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Energy decay for Korteweg-de Vries-Burgers equations with delay feedback

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KdV-Burgers equation

The KdV-Burgers equation

$$u_t + u_{xxx} - u_{xx} + u\,u_x = 0, \quad \text{in } \mathbb{R} \times (0, +\infty), \quad \textbf{(K)}$$

arises in modelling unidirectional propagation of planar waves.

u may represent a displacement of the underlying medium or a velocity.

The solutions of **(K)** approach zero for $t \to +\infty$. A natural question is about the rate at which $\|u(t)\|$ approaches zero, for some norm $\|\cdot\|$.

In [Amick, Bona and Shonbek, 1989] it is proved that

$$\|u(\cdot,t)\|_{L^2(\mathbb{R})} \leq C t^{-\frac{1}{4}}, \quad t > 0,$$

for initial data $u_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$.

Damped KdV-Burgers equation

In [Cavalcanti, Domingos Cavalcanti, Komornik and Rodrigues, 2014] the damped KdV-Burgers equation is considered

$$\begin{cases} u_t(x,t) + u_{xxx}(x,t) - u_{xx}(x,t) + \lambda_0 u(x,t) \\ \qquad\qquad\qquad + u(x,t)u_x(x,t) = 0, & \text{in } \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}, \end{cases}$$

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with $\lambda_0 \in L^\infty(\mathbb{R})$, together with its linear version, i.e., without the term $u\,u_x$.

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with $\lambda_0 \in L^\infty(\mathbb{R})$, together with its linear version, i.e., without the term $u\, u_x$.

The authors investigated the well-posedness and exponential stability for an indefinite damping $\lambda_0(x)$, giving exponential decay estimates for $\|u(\cdot,t)\|_{L^2(\mathbb{R})}$ if λ_0 satisfies suitable assumptions.

KdV-Burgers equations with delay

Here, we deal with the model

$$\begin{cases} u_t(x,t) + u_{xxx}(x,t) - u_{xx}(x,t) + h(x)u(x,t) \\ \qquad\quad +\lambda(x)u(x,t-\tau) + a(u(x,t))u_x(x,t) = 0, \quad \text{in } \mathbb{R} \times (0,+\infty) \\ u(x,s) = u_0(x,s), \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathbb{R} \times [-\tau, 0], \end{cases} \quad (\mathbf{P})$$

where $\tau > 0$ is the time delay, λ and h belong to $L^\infty(\mathbb{R})$ and satisfy suitable assumptions,

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where $\tau > 0$ is the time delay, λ and h belong to $L^\infty(\mathbb{R})$ and satisfy suitable assumptions,

and its linear version

$$\begin{cases} u_t(x,t) + u_{xxx}(x,t) - u_{xx}(x,t) \\ \qquad +hu(x,t) + \lambda u(x,t-\tau) = 0, & \text{in } \mathbb{R}\times(0,+\infty) \\ u(x,s) = u_0(x,s), & \text{in } \mathbb{R}\times[-\tau,0]. \end{cases} \quad (\mathbf{L})$$

Delay effects

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- Nevertheless, appropriate feedback laws may restitute stability properties (cf. [Nicaise and P., 2006]) as well as appropriate choices of the time delay (cf. [Gugat, 2010]).

- Our aim here is to furnish sufficient conditions on the coefficients λ, h in order to have well-posedness of the models **(P)** and **(L)** and exponential decay estimates. We emphasize the fact that the results here obtained cannot be deduced from the general approaches of [Nicaise and P. (2006, 2015)]. Indeed, the methods there proposed would require a smallness assumption on the $L^\infty-$ norm of the delay feedback coefficient λ.

Linear model: well–posedness

First, we consider the linear model

$$\begin{cases} u_t(x,t) + u_{xxx}(x,t) - u_{xx}(x,t) & \\ \qquad\qquad +h(x)u(x,t) + \lambda(x)u(x,t-\tau) = 0, & \text{in } \mathbb{R}\times(0,+\infty) \\ u(x,s) = u_0(x,s), & \text{in } \mathbb{R}\times[-\tau,0], \end{cases} \quad (\mathbf{L})$$

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It is easy to show the following result:

PROPOSITION

If $h \in L^\infty(\mathbb{R})$, then the operator A_h defined by the formula $A_h u := -u_{xxx} + u_{xx} - hu$ on $\mathcal{D}(A_h) := H^3(\mathbb{R})$ generates a strongly continuos semigroup in the Hilbert space $H := L^2(\mathbb{R})$.

