UNIVERSITÀ
DEGLI STUDI
DELL'AQUILA

DISIM
Dipartimento di Ingegneria
e Scienze dell'Informazione
e Matematica

Dynamic Stabilization of Connected-Strings System with Dynamical Interior Mass: Unveiling the Role of Higher-Order Nodal Damping

(Joint work with: M. Akil (Université Polytechnique Hauts-de-France), A. O. Özer (Western Kentucky University) and C. Pignotti (UnivAQ))

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Calais, France

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Outline

1. Literature and Motivation
2. Physical Significance
3. Stability Results
4. Numerical Experiments
5. (N+1)-strings with N-masses

Two serially connected strings with an intermediate dynamic mass

$$\begin{cases} \rho_1 w^1_{tt} - \alpha_1 w^1_{xx}(x,t) = 0, & (-\ell_1, 0) \times \mathbb{R}^+_*, \\ \rho_2 w^2_{tt} - \alpha_2 w^2_{xx}(x,t) = 0, & (0, \ell_2) \times \mathbb{R}^+_*, \end{cases} \tag{1}$$

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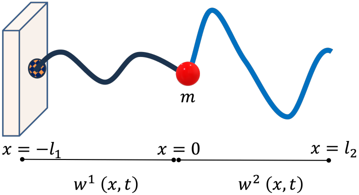

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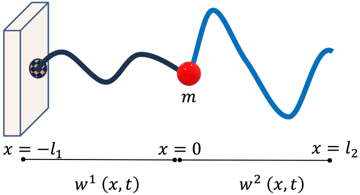

Two strings with a dynamic interior mass

Defining the dynamic coupling of m at the transmission point $x = 0$

$$\eta(t) := b_0(\alpha_1 w^1_x - \alpha_2 w^2_x)(0,t) + m z_t(t).$$

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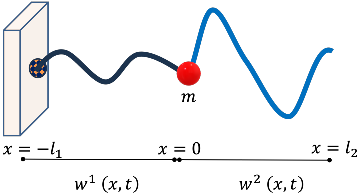

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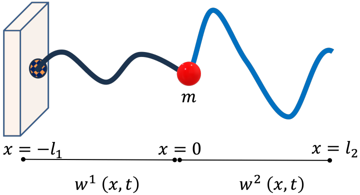

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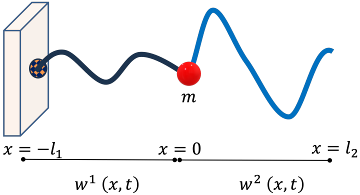

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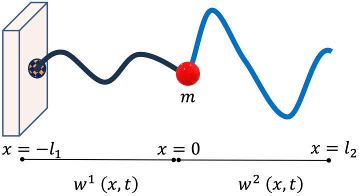

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- ρ_i and α_i: the mass density and stiffness constants of the ith string respectively.
- The coupling at $x = 0$ is characterized by displacement continuity and the shared dynamic variable $z(t)$, which represents the displacement of an attached mass $m > 0$.

Literature and Motivation

- For a finite vibrating string with zero Dirichlet condition at the left end point, velocity feedback at the right end point, and no point mass, it is known that the energy decays exponentially.

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- **W. Littman and S. W. Taylor 2002** [3]:
  They refined the stability analysis where it was shown that the decay rate is polynomial, specifically of order $\frac{1}{t}$.

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→ Exponential stability was established using Lyapunov function and the energy multiplier method.
→ When $b_1 = \frac{m}{b_0} = 1$, an exact decay rate of $\frac{1}{b_0}$ is obtained.
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- **B. Guo and C.-Z. Xu 2000** [5]:
  Resolved this open problem, using essential spectral analysis, the authors demonstrated that the spectrum-determined growth condition always holds under an output feedback control law.

