On the approximation of the controls for the 2-D wave equation

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Controlled wave equation

Let $\Omega \subset \mathbb{R}^n$ and divide its boundary in two parts: $\partial\Omega = \Gamma_0 \cup \Gamma_1$. We consider the linear (non homogeneous) wave equation

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) = 0 & t>0, \ x\in\Omega \\ u(t,x) = v(t,x) & t>0, \ x\in\Gamma_0 \\ u(t,x) = 0 & t>0, \ x\in\Gamma_1 \\ u(0,x) = u^0(x), \quad u_t(0,x) = u^1(x) & x\in\Omega. \end{cases} \tag{1}$$

Given $T>0$ equation (1) is exactly controllable in time T if for each initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H} := L^2(\Omega)\times H^{-1}(\Omega)$ there exists a function $v\in L^2((0,T)\times\Gamma_0)$, called control, such that the corresponding solution of (1) verifies

$$u(T,x) = u_t(T,x) = 0 \qquad (x\in\Omega). \tag{2}$$

Variational characterization

The function $v \in L^2((0,T) \times \Gamma_0)$ is a control which drives to zero the solution of (1) in time T if and only if, the following relation holds

$$\int_0^T \int_{\Gamma_0} v(t,s) \frac{\partial \overline{\varphi}}{\partial \nu}(t,s) ds dt$$
$$= \underbrace{\langle u^1, \varphi(0)\rangle_{H^{-1},H_0^1} - \int_\Omega u^0(x)\overline{\varphi_t}(0,x)dx}_{\langle (u^0,u^1),(\varphi(0),\varphi_t(0))\rangle_D}, \quad (3)$$

for every $\begin{pmatrix}\varphi^0\\ \varphi^1\end{pmatrix} \in H_0^1(\Omega) \times L^2(\Omega)$, where $\begin{pmatrix}\varphi\\ \varphi_t\end{pmatrix}$ is the solution of the following adjoint backward problem

$$\begin{cases} \varphi_{tt}(t,x) - \Delta\varphi(t,x) = 0 & t > 0,\ x \in \Omega \\ \varphi(t,x) = 0 & t > 0,\ x \in \partial\Omega \\ \varphi(T,x) = \varphi^0(x), \quad \varphi_t(T,x) = \varphi^1(x) & x \in \Omega. \end{cases} \quad (4)$$

Variational characterization

Relation (3) may be seen as an optimality condition for the critical points of the functional $\mathcal{J} : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$,

$$\mathcal{J}(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial \nu}(t,s) \right|^2 dsdt - \left\langle \left(u^0, u^1\right), (\varphi(0), \varphi_t(0)) \right\rangle_D, \tag{5}$$

where φ is the solution of (4). We have:

Theorem

Let $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and suppose that $\begin{pmatrix} \widehat{\varphi}^0 \\ \widehat{\varphi}^1 \end{pmatrix} \in H_0^1(\Omega) \times L^2(\Omega)$ is a minimizer of $\mathcal{J}$. If $\widehat{\varphi}$ is the corresponding solution of (4) with initial data $\begin{pmatrix} \widehat{\varphi}^0 \\ \widehat{\varphi}^1 \end{pmatrix}$ then

$$v = \left. \frac{\partial \widehat{\varphi}}{\partial \nu} \right|_{\Gamma_0} \tag{6}$$

is a control which leads $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix}$ to zero in time T.

Variational characterization: inverse inequality

The functional $\mathcal{J}$ has a unique minimizer in $H_0^1(\Omega) \times L^2(\Omega)$ if it is coercive, i.e. there exits a positive constant $C > 0$ such that the following inequality holds

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial \nu}(t,s) \right|^2 ds dt \geq C \left\| \begin{pmatrix} \varphi^0 \\ \varphi^1 \end{pmatrix} \right\|_{H_0^1 \times L^2}^2, \qquad (7)$$

for every $\begin{pmatrix} \varphi^0 \\ \varphi^1 \end{pmatrix} \in H_0^1(\Omega) \times L^2(\Omega)$, where $\begin{pmatrix} \varphi \\ \varphi_t \end{pmatrix}$ is the solution of (4).