Linear model: well–posedness

Now, using an iterative procedure (see e.g. [Nicaise and P., 2015]) and standard semigroup arguments (see e.g. [Pazy, 1983]), we can prove a well–posedness result for the linear problem with delay (L).

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PROPOSITION

Assume $h, \lambda \in L^\infty(\mathbb{R})$ and $u_0 \in C([-\tau, 0]; H)$. Then, there exists a unique solution $u \in C([0, +\infty); H)$ of problem (L).

Proof. First, we argue on the interval $[0, \tau]$. Then problem (L) can be regarded as an inhomogeneous Cauchy problem of the form

$$\begin{cases} u_t(t) - A_h u(t) = g_0(t), & \text{in } (0, \tau) \\ u(0) = u_0, \end{cases}$$

where $g_0(t) := -\lambda u(t-\tau) = -\lambda u_0(t-\tau)$, for $t \in [0, \tau]$. This problem admits a unique solution $u(\cdot) \in C([0, \tau), H)$.

Linear model: well–posedness

Now, consider $t \in [\tau, 2\tau]$. Thus, problem (L) can be rewritten as

$$\begin{cases} u_t(t) - A_h u(t) = g_1(t), & \text{in } (\tau, 2\tau) \\ u(\tau) = u(\tau_-), \end{cases}$$

with $g_1(t) = -\lambda u(t-\tau)$. We then deduce the existence of a solution $u(\cdot) \in C([0, 2\tau], H)$.

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Before presenting our exponential decay estimate, assuming $\lambda \in L^\infty(\mathbb{R})$, we introduce the following Lyapunov functionals:

Energy functionals

$$E(t) := E(u(t)) = \frac{1}{2}\int_{\mathbb{R}} u^2(x,t)dx + \frac{1}{2}\int_{t-\tau}^{t}\int_{\mathbb{R}} |\lambda(x)|u^2(x,s)dxds, \qquad \textbf{(E)}$$

and

$$\mathcal{E}(t) = E(t) + c_0\mathcal{F}(t), \qquad (\mathbf{E_{cal}})$$

where

$$\mathcal{F}(t) = \int_{t-\tau}^{t}\int_{s}^{t}\int_{\mathbb{R}} |\lambda(x)|u^2(x,\sigma)dxd\sigma ds,$$

and c_0 is a positive constant to be determined later on.

Damping coefficients

Under the assumption

$$h(x) > \alpha_0 \quad \text{for a.e. } x \in \mathbb{R}, \qquad \textbf{(H1)}$$

for some positive constant α_0, we could easily obtain an exponential decay estimate when the coefficient of the delay term $\lambda(x)$ satisfies

$$\|\lambda\|_\infty < \alpha_0 \, .$$

Indeed, in this case, one could compensate the delay effect with the undelayed damping term (cf. [Nicaise and P., 2006]).

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However, we will deal here in a more general setting. For the sake of clearness, we restrict ourselves to the case of h bounded from below by a positive constant, i.e. assume (H1), but it may be $|\lambda(x)| \geq h(x)$ in some part of the domain. Our results can be extended to the case in which the coefficient of the undelayed feedback h is also indefinite.

Assumptions

For our arguments, we will need that the functions h and λ satisfy suitable conditions.