- Recent Papers: **(S. Avdonin and J. Edward, 2019, MCRF)**, **(J. B. Amara and W. Boughamda, 2020, SCL)**

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$$\eta(t) := b_0(\alpha_1 w^1_x - \alpha_2 w^2_x)(0,t) + m z_t(t).$$

The system can be rewritten as

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with the initial data

$$\begin{cases} \eta(0) = \eta_0, \\ (w^1, w^1_t)(x,0) = (w^1_0, w^1_1)(x), \quad x \in [-\ell_1, 0], \\ (w^2, w^2_t)(x,0) = (w^2_0, w^2_1)(x), \quad x \in [0, \ell_2]. \end{cases} \tag{6}$$

- $b_0 > 0$: the coefficient for higher-order damping, which represents dissipation due to angular velocity at the interface.
- $b_1 > 0$: the coefficient for lower-order damping, associated with energy dissipation caused by tip velocity at the same interface.
- $d_1 > 0$: models boundary damping at the free end of the second string ($x = \ell_2$), introducing further energy dissipation.

Physical Significance :

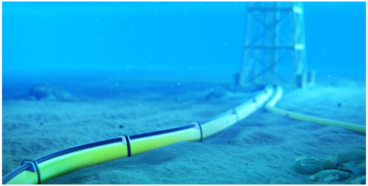
- Suspended transmission lines

- Stockbridge Dampers

In some cases, additional masses are attached to power lines, such as Stockbridge dampers, which help mitigate wind-induced vibrations.

- Insulator strings, particularly in high-voltage power lines, also act as localized masses affecting the cable's shape and dynamics.

- Under water cables



Let $(w^1, v^1, w^2, v^2, \eta)$ be a regular solution of (5). The energy, $\mathscr{E}(t)$, consists of kinetic, potential, and interface contributions, given as follows

$$\mathscr{E}(t) = \mathscr{E}_{\text{kinetic}}(t) + \mathscr{E}_{\text{potential}}(t) + \mathscr{E}_{\text{interface}}(t),$$

where

$$\begin{aligned}
\mathscr{E}_{\text{kinetic}}(t) &= \frac{1}{2}\sum_{j=1}^{2}\int_{(-1)^j \ell_{2-j}}^{2j-2} \rho_j |w_t^j(x,t)|^2 \, dx, \\
\mathscr{E}_{\text{potential}}(t) &= \frac{1}{2}\sum_{j=1}^{2}\int_{(-1)^j \ell_{2-j}}^{2j-2} \alpha_j |w_x^j(x,t)|^2 \, dx, \\
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\mathscr{E}_{\text{interface}}(t) &= \frac{1}{2}\frac{1}{m+b_0 b_1}|\eta(t)|^2.
\end{aligned}$$

The energy dissipation rate is given by

$$\frac{d\mathscr{E}(t)}{dt} = -\frac{b_0}{m+b_0 b_1}\left|(\alpha_1 w_x^1 - \alpha_2 w_x^2)(0,t)\right|^2 - \frac{m b_1}{m+b_0 b_1}|z_t(t)|^2 - d_1 |w_t^2(\ell_2,t)|^2.$$

Now, in order to have a unique solution, we choose a proper energy space. Let us first introduce the following Hilbert space:

$$H_L^1(-\ell_1,0) = \left\{ f \in H^1(-\ell_1,0) \,:\, f(0) = 0 \right\}.$$

Now, we define the energy space $\mathscr{H}$ as follows

$$\mathscr{H} = H_L^1(-\ell_1,0) \times L^2(-\ell_1,0) \times H^1(0,\ell_2) \times L^2(0,\ell_2) \times \mathbb{C}.$$

Now, we introduce the functions $w_t^1 = v^1$, $w_t^2 = v^2$, and we rewrite the system (5) as the following initial value problem

$$\frac{d}{dt}\mathbb{U}(t) = \mathscr{A}\mathbb{U}(t), \quad \mathbb{U}(0) = \mathbb{U}_0, \quad \forall t > 0,$$

where $\mathbb{U} = (w^1, v^1, w^2, v^2, \eta)^\top$ and $\mathbb{U}_0 = (w_0^1, w_1^1, w_0^2, w_1^2, \eta_0)^\top$, and the operator $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ is given by

$$\mathscr{A}\begin{pmatrix} w^1 \\ v^1 \\ w^2 \\ v^2 \\ \eta \end{pmatrix} = \begin{pmatrix} v^1 \\ \frac{\alpha_1}{\rho_1} w_{xx}^1 \\ v^2 \\ \frac{\alpha_2}{\rho_2} w_{xx}^2 \\ -(A_x^1(0) - A_x^2(0)) - b_1 v(0) \end{pmatrix}$$

with the domain

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{array}{l} U \in \mathscr{H} : w^1 \in \left(H_L^1 \cap H^2\right)(-\ell_1, 0), w^2 \in H^2(0, \ell_2), v^1 \in H_L^1(-\ell_1, 0), v^2 \in H^1(0, \ell_2), \\ \alpha_2 w_x^2(l_2) = -d_1 v^2(l_2),\ b_0(\alpha_1 w_x^1(0) - \alpha_2 w_x^2(0)) + m v^1(0) = \eta(t), v^1(0) = v^2(0) \end{array} \right\}.$$