Methods to prove (7):

- Multipliers
- Carleman inequalities
- Spectral methods: 1-D, n-D (Special Geometries)
  We can address (7) or we can go back to the variational equality and show that it holds for the eigenvectors of the corresponding differential operator (problem of moments).

Geometric condition

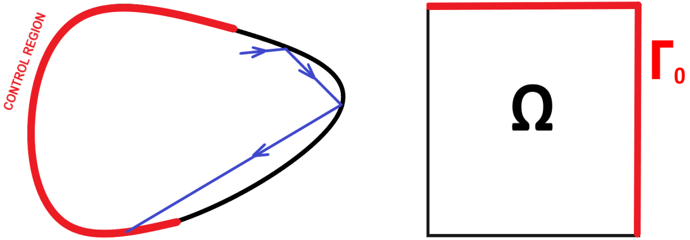

Figure: Geometric condition for the control region.

A necessary and sufficient condition is that the region of control must meet every ray of geometric optics in a nondiffractive point in time T.

C. Bardos, G. Lebeau, and J. Rauch: *Sharp Sufficient Conditions for the Observation, Control, and Stabilization of Waves from the Boundary*, SIAM J. Cont. Opt., 30 (1992), pp. 1024–1065.

- H.O. Fattorini and D.L. Russell: Exact controllability theorems for linear parabolic equation in one space dimension, Arch. Rat. Mech. Anal. 43 (1971), 272- 292.
- H.O. Fattorini and D.L. Russell: Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations, Quarterly J. Appl. Math. 32 (1974), 45–69.
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- K. Ramdani, T. Takahashi, G. Tenenbaum, M. Tucsnak: A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator, Journal of Functional Analysis 226 (1) (2005), 193-229.
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Particular case: Spectral analysis

$$\Omega = (0,a) \times (0,b) \subset \mathbb{R}^2,$$

$$\partial\Omega = \Gamma_0 \cup \Gamma_1, \quad \Gamma_0 = \{(a,y) \,|\, 0 \le y \le b\} \cup \{(x,b) \,|\, 0 \le x \le a\}.$$

- Eigenvalues of the differential operator: $(i\lambda_{mn}^{\pm})_{(m,n)\in\mathbb{N}^*\times\mathbb{N}^*}$, where

$$\lambda_{mn}^{\pm} = \pm\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

- Eigenfunctions of the differential operator: $(\Phi_{mn}^{\pm})_{(m,n)\in\mathbb{N}^*\times\mathbb{N}^*}$ form an orthonormal basis in $H_0^1(\Omega) \times L^2(\Omega)$, where

$$\Phi_{mn}^{\pm} = \sqrt{2}\begin{pmatrix} \frac{1}{i\lambda_{mn}^{\pm}} \\ -1 \end{pmatrix} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right).$$

The controllability of the wave equation is equivalent to solve the following moment problem:

$$\int_0^T \int_0^b e^{-i\lambda_{mn}^{\pm} t} \sin\left(\frac{n\pi y}{b}\right) v^1(t,y)dydt + \int_0^T \int_0^a e^{-i\lambda_{mn}^{\pm} t} \sin\left(\frac{m\pi x}{a}\right) v^2(t,x)dxdt = \widetilde{a}_{mn}^{\pm}, \quad (8)$$

where $\widetilde{a}_{mn}^{\pm}$ are, essentially, the Fourier coefficients of the initial data.

A solution (v^1, v^2) of the moment problem is constructed by means of two biorthogonal sequences to the family $\left(e^{i\lambda_{mn}^{\pm} t}\right)_{(m,n)\in\mathbb{N}^*\times\mathbb{N}^*}$.