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Let us denote for $1 \le p < +\infty$,

$$c(p) := \left(1 - \frac{1}{2p}\right)\left(\frac{2}{p}\right)^{\frac{1}{2p-1}} .$$

Besides

$$h(x) > \alpha_0 \quad \text{for a.e. } x \in \mathbb{R}, \qquad \textbf{(H1)}$$

for some positive constant α_0, we assume that there exist a constant α and a function $g \in L^p(\mathbb{R})$, for $1 \le p < \infty$, such that

$$|\lambda(x)| \le \alpha + g(x) \quad \text{for a.e.} \quad x \in \mathbb{R}, \qquad \textbf{(H2)}$$

with

$$0 \le \alpha < \alpha_0 \quad \text{and} \quad \|g\|_p < \left(\frac{\alpha_0 - \alpha}{c(p)}\right)^{1-\frac{1}{2p}} .$$

Stability result for the linear model

THEOREM [I.Issa & P., 2025]

Assume that $\lambda, h \in L^\infty(\mathbb{R})$ and h satisfies (H1). Moreover, assume that there exist a positive constant α and a function $g \in L^p(\mathbb{R})$, for some $1 \leq p < \infty$, such that λ satisfies (H2). Then, there exists a constant $c_0 > 0$ such that the solution u of the problem (L) satisfies the exponential decay estimate

$$E(t) \leq C(u^0, c_0)e^{-\delta_1(c_0)t},$$

for a suitable constant $\delta_1(c_0) > 0$, with

$$\begin{aligned} C(u^0, c_0) = {} & \tfrac{1}{2}\|u^0\|_2^2 + \tfrac{1}{2}\int_{-\tau}^0 \int_{\mathbb{R}} |\lambda(x)|u^2(x,s)\,dxds \\ & + c_0 \int_{-\tau}^0 \int_s^0 \int_{\mathbb{R}} |\lambda(x)|u^2(x,\sigma)\,dxd\sigma ds. \end{aligned}$$

Remarks

- The KdV-Burgers model with time delay has been previously analyzed in [Komornik & P., 2020] (linear + nonlinear $a(u)u_x = uu_x$).

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- Note that our above conditions significantly improve upon those in [Komornik & P., 2020]. Indeed, we assume (H2), which is more general than the key assumption there:

$$\frac{e^{\tau}+1}{2}|\lambda(x)| \leq \alpha + g(x), \text{ for a.e. } x \in \mathbb{R}, \qquad (\textbf{CondKP})$$

where α is a suitable positive constant and $g \in L^p(\mathbb{R})$ has to satisfy some conditions. In particular, our assumption is independent of the time delay, whereas (CondKP), for fixed damping coefficients, can be regarded as a smallness restriction on the time delay size.

Stability result for the linear model

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By differentiating $\mathcal{E}(t)$ we obtain

$$\begin{aligned}\frac{d\mathcal{E}}{dt}(t) = & \int_{\mathbb{R}} u(t)(u_{xx}(t) - hu(t) - \lambda u(t-\tau))dx + \frac{1}{2}\int_{\mathbb{R}} |\lambda| u^2(t)dx \\ & -\frac{1}{2}\int_{\mathbb{R}} |\lambda| u^2(t-\tau)dx - c_0 \int_{t-\tau}^{t}\int_{\mathbb{R}} |\lambda| u^2(x,s)dxds \\ & +c_0\tau \int_{\mathbb{R}} |\lambda| u^2(x,s)dxds,\end{aligned}$$

where we used the equation and the fact that, for $u \in H^3$,

$$\int_{\mathbb{R}} u\, u_{xxx}\, dx = 0\,.$$

Stability result for the linear model

Then, integrating by parts and using the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}\frac{d\mathcal{E}}{dt}(t) \leq & -\int_{\mathbb{R}} u_x^2(t)dx - \int_{\mathbb{R}} h(x)u^2(t)dx + \int_{\mathbb{R}} |\lambda(x)|u^2(t)dx \\ & -c_0 \int_{t-\tau}^{t} \int_{\mathbb{R}} |\lambda(x)|u^2(x,s)dxds + c_0\tau \int_{\mathbb{R}} |\lambda(x)|u^2(x,s)dxds.\end{aligned}$$

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Recalling (H1) and (H2), we obtain

$$\begin{aligned}\frac{d\mathcal{E}}{dt}(t) \leq & -\int_{\mathbb{R}} u_x^2(t)dx - \alpha_0 \int_{\mathbb{R}} u^2(t)dx + (1+c_0\tau)\alpha \int_{\mathbb{R}} u^2(t)dx \\ & +(1+c_0\tau) \int_{\mathbb{R}} g(x)u^2(x,t)dxdt - c_0 \int_{t-\tau}^{t} \int_{\mathbb{R}} |\lambda|u^2(x,s)dxds.\end{aligned}$$

