 0$*, such that in all solutions for small* $\varepsilon > 0$*, the vorticity does not blow up in* $[0, T_0]$ *and in the* $\varepsilon \searrow 0$ *limit, we have*

$$p^\varepsilon(t) \cdot e_3 \geqslant \varepsilon^{-\frac{1}{10}},$$

for all $t \in (\varepsilon, T_0)$ *for all small enough* ε*. In particular, the body travels a distance* $\gtrsim \varepsilon^{-\frac{1}{10}}$ *in the original frame.*

Difficulties

$$m_i\, h_i''(t) = \int_{\partial \mathcal{S}_i(t)} p\, n \; ds,$$

$$(\mathcal{J}_i(t)\, R_i)(t) = \int_{\partial \mathcal{S}_i(t)} p\,(x - h_i(t)) \wedge n \; ds.$$

- We have to study the pressure in details. It contains in particular acceleration terms which makes the system quasilinear and the solid equations strongly coupled.
- The problem is singular in space since $\mathcal{S}_i^\varepsilon$ shrinks to a curve and the circulation remains constant.
- The energy is not bounded as $\varepsilon \to 0$ ($\rightsquigarrow$ modulated energy estimates is needed).

Some ideas of the proof (case of a single solid)

1. The added mass effect (2D)

- One introduces Kirchhoff's potentials $\Phi_j = \Phi_j(h, \vartheta, x)$ for $j = 1, 2, 3$:

$$\begin{aligned} \Delta\Phi_j &= 0 \ \text{ in } \ \mathcal{F}, \\ \partial_n\Phi_j &= 0 \text{ on } \partial\Omega \\ \partial_n\Phi_j &= \left\{ \begin{array}{l} n_j \quad (j = 1, 2), \\ (x - h(t))^\perp \cdot n \quad (j = 3), \end{array} \right. \qquad \text{on } \ \partial\mathcal{S}. \end{aligned}$$

- The solid equations become

$$\begin{aligned} \begin{pmatrix} m\mathrm{Id}_2 & 0 \\ 0 & \mathcal{J} \end{pmatrix} \begin{pmatrix} h'' \\ \vartheta'' \end{pmatrix} &= \left(\int_{\mathcal{F}} \nabla p \cdot \nabla\Phi_j \, dx \right)_{j=1,2,3} \\ &= -\left(\int_{\mathcal{F}} (\partial_t u + (u \cdot \nabla)u) \cdot \nabla\Phi_j \, dx \right)_{j=1,2,3} \end{aligned}$$

- One decomposes u as

$$u = \underbrace{h_1' \nabla \Phi_1 + h_2' \nabla \Phi_2 + \vartheta' \nabla \Phi_3}_{u^{pot}} + u^{tan},$$

  and injects in the equation.

- We obtain:

$$\begin{pmatrix} m\mathrm{Id}_2 & 0 \\ 0 & \mathcal{J} \end{pmatrix} \begin{pmatrix} h'' \\ \vartheta'' \end{pmatrix} = - \left(\int_{\mathcal{F}} \nabla \Phi_j \cdot \nabla \Phi_k \right)_{j,k=1,2,3} \begin{pmatrix} h'' \\ \vartheta'' \end{pmatrix}$$

$+$ terms with shape-derivatives of Kirchhoff's potentials

$+$ lower-order terms (in time).

- Hence we get
$$\mathcal{M}\begin{pmatrix} h'' \\ \vartheta'' \end{pmatrix} = \text{ lower-order terms,}$$
where
$$\mathcal{M} := \begin{pmatrix} m\mathrm{Id}_2 & 0 \\ 0 & \mathcal{J} \end{pmatrix} + \underbrace{\left(\int_{\mathcal{F}(t)} \nabla\Phi_i \cdot \nabla\Phi_j \; dx \right)_{i,j=1,2,3}}_{=:\mathcal{M}_a(h,\vartheta)} .$$
- The matrix $\mathcal{M}_a$ is a matrix of added inertia, expressing how the fluid opposes the movement of the solid. It is positive, and even positive definite when $\mathcal{S}_0$ is not a disk, as a Gram matrix of independent functions.

2. Treat the space singularity

- One defines the circulation vector field H^ε in $\mathbb{R}^d \backslash \mathcal{S}$ as the solution of

$$\begin{gathered}
\operatorname{div} H^\varepsilon = 0 \text{ in } \mathbb{R}^d \backslash \mathcal{S}, \\
\operatorname{curl}\ H^\varepsilon = 0 \text{ in } \mathbb{R}^d \backslash \mathcal{S}, \\
\nabla H^\varepsilon \to 0 \text{ at infinity}, \\
\int_{\partial \Gamma^\varepsilon(t)} H^\varepsilon \cdot \tau \, ds = 1,
\end{gathered}$$

  where $\Gamma^\varepsilon(t)$ generates the fondamental group of $\mathbb{R}^d \backslash \mathcal{S}^\varepsilon(t)$. One observes that $H^\varepsilon = \mathcal{O}(1/\varepsilon)$.

- One decomposes u^ε in

$$u^\varepsilon = \gamma H^\varepsilon + u^{reg}.$$

  and injects in the equation.