• By Lumer-Phillips theorem, $\mathscr{A}$ generates a C_0-semigroup of contractions $(e^{t\mathscr{A}})_{t\geq 0}$. Therefore, the solution admits the following representation $U(t) = e^{t\mathscr{A}} U_0, \quad t \geq 0$, which leads to the **well-posedness**.

Summary of Stability Cases

Case	d_1	b_0	b_1	Strong Stability	Type
1	✓	✓	X	No Condition	Exp
2	✓	X	✓		Pol. type t^{-1}
3	✓	X	X		
4	X	✓	✓	$\frac{\ell_2}{\ell_1}\sqrt{\frac{\alpha_1\rho_2}{\rho_1\alpha_2}} \neq \frac{2m+1}{2n}$	Exp
5	X	✓	X		
6	X	X	✓		Pol. type t^{-1}

The proofs of the above cases rely on one of the methods:

- Energy Multiplier Method.
- Borichev–Tomilov Theorem.
- Huang-Prüss Theorem.

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[8] V. Komornik, Exact Controllability and Stabilization: The Multiplier Method, J. Wiley. 1994

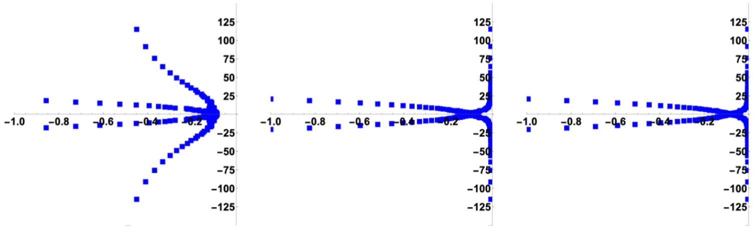
[9] JA. Borichev and Y. Tomilov. Optimal polynomial decay of functions and operator semigroups. Math. Ann., 347(2):455–478, 2010.

Numerical Experiments 1 & 2

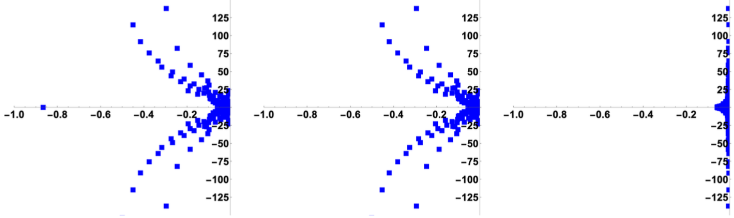
Identical and Varying Wave Propagation Speeds: Effects of the arithmetic condition (SC)

Group	**Parameter**	**Experiment 1**	**Experiment 2**
Common	Feedback gains	$b_0 = 1$, $b_1 = 1$, $d_1 = 1$	
	Interior mass	$m = 0.6$	
Varying	Densities	$\rho_1 = 3$, $\rho_2 = 4$	$\rho_1 = 3$, $\rho_2 = 3$
	Stiffness coeffs.	$\alpha_1 = \frac{9}{16}$, $\alpha_2 = \frac{1}{3}$	$\alpha_1 = 1$, $\alpha_2 = 1$
	Geometry	$l_1 = 2$, $l_2 = 1$	$l_1 = 1$, $l_2 = 1$

Eigenvalue spectra for Experiment 1 (arithmetic condition (SC) violated)