Stability result for the linear model

Using Hölder inequality we then obtain, for $q = \frac{p}{p-1}$,

$$\frac{d\mathcal{E}}{dt}(t) \leq -\|u_x(t)\|_2^2 - (\alpha_0 - (1 + c_0\tau)\alpha)\|u(t)\|_2^2$$

$$+(1 + c_0\tau)\|g\|_p\|u\|_{2q}^2 - c_0 \int_{t-\tau}^{t} \int_{\mathbb{R}} |\lambda(x)|u^2(x,s)dxds.$$

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Now, observe that

$$\|u\|_{2q}^2 = \left(\int_{\mathbb{R}} (u(t))^{2q} dx\right)^{\frac{1}{q}} = \left(\int_{\mathbb{R}} u^2(t)(u(t))^{\frac{2}{p-1}} dx\right)^{\frac{1}{q}}$$
$$\leq \|u\|_2^{\frac{2}{q}} \|u\|_\infty^{\frac{2}{q(p-1)}} = \|u\|_2^{\frac{2}{q}} \|u\|_\infty^{\frac{2}{p}}.$$

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Then, we deduce

$$\begin{aligned}\frac{d\mathcal{E}}{dt}(t) \leq & -\|u_x(t)\|_2^2 - (\alpha_0 - (1 + c_0\tau)\alpha)\|u(t)\|_2^2 \\ & +(1 + c_0\tau)\|g\|_p\, \|u(t)\|_\infty^{\frac{2}{p}}\, \|u\|_2^{\frac{2}{q}} - c_0 \int_{t-\tau}^{t}\int_{\mathbb{R}} |\lambda|u^2(x,s)dxds,\end{aligned}$$

Stability result for the linear model

and so, observing that for all $v \in H^1(\mathbb{R})$,

$$\|v\|_\infty^2 \le 2\|v\|_2\|v_x\|_2,$$

we have

$$\frac{d\mathcal{E}}{dt}(t) \le -\|u_x(t)\|_2^2 - (\alpha_0 - (1 + c_0\tau)\alpha)\|u(t)\|_2^2$$
$$+2^{\frac{1}{p}}(1 + c_0\tau)\|g\|_p\|u\|_2^{\frac{2p-1}{p}}\|u_x\|_2^{\frac{1}{p}} - c_0\int_{t-\tau}^{t}\int_{\mathbb{R}}|\lambda|u^2(x,s)dxds$$

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Then, by using the Young inequality, we obtain for every fixed $\delta > 0$,

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Then, by using the Young inequality, we obtain for every fixed $\delta > 0$,

$$\frac{d\mathcal{E}}{dt}(t) \leq -\|u_x(t)\|_2^2 - (\alpha_0 - (1 + c_0\tau)\alpha)\|u(t)\|_2^2$$
$$+\frac{\left(\frac{1}{\delta}(1 + c_0\tau)\|g\|_p\|u\|_2^{\frac{2p-1}{p}}\right)^{\frac{2p}{2p-1}}}{\frac{2p}{2p-1}} + \frac{\left(\delta 2^{\frac{1}{p}}\|u_x\|_2^{\frac{1}{p}}\right)^{2p}}{2p}$$
$$-c_0\int_{t-\tau}^{t}\int_{\mathbb{R}}|\lambda|u^2(x,s)dxds.$$

Stability result for the linear model

Now, let us fix δ such that $4\delta^{2p} = 2p$. With this choice we obtain

$$
\begin{aligned}
\frac{d\mathcal{E}}{dt}(t) \leq\; & \\
& -\left(\alpha_0 - (1+c_0\tau)\alpha - \frac{2p-1}{2p}\left(\frac{2}{p}\right)^{\frac{1}{2p-1}}(1+c_0\tau)^{\frac{2p}{2p-1}}\|g\|_p^{\frac{2p}{2p-1}}\right)\|u(t)\|_2^2 \\
& \qquad -c_0\int_{t-\tau}^{t}\int_{\mathbb{R}}|\lambda|u^2(x,s)\,dxds.
\end{aligned}
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From the assumption (H2), we can deduce that for a sufficiently small c_0, the following inequalities hold:

$$0 \leq (1+c_0\tau)\alpha < \alpha_0,$$

and

$$\|g\|_p < \left(\frac{\alpha_0 - (1+c_0\tau)\alpha}{c(p)(1+c_0\tau)^{\frac{2p}{2p-1}}}\right)^{1-\frac{1}{2p}}.$$