- The most singular terms in the right-hand side are

$$-\gamma \int_{\mathcal{F}^\varepsilon} \partial_t H^\varepsilon \cdot \nabla \Phi_j \, dx \quad \text{and} \quad -\frac{1}{2}\gamma^2 \int_{\mathcal{F}^\varepsilon} \nabla |H^\varepsilon|^2 \cdot \nabla \Phi_j \, dx$$

- One uses the relations

$$\partial_t H^\varepsilon + \nabla(v_{\mathcal{S}} \cdot \nabla^\perp \widehat{\psi}) = 0 \text{ and } \int_{\partial \mathcal{S}} |H^\varepsilon|^2 K_i \, ds = 0,$$

  which allow to get rid of the most singular terms.

- However there remains linear terms in H^ε which still give too a strong contribution (think of light solids).

- One would like to perform an energy estimate in which these terms give only a mild strong contribution (we will say that they are either weakly nonlinear or gyroscopic).
- This is connected to Berkowitz and Gardner's analysis of a light charged particle in a strong electromagnetic field.
- The idea to consider a modulated variable

  $$\tilde{p} = (h', \vartheta') - \mathsf{modulation}(\varepsilon, q, p, u^{\varepsilon}),$$

  where

  $$q = (h, \vartheta) \quad \text{and} \quad p = (h', \vartheta'),$$

  which makes the equation more suited to (modulated) energy estimates. This requires to give a normal form to the equation.

A *very* simple example of modulated energy estimate

- Consider a light 2D charged particle in a steady electromagnetic field (B is orthogonal to the plane of study, E is in it):

$$\varepsilon h'' = E(h) + b(h) h'^{\perp}.$$

- A *direct* energy estimate (multiplying by h') does not work.
- Set $\tilde{p} := h' - \frac{E(h)^{\perp}}{b(h)}$. You get

$$\varepsilon \tilde{p}' = b(h) \tilde{p}^{\perp} - \varepsilon (h' \cdot \nabla) \left(\frac{E(h)^{\perp}}{b(h)} \right).$$

- Here multiplying by $\tilde{p}$ (and using Gronwall's inequality) gives you uniform estimates on $\tilde{p}$ and allows you to find a limit equation!

3. About the the normal form

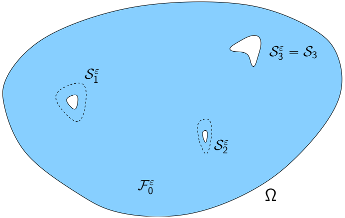
- One decomposes
$$u^{\varepsilon}(t,x) = u^{pot} + \gamma \nabla^{\perp} \widehat{\psi} + u^{ext},$$
- One injects in the equation and develops in powers of ε the Kirchhoff potentials and the Biot-Savart operator.

  (One also need some information concerning their shape-derivatives.)

- The exterior field u^{ext} is the natural source of the modulation.
- Then one has to analyze precisely the various terms arising in the equation. . .

- The same normal form will be used to pass to the limit.

4. Expansions of the potentials



Fluid state in $\Omega \setminus \cup_{i=1}^{N} \mathcal{S}_i^\varepsilon(t)$

$=$ Fluid state as if ~~$\partial\Omega$~~ and $\mathcal{S}_1^\varepsilon, \dots, \mathcal{S}_N^\varepsilon$ were invisible one to another

$+$ Correction as if ~~$\mathcal{S}_i^\varepsilon(t)$~~

$+$ Correction(Correction) as if ~~$\partial\Omega$~~ and $\mathcal{S}_i$ were invisible one to another

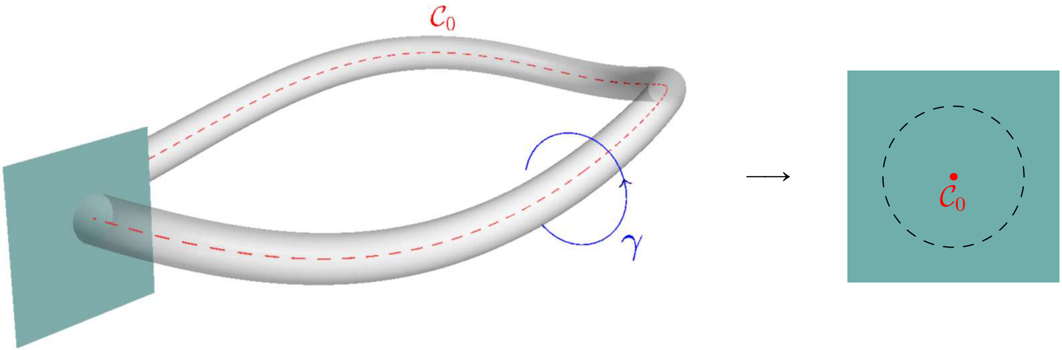
$+$ ...

Additional difficulty when there are solids of fixed-size. . .

Circulation stream function in the 3D case

An important argument in the 3D case is that in each section orthogonal to the center curve, the circulation vector field H^ε can be well approximated by its 2D counterpart

$$H^\varepsilon(x) \simeq \frac{1}{2\pi}\frac{(x-h)^\perp}{|x-h|^2} \text{ on the section passing through } h \in \mathcal{C}_0.$$



Thank you for your attention!