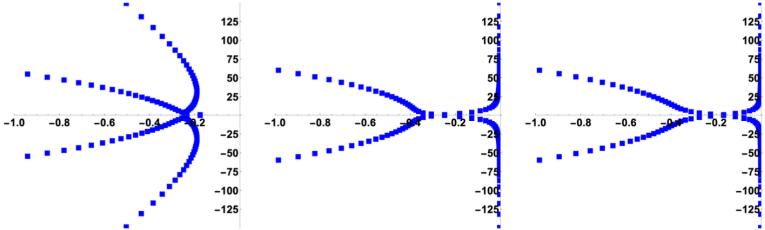
(a) Case 1 (b) Case 2 (c) Case 3



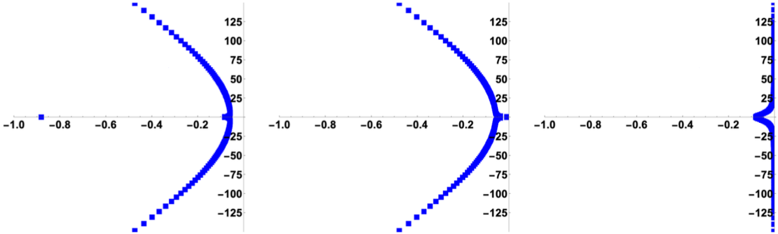
(d) Case 4 (e) Case 5 (f) Case 6

Eigenvalue spectra for Experiment 1

Eigenvalue spectra for Experiment 2 (arithmetic condition (SC) satisfied)

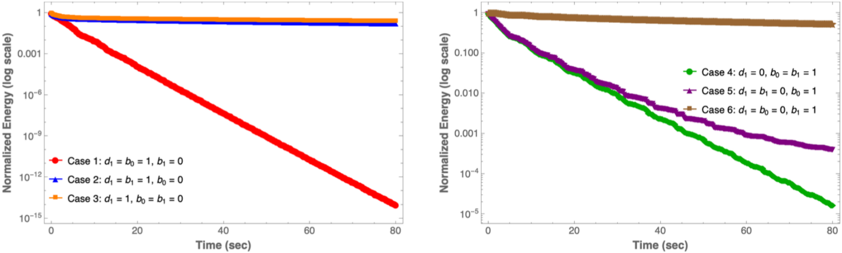


(a) Case 1 (b) Case 2 (c) Case 3



(d) Case 4 (e) Case 5 (f) Case 6

Eigenvalue spectra for Experiment 1



Normalized energy decay in Experiment 2 (arithmetic condition satisfied), plotted on a logarithmic scale.
(Cases 1–3, $d_1 = 1$): Rapid, exponential decay occurs only in Case 1 ($b_0 = 1$); Cases 2–3 endorse the polynomial decay.
(Cases 4–6, $d_1 = 0$): The interface pair (b_0, b_1) in Case 4 restores exponential decay; b_0 alone (Case 5) produces mild decay; b_1 alone (Case 6) remains ineffective.

Videos - Numerical Experiments

Experiment I:
Unit Lengths
Identical Wave propagation speed

Experiment II:
Unit Lengths
Non-Identical Wave propagation speed

(N+1)–Connected strings by N-masses

We consider the following model which consists of $(N+1)-$ strings:

$$\begin{cases} \rho_j w^j_{tt} - \alpha_i w^j_{xx} = 0, & (x,t) \in I_j \times (0,\infty), j \in \{1,...,N+1\}, \\ w^1(0,t) = 0, & t \in (0,\infty), \\ w^j(\ell_j,t) = w^{j+1}(\ell_j,t) = z^j(t), & t \in (0,\infty), j \in \{1,2,...,N\}, \\ \alpha_j w^j_x(\ell_j,t) - \alpha_{j+1} w^{j+1}_x(\ell_j,t) + m^j z^j_{tt}(t) = g_1(t), & t \in (0,\infty), j \in \{1,2,...,N\}, \\ \alpha_{N+1} w^{N+1}_x(\ell_{N+1},t) = g_{N+1}(t), & t \in (0,\infty), \end{cases} \tag{7}$$

where $I_j = (\ell_{j-1}, \ell_j)$ and with the following initial data: $(w^j, w^j_t)(x,0) = (w^j_0, w^j_1)(x), x \in [\ell_{j-1}, \ell_j]$. The control inputs are subsequently chosen in the form of feedback laws as defined below

$$\begin{cases} g_1(t) := -b^j_0 \left(w^1_{xt}(\ell_j,t) - w^2_{xt}(\ell_j,t) \right) - b^j_1 w^1_t(\ell_i,t), \quad j \in \{1,2,...,N\} \\ g_{N+1}(t) := -d_1 w^{N+1}_t(\ell_{N+1},t). \end{cases} \tag{8}$$