Stability result for the linear model

Then,

$$\alpha_0 - (1+c_0\tau)\alpha - c(p)(1+c_0\tau)^{\frac{2p}{2p-1}}\|g\|_p^{\frac{2p}{2p-1}} > 0,$$

and so, from previous inequality we obtain

$$\frac{d}{dt}\mathcal{E}(t) \leq -\delta_0(c_0)E(t),$$

with

$$\delta_0(c_0) = \min\left\{2\left(\alpha_0 - (1+c_0\tau)\alpha - c(p)(1+c_0\tau)^{\frac{2p}{p-1}}\|g\|_p^{\frac{2p}{2p-1}}\right), 2c_0\right\}$$

.

Stability result for the linear model

Furthermore, we can observe that

$$
\begin{aligned}
\mathcal{E}(t) &= E(t) + c_0 \int_{t-\tau}^{t} \int_{s}^{t} \int_{\mathbb{R}} |\lambda(x)| u^2(x,\sigma)\, dx\, d\sigma\, ds \\
&\leq E(t) + c_0 \tau \int_{t-\tau}^{t} \int_{\mathbb{R}} |\lambda(x)| u^2(x,\sigma)\, dx\, d\sigma \\
&\leq \frac{1}{2} \int_{\mathbb{R}} u^2(x,t)\, dx + \frac{1}{2}(1 + 2c_0\tau) \int_{t-\tau}^{t} \int_{\mathbb{R}} |\lambda(x)| u^2(x,\sigma)\, dx\, d\sigma \\
&\leq (1 + 2c_0\tau) E(t).
\end{aligned}
$$

Thus, we conclude that

$$\frac{d}{dt}\mathcal{E}(t) \leq -\delta_1(c_0)\mathcal{E}(t).$$

where $\delta_1(c_0) = \frac{\delta_0(c_0)}{1+2c_0\tau}$. Thus, using Gronwall's Lemma and the fact that $E(t) \leq \mathcal{E}(t)$, we obtain the exponential decay estimate for $E(t)$ with $C(u^0, c_0) = \mathcal{E}(0)$.

Linear inhomogeneous model

In order to prove the well–posedness of the nonlinear model (P) we first consider the corresponding linear inhomogeneous initial value problem

$$\begin{cases} u_t(x,t) + u_{xxx}(x,t) - u_{xx}(x,t) + hu(x,t) & \\ \qquad\qquad + \lambda u(x,t-\tau) = f(x,t), & \text{in } \mathbb{R} \times (0,T), \\ u(x,s) = u_0(x,s), & \text{in } \mathbb{R} \times [-\tau, 0], \end{cases} \qquad \textbf{(LI)}$$

for some $T > 0$.

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for some $T > 0$.

Setting

$$A_h := -\partial_x^3 + \partial_x^2 - hI, \quad \mathcal{D}(A_h) = H^3(\mathbb{R}),$$

we can rewrite (LI) in the form

$$\begin{cases} u_t(x,t) + \lambda u(x,t-\tau) = A_h u(x,t) + f(x,t), & \text{in } \mathbb{R} \times (0,T), \\ u(x,s) = u_0(x,s), & \text{in } \mathbb{R} \times [-\tau, 0]. \end{cases}$$

Linear inhomogeneous model

We know that A_h generates a strongly continuous semigroup of contractions in $L^2(\mathbb{R})$ (see [Cavalcanti, Domingos Cavalcanti, Komornik & Rodrigues, 2014]).

Then, for any datum $u_0 \in C([-\tau, 0], H)$ and $f \in L^1(0, T; L^2(\mathbb{R}))$, problem **(LI)** has a unique mild solution $u \in C([-\tau, T]; L^2(\mathbb{R}))$, satisfying the representation formula

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$$\begin{aligned} u(t) = S(t)u_0(0) &- \int_0^t S(t-s)\lambda u(s-\tau)\,ds \\ &+ \int_0^t S(t-s)f(s)\,ds, \quad t \in [0, T]. \end{aligned}$$

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One can show that the mild solution depends continuously on the initial data and the inhomogeneous term.