Biorthogonal sequence

Definition

Let $m \in \mathbb{N}^*$ be fixed. The sequence $\left(\theta_{mn}^{1\pm}\right)_{n\in\mathbb{N}^*} \in L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ is $(1, m)$-biorthogonal to the family $\left(e^{i\lambda_{mn}^{\pm} t}\right)_{n\in\mathbb{N}^*}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ if

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_{mn}^{1\pm}(t) e^{-i\lambda_{mq}^{\pm} t} dt = \delta_{nq}, \quad \int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_{mn}^{1\mp}(t) e^{-i\lambda_{mq}^{\pm} t} dt = 0 \quad (n, q \in \mathbb{N}^*).$$

Let $n \in \mathbb{N}^*$ be fixed. The sequence $\left(\theta_{mn}^{2\pm}\right)_{m\in\mathbb{N}^*} \in L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ is $(2, n)$-biorthogonal to the family $\left(e^{i\lambda_{mn}^{\pm} t}\right)_{m\in\mathbb{N}^*}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ if

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_{mn}^{2\pm}(t) e^{-i\lambda_{pn}^{\pm} t} dt = \delta_{mp}, \quad \int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_{mn}^{2\mp}(t) e^{-i\lambda_{pn}^{\pm} t} dt = 0 \quad (m, p \in \mathbb{N}^*).$$

If $\left(\theta^{1\pm}_{mn}\right)_{n\in\mathbb{N}^*}$ is $(1,m)$-biorthogonal to the family $\left(e^{i\lambda^{\pm}_{mn}t}\right)_{n\in\mathbb{N}^*}$ and $\left(\theta^{2\pm}_{mn}\right)_{m\in\mathbb{N}^*}$ is $(2,n)$-biorthogonal to the family $\left(e^{i\lambda^{\pm}_{mn}t}\right)_{m\in\mathbb{N}^*}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ then a "formal" solution of the moment problem is given by $\begin{pmatrix} v^1(t,y) \\ v^2(t,x) \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} v^1_n(t)\sin\left(\frac{n\pi y}{b}\right) \\ \sum_{m=1}^{\infty} v^2_m(t)\sin\left(\frac{m\pi x}{a}\right) \end{pmatrix}$, where

$$\begin{cases} v^1_n(t) = \sum\limits_{m>\frac{a}{b}n}^{\infty} \widetilde{a}^{\pm}_{mn}\theta^{2\pm}_{mn}\left(\frac{T}{2}-t\right) & (n\in\mathbb{N}^*) \\ v^2_m(t) = \sum\limits_{n\geq\frac{b}{a}m}^{\infty} \widetilde{a}^{\pm}_{mn}\theta^{1\pm}_{mn}\left(\frac{T}{2}-t\right) & (m\in\mathbb{N}^*). \end{cases} \tag{9}$$

The control $v^1(t,y)$ is in charge of the frequencies (m,n) with $m>\frac{a}{b}n$ and left untouched the rest of the frequencies. Similarly, The control $v^2(t,x)$ is in charge of the frequencies (m,n) with $n\geq\frac{b}{a}m$ and left untouched the rest of the frequencies. In the end, the entire spectrum is controlled!

Main problems

- the existence of the biorthogonal sequences $\left(\theta_{mn}^{1\pm}\right)_n$ and $\left(\theta_{mn}^{2\pm}\right)_m$ to the family $(e^{i\lambda_{mn}^{\pm}t})_{(m,n)}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$
- evaluation of the norms of $\left(\theta_{mn}^{1\pm}\right)_n$ and $\left(\theta_{mn}^{2\pm}\right)_m$

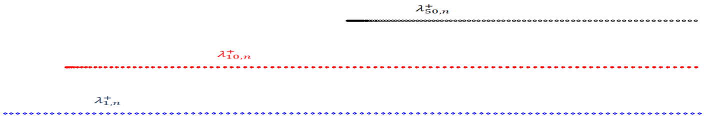
This estimates are needed to show the convergence of the series in (9) and to have a bound of the norms of v_n^1 and v_m^2.