Specifically, $g_1(t)$ combines a higher-order (slope-velocity) feedback term and a lower-order (velocity) feedback term at the mass interface, while $g_2(t)$ applies a lower-order velocity feedback at the right end. We introduce an auxiliary dynamic variable $\eta^j(t)$ by

$$\eta^j(t) := b^j_0(\alpha_j w^j_x - \alpha_{j+1} w^{j+1}_x)(\ell_j,t) + m^j z^j_t(t), j \in \{1,...,N\}. \tag{9}$$

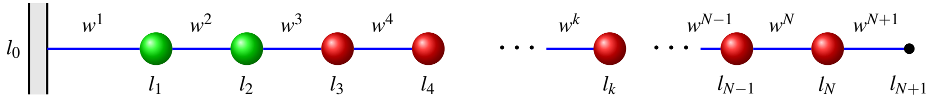
Case with Only boundary damping

Here we consider $d_1 \neq 0, b_0^j = b_1^j = 0, \forall j \in \{1, \ldots, N\}$

- Strong Stability without any condition
- Polynomial Stability of type $t^{\frac{-2}{2N}}$

Cases with Only Two Dampings

Case 2.1: $b_0^1.b_0^2 \neq 0$ and $b_0^j = 0$, $j \in \{3,\dots,N\}, b_j^1 = 0, j \in \{1,\dots,N\}$, $d_1 = 0$.



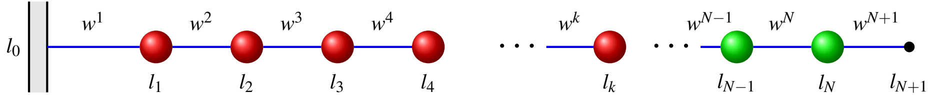
The first two nodes are damped

Stability under the condition:

$$\frac{l_2 - l_1}{l_1}\sqrt{\frac{\alpha_1\rho_2}{\alpha_2\rho_1}} \notin \mathbb{Q}. \tag{SC1}$$

If only the damping is changed while keeping the same configuration:

b_0^1	b_0^2	b_1^1	b_1^2	Strong stability
✓	✓	X	X	$\frac{l_2-l_1}{l_1}\sqrt{\frac{\alpha_1\rho_2}{\alpha_2\rho_1}} \notin \mathbb{Q}$
X	X	✓	✓	
✓	X	X	✓	
X	✓	✓	X	



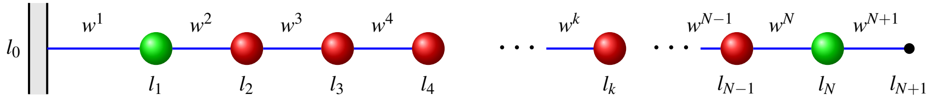
The last two nodes are damped

We get stability under condition

$$\frac{l_N - l_{N-1}}{l_{N+1} - l_N}\sqrt{\frac{\alpha_{N+1}\rho_M}{\alpha_M\rho_{N+1}}} \neq \frac{even}{odd}. \tag{SC2}$$

If only the damping is changed while keeping the same configuration:

b_0^{N-1}	b_0^N	b_1^{N-1}	b_1^N	Strong stability
✓	✓	X	X	$\frac{l_N-l_{N-1}}{l_{N+1}-l_N}\sqrt{\frac{\alpha_{N+1}\rho_M}{\alpha_M\rho_{N+1}}} \neq \frac{even}{odd}.$
X	X	✓	✓	
✓	X	X	✓	
X	✓	✓	X	



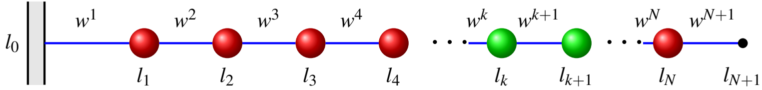
The first and last nodes are damped

We get stability under condition

$$\frac{l_1}{l_{N+1}-l_N}\sqrt{\frac{\alpha_{N+1}\rho_1}{\alpha_1\rho_{N+1}}} \neq \frac{even}{odd}. \tag{SC3}$$