Linear inhomogeneous model

PROPOSITION

Let $u_0 \in C([-\tau, 0], L^2(\mathbb{R}))$ and $f \in L^1(0, T; L^2(\mathbb{R}))$, then the solution of problem (LI) satisfies the following estimate:

$$\|u(t)\|_{C([0,T];L^2(\mathbb{R}))} \leq e^{\|\lambda\|_\infty T}\left(\|u(0)\|_{L^2(\mathbb{R})} + \|f\|_{L^1(0,T;L^2(\mathbb{R}))} + \|\lambda\|_\infty \int_{-\tau}^{0} \|u(s)\|_{L^2(\mathbb{R})} ds\right).$$

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Actually, the solution of **(LI)** has an additional regularity. Let us introduce the Banach space

$$\mathcal{B}_T := C([0, T]; L^2(\mathbb{R})) \cap L^2(0, T; H^1(\mathbb{R}))$$

with the norm

$$\|u\|_{\mathcal{B}_T} = \|u\|_{C([0,T];L^2(\mathbb{R}))} + \|\partial_x u\|_{L^2(0,T;L^2(\mathbb{R}))}.$$

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PROPOSITION

If $u_0 \in C([-\tau, 0], L^2(\mathbb{R}))$ and $f \in L^1(0, T; L^2(\mathbb{R}))$, then the solution of problem **(LI)** belongs to $\mathcal{B}_T$ and satisfies the estimate

$$\begin{aligned}\|u\|_{\mathcal{B}_T} \le C_T \Big\{ &\|u(0)\|_{L^2(\mathbb{R})} + \|f\|_{L^1(0,T;L^2(\mathbb{R}))} \\ &+ \left(\|\lambda\|_\infty \tau^{1/2} + \|\lambda\|_\infty^{1/2} \right) \|u\|_{L^2(-\tau,0;L^2(\mathbb{R}))} \Big\},\end{aligned}$$

with

$$C_T = \sqrt{\frac{3}{2}} \left(1 + e^{2\|\lambda\|_\infty T} \right)^{1/2} e^{(\|\lambda\|_\infty + \|h\|_\infty) T}.$$

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with

$$C_T = \sqrt{\frac{3}{2}} \left(1 + e^{2\|\lambda\|_\infty T}\right)^{1/2} e^{(\|\lambda\|_\infty + \|h\|_\infty)T} .$$

Moreover, for all $t \in [0,T]$ the following identity holds:

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|u_x\|^2_{L^2(\mathbb{R})} ds + \int_0^t \int_{\mathbb{R}} h u^2(s) \, dx \, ds$$

$$+ \int_0^t \int_{\mathbb{R}} \lambda u(s-\tau) u(s) \, dx \, ds = \frac{1}{2}\|u(0)\|^2_{L^2(\mathbb{R})} + \int_0^t \int_{\mathbb{R}} f(x,s) u(s) \, dx \, ds .$$

The nonlinear model: well-posedness

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Then, by a mild solution of **(P)** we mean a function $u \in \mathcal{B}_T$, $T > 0$, which satisfies

$$\begin{aligned} u(t) = S(t)u_0(0) &- \int_0^t S(t-s)\lambda u(s-\tau)\, ds \\ &- \int_0^t S(t-s)a(u(s))\partial_x u(s)\, ds, \quad t \in [0, T]. \end{aligned}$$

By a global mild solution of **(P)** we mean a function $u : [0, +\infty) \to H^1(\mathbb{R})$ whose restriction to every bounded interval $[0, T]$ is a mild solution of **(P)**.

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We can prove a well–posedness result under appropriate conditions on the nonlinear term $a(u)u_x$, i.e., on the function $a(\cdot)$.

Preliminary estimates

[Gallego & Pazoto (2019)]

- Let $a \in C^0(\mathbb{R})$ be a function satisfying

$$|a(\xi)| \leq C\left(1 + |\xi|^r\right), \quad \forall \xi \in \mathbb{R},$$

with $0 \leq r < 2$. Then, there exists a positive constant C such that for any $T > 0$ and $u, v \in \mathcal{B}_T$, we have

$$\|a(u)v_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leq 2^{\frac{r}{2}} C T^{\frac{2-r}{4}} \|u\|^r_{\mathcal{B}_T} \|v\|_{\mathcal{B}_T} + C T^{\frac{1}{2}} \|v\|_{\mathcal{B}_T}.$$

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- Let $a \in C^1(\mathbb{R})$ is a function satisfying

$$|a(\xi)| \leq C\left(1 + |\xi|^r\right) \quad \text{and} \quad |a'(\xi)| \leq C\left(1 + |\xi|^{r-1}\right), \quad \forall\, \xi \in \mathbb{R},$$

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Preliminary estimates

the map $\mathcal{M} : B_T \to L^1(0,T;L^2(\mathbb{R}))$ defined by $\mathcal{M}u := a(u)u_x$ is locally Lipschitz continuous and we have,

$$\begin{aligned} &\|\mathcal{M}u - \mathcal{M}v\|_{L^1(0,T;L^2(\mathbb{R}))} \\ &\leq C\left(\sqrt{2}T^{\frac{1}{4}}\|u\|_{\mathcal{B}_T} + 2^{\frac{r}{2}}T^{2-r}4\left(\|u\|^r_{\mathcal{B}_T} + \|u\|_{\mathcal{B}_T}\|v\|^{r-1}_{\mathcal{B}_T} + \|v\|^r_{\mathcal{B}_T}\right) + T^{\frac{1}{2}}\right) \times \\ &\qquad \times \|u-v\|_{\mathcal{B}_T}, \quad \forall\, u,v \in \mathcal{B}_T. \end{aligned}$$

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&\qquad \times \|u - v\|_{\mathcal{B}_T}, \quad \forall\, u, v \in \mathcal{B}_T.
\end{aligned}$$

Under the above assumptions on the nonlinearity $a(u)$ and assumptions (H1)-(H2) on the feedback coefficients we are able to prove well-posedness and stability for our problem (P).

The nonlinear model: local well-posedness

THEOREM [I. Issa & P., 2025]

Let $a \in C^1(\mathbb{R})$ be a function satisfying

$$|a(\xi)| \leq C\,(1+|\xi|^r) \quad \text{and} \quad |a'(\xi)| \leq C\left(1+|\xi|^{r-1}\right), \quad \forall\, \xi \in \mathbb{R},$$

with $1 \leq r < 2$. Let $h, \lambda \in L^\infty(\mathbb{R})$ and $u_0 \in C([-\tau, 0]; L^2(\mathbb{R}))$. Then , the problem (P) has a unique mild solution on $[0, T)$ for some $T > 0$. Moreover,

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$$\begin{aligned} \|u(t)\|_2^2 + 2\int_0^t \|u_x(s)\|_2^2 ds + 2\int_0^t \int_{\mathbb{R}} h(x)u^2(x,s)dxds \\ +2\int_0^t \int_{\mathbb{R}} \lambda(x)u(x,s-\tau)u(x,s)dxds = \|u_0\|_2^2, \quad \forall\, t \in [0,T). \end{aligned}$$

Proof. Banach fixed point Theorem and multiplier identities.

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In order to prove that the solution is global we need to show that its norm remains bounded in the existence time interval.

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For this purpose, we consider the functional $\mathcal{E}(\cdot)$ previously defined. By differentiating, with similar computations as before, assuming (H1) and (H2), we obtain

$$\frac{d\mathcal{E}}{dt}(t) \leq 0.$$

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This ensures that $\|u(t)\|_{L^2(\mathbb{R})}$ remains bounded for $t \in [0, T]$.
From previous identity

$$\begin{aligned} \|u(t)\|_2^2 + 2\int_0^t \|u_x(s)\|_2^2 ds + 2\int_0^t \int_{\mathbb{R}} h(x)u^2(x,s)dxds \\ +2\int_0^t \int_{\mathbb{R}} \lambda(x)u(x,s-\tau)u(x,s)dxds = \|u_0\|_2^2, \quad \forall\, t \in [0,T), \end{aligned}$$

we then deduce that $\|u\|_{\mathcal{B}_T}$ remains bounded for $t \in [0, T]$.