If the the first and last nodes are damped:

b_0^1	b_0^N	b_1^1	b_1^N	Strong stability
✓	✓	X	X	$\frac{l_1}{l_{N+1}-l_N}\sqrt{\frac{\alpha_{N+1}\rho_1}{\alpha_1\rho_{N+1}}} \neq \frac{even}{odd}$
X	X	✓	✓	
✓	X	X	✓	
X	✓	✓	X	



Any two consecutive nodes at l_k and l_{k+1} are damped

The system is Not Stable

Hypothesis

Following [10], the stability of the system is closely linked to the arithmetic nature of the ratio q :

- **(H1)**: The ratio $q \in \mathbb{Q}$, where $q = \frac{\xi_1}{\xi_2}$ with $\gcd(\xi_1, \xi_2) = 1$, and integers ξ_1 odd and ξ_2 is (odd or even).

- **(H2)**: The ratio $q \in \mathbb{R} \setminus \mathbb{Q}$. Suppose there exists a number $\mu(q) \geq 2$, depending on q such that for every sequence $\Lambda = (\xi_{1,n}, \xi_{2,n})_{n \in \mathbb{N}} \in (\mathbb{N} \times \mathbb{N}^*)^{\mathbb{N}}$ satisfying $c_1 \leq \frac{\xi_{1,n}}{\xi_{2,n}} \leq c_2$, for some positive constants c_1, c_2, there exist a positive constant $c\,(q, \Lambda)$ and a positive integer $N\,(q, \Lambda)$, depending on q and the sequence Λ, such that $\left| q - \frac{\xi_{1,n}}{\xi_{2,n}} \right| > \frac{c(q,\Lambda)}{\xi_{2,n}^{\mu(q)}}$ for all $n \geq N\,(q, \Lambda)$.

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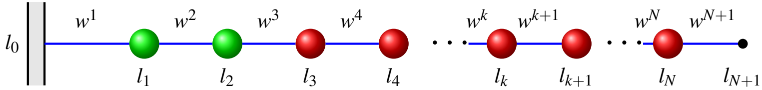
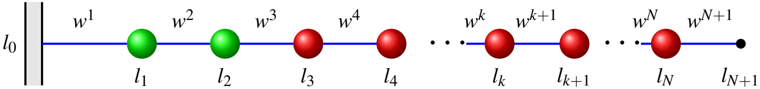
[10] Akil, M., Nicaise, S., Özer, A.Ö. et al. Stability Results for Novel Serially-Connected Magnetizable Piezoelectric and Elastic Smart-System Designs. Appl Math Optim

Polynomial Stability-2 dampings



Polynomial Stability of type $t^{\frac{-2}{2\mu(q)-2+2(N-2)}}$ under (H2).

Polynomial Stability-2 dampings



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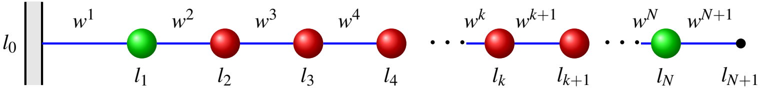
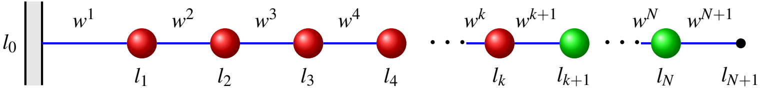
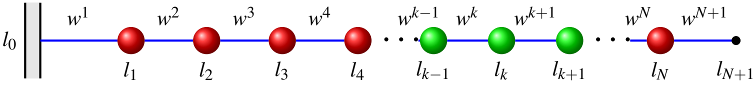




Polynomial Stability of type $t^{\frac{-2}{2(N-2)}}$ under (H1).

Cases with Three Dampings

Stability in Case of 3-dampings



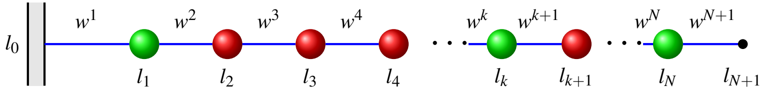
Three consecutive damped nodes

- Three consecutive damped nodes: stability under the condition

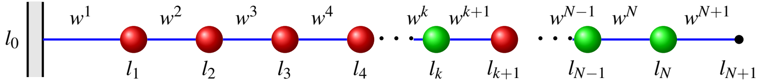
$$\frac{l_{k-1}-l_{k+1}}{l_{k+1}-l_k}\sqrt{\frac{\alpha_{k+1}\rho_{k+2}}{\rho_{k+1}\alpha_{k-1}}} \neq \frac{n_{k+2}}{n_{k+1}}, n_{k-1}, n_{k+1} \in \mathbb{N}.$$

- The presence of at least one undamped node between the three damped nodes leads to instability.