The nonlinear model: stability

Therefore, the local solution u obtained with the point fix argument can be extended on $[-\tau, +\infty)$ under the assumptions (H1) and (H2) on the damping coefficients h, λ.

THEOREM [I. Issa & P., 2025]

Let $h, \lambda \in L^\infty(\mathbb{R})$ and $u_0 \in C([-\tau, 0]; L^2(\mathbb{R}))$ such that h satisfies (H1) and λ satisfies (H2). Let $a \in C^1(\mathbb{R})$ be a function satisfying

$$|a(\xi)| \leq C\,(1 + |\xi|^r) \quad \text{and} \quad |a'(\xi)| \leq C\left(1 + |\xi|^{r-1}\right), \quad \forall\, \xi \in \mathbb{R},$$

with $1 \leq r \leq 2$. Then, the model (P) is exponentially stable. In particular, the solution u satisfies the following estimation

$$E(t) \leq C(u^0, c_0)e^{-\delta_1(c_0)t},$$

for a suitable constant $\delta_1(c_0)$ and

$$C(u^0, c_0) = \tfrac{1}{2}\|u^0\|_2^2 + \tfrac{1}{2}\int_{-\tau}^{0}\int_{\mathbb{R}} |\lambda(x)|u^2(x,s)dxds$$
$$+ c_0 \int_{-\tau}^{0}\int_{s}^{0}\int_{\mathbb{R}} |\lambda(x)|u^2(x,\sigma)dxd\sigma ds.$$

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- The proof is obtained analogously to the one of the linear case , by using the Lyapunov functional $\mathcal{E}(\cdot)$.

- If h and λ are constant with $|\lambda| < h$, then the preceding theorem ensures the exponential stability, according to the feedback strategy introduced in [Nicaise and P., 2006] for wave equations.
The same remark applies if $\lambda, h \in L^\infty(\mathbb{R})$ with h satisfying (H1), and $\|\lambda\|_\infty < \alpha_0$.

Sign-changing undelayed damping

We can also consider a broader framework for the damping coefficient, permitting it to change sign.

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First, we assume that there exist a positive constant $\alpha_0 > 0$ and a function $g_1 \in L^p(\mathbb{R})$ for $1 \leq p < \infty$, such that

$$h(x) \geq \alpha_0 - g_1(x) \quad \text{for a.e} \quad x \in \mathbb{R},$$

where the function g_1 satisfies

$$\|g_1\|_p < \left(\frac{\alpha_0}{c(p)} \right)^{1 - \frac{1}{2p}},$$

with $c(p)$ defined as above, i.e.,

$$c(p) := \left(1 - \frac{1}{2p}\right) \left(\frac{2}{p}\right)^{\frac{1}{2p-1}}.$$

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$$c(p) := \left(1 - \frac{1}{2p}\right)\left(\frac{2}{p}\right)^{\frac{1}{2p-1}}.$$

Moreover, we assume that the function λ satisfies the following

$$|\lambda(x)| \leq \alpha + g(x) \quad \text{for a.e.} \quad x \in \mathbb{R},$$

for some constant α and some function $g \in L^p(\mathbb{R})$ such that

Sign-changing undelayed damping

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$$0 \leq \alpha < \alpha_0 \quad \text{and} \quad \|g + g_1\|_p < \left(\frac{\alpha_0 - \alpha}{c(p)}\right)^{1-\frac{1}{2p}}.$$

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- Also in this case, we can obtain exponential stability without any restrictions on the time delay size.
- In both cases, some explicit values of the decay rates in terms of the problem's parameters can be obtained.

Further extensions and recent results

Possible extensions:

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Some recent results:

- Stabilization for higher-order dispersive systems [Capistrano-Filho, Gallego & Komornik, 2023];
- Boundary stabilization of delayed KdV-B in bounded domain (stochastic delay impulsive in [Liang, Xue & Wu, 2025], nonlinear boundary control in [Cheng, Wu, Wu & Guo, 2025]).

Thank you for your attention!