Polynomial Stability in 3 damping cases

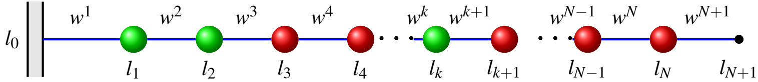


The first and the last nodes and any arbitrary node are damped



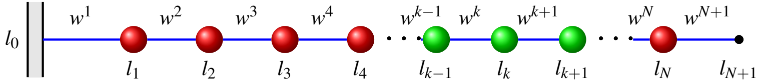
The last two nodes and one arbitrary node are damped

Polynomial stability of type $t^{\frac{-2}{2(N-3)}}$.



First two nodes and one arbitrary node are damped

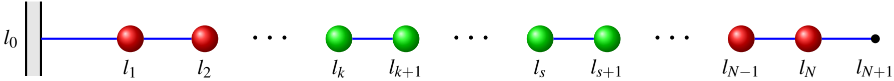
Polynomial stability of type $t^{\frac{-2}{2\mu(q)-2+2(N-3)}}$.



Any Three consecutive damped nodes

Polynomial stability of type $t^{\frac{-2}{2\mu(q)-2+2\max\{k-2,N-k-1\}}}$.

Cases with Four Dampings



Four damped nodes arranged in two consecutive pairs.

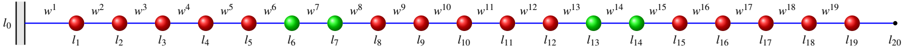
Polynomial Stability of type $t^{\frac{-2}{2\mu(q_{k,k+1}^{s,s+1})-2+2\max\{k+1,s-k-2,N-s-1\}}}$ under condition (H2), where $\mu(q_{k,k+1}^{s,s+1}) = \frac{l_{s+1}-l_{k+1}}{l_s-l_k}\sqrt{\frac{\alpha_{k+1}\rho_{s+1}}{\rho_{k+1}\alpha_{s+1}}}$.

Example for best decay rate : $N = 19$

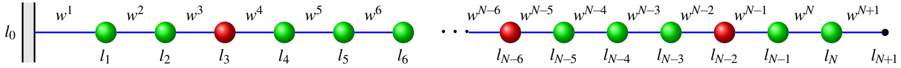
Here we need to take

$$k = 6 \quad \text{and} \quad s = 13$$

to get the best decay rate in this case.



k damped nodes Case:
Best Scenario to Achieve polynomial stability with a rate independent from the number of the undamped nodes



The number of the damped nodes $= \frac{3N+1}{4}$
The number of the undamped nodes $= \frac{N-1}{4}$
such that **$N-5$ is a multiple of 4**. Here we have polynomial stability of type

$$t^{\frac{-2}{2\max\{\mu(q_{1,2}),\mu(q_{4,5,6}),\dots,\mu(q_{N-5,N-4,N-3})\}-2}},$$

where $\mu(q_{1,2}),\mu(q_{4,5,6}),\dots,\mu(q_{N-5,N-4,N-3})$ are the ratios that depends on $l_1,l_2,l_4,l_5,l_6,\dots,l_{N-3}$, the densities and the stiffness coefficients.

Exponential Stability

Remarks:

- The exponential stability is attained only in the case when all the nodes are damped and satisfaction of condition at (H1) at the node l_N.

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- Violation of (H1) at l_N while all nodes are damped, gives polynomial stability of type $t^{\frac{-2}{2\mu(q)-2}}$.
- If we remove only one damping, other than the one at l_N, we get polynomial stability of type t^{-1}.
- If we remove the damping only at the node l_N, then we obtain polynomial stability of type $t^{\frac{-2}{2\mu(q)}}$.
- If we remove only one damping, other than the one at l_N, and the condition (H1) at l_N is not satisfied, we get polynomial stability of type $t^{\frac{-2}{2\mu(q)}}$.

Thank you for your Attention!