S. M., L. Teresa, Asymptotic Analysis, 2010

$$\|\theta_{mn}^{1\pm}\|_{L^2(0,T)} \leq C \qquad (m\in\mathbb{N}^*,\ n\geq \frac{b}{a}m),$$

$$\|\theta_{mn}^{2\pm}\|_{L^2(0,T)} \leq C \qquad (n\in\mathbb{N}^*,\ m> \frac{a}{b}n).$$

A few comments



- The asymptotic gap is uniform $\frac{\pi}{b}$, for each family $(\lambda^{\pm}_{mn})_{n\in\mathbb{N}^*}$ and each $m \in \mathbb{N}$.
- At the beginning of the sequence $(\lambda^{\pm}_{mn})_{n\in\mathbb{N}^*}$ the gap depends of m, being of order $\frac{1}{m}$.
- The gap becomes uniform on m if we consider $n > m$.

A few comments

- H.O. Fattorini: Estimates for sequences biorthogonal to certain exponentials and boundary control of the wave equation, in: New Trends in Systems Analysis, 1977, 111-124.
- A. Haraux: A generalized internal control for the wave equation in a rectangle, Journal of Mathematical Analysis and Applications, 153(1) (1990), 190-216.

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{n\in\mathbb{N}^*} a_{mn}^{\pm} e^{i\lambda_{mn}^{\pm} t} \right|^2 \mathrm{d}t \geq C e^{\omega m} \sum_{n\in\mathbb{N}^*} \left| a_{mn}^{\pm} \right|^2$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{n\in\mathbb{N}^*} a_{mn}^{\pm} e^{i\lambda_{mn}^{\pm} t} \right|^2 \mathrm{d}t \geq C e^{\omega m} \sum_{n<\frac{b}{a}m} \left| a_{mn}^{\pm} \right|^2 + C' \sum_{n\geq\frac{b}{a}m} \left| a_{mn}^{\pm} \right|^2 .$$

$$\|\theta_{mn}^{1\pm}\|^2_{L^2(-\frac{T}{2},\frac{T}{2})} \leq C e^{\omega m} \quad (n \in \mathbb{N}^*), \qquad \|\theta_{mn}^{1\pm}\|^2_{L^2(-\frac{T}{2},\frac{T}{2})} \leq C' \quad (n \geq \frac{b}{a} m).$$

A constructive way to obtain a biorthgonal sequence

- R.E.A.C. Paley and N.Wiener: Fourier Transforms in Complex Domains, AMS Colloq. Publ., Vol. 19, Amer. Math. Soc., New York, 1934.

- $(\Psi_{mn}^{\pm})_{(m,n)\in\mathbb{N}^*\times\mathbb{N}^*}$ entire functions.

H1 ▷ $|\Psi_{mn}^{\pm}(z)| \leq Ae^{\frac{T}{2}|z|}$,

H2 ▷ $\Psi_{mn}^{\pm} \in L^2(\mathbb{R})$,

H3 ▷ $\begin{cases} \Psi_{mn}^{+}\left(\lambda_{mq}^{+}\right) = \delta_{nq} & \Psi_{mn}^{+}\left(\lambda_{mq}^{-}\right) = 0 \\ \Psi_{mn}^{-}\left(\lambda_{mq}^{-}\right) = \delta_{nq} & \Psi_{mn}^{-}\left(\lambda_{mq}^{+}\right) = 0 \end{cases}$

Paley–Wiener Theorem

$$\theta_{mn}^{\pm} \in L^2\left(-\frac{T}{2}, \frac{T}{2}\right) \text{ such that } \Psi_{mn}^{\pm}(z) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_{mn}^{\pm}(t)e^{-izt}\,dt.$$

Plancherel's Theorem

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left|\theta_{mn}^{\pm}(t)\right|^2 dt = \frac{1}{2\pi}\int_{\mathbb{R}} \left|\Psi_{mn}^{\pm}(x)\right|^2 dx.$$

Finite differences for the 2-D wave equation

Let $J, K \in \mathbb{N}^*$, $h_1 = \frac{a}{J+1}$, $h_2 = \frac{b}{K+1}$ and $x_{j,k} = (jh_1, kh_2)$, $0 \leq j \leq J+1$, $0 \leq k \leq K+1$, $\sigma = \frac{h_1}{h_2}$ is uniformly bounded,
$\Gamma_h^1 = \{(0, kh_2) \,|\, 0 \leq k \leq K+1\} \cup \{(jh_1, 0) \,|\, 0 \leq j \leq J+1\}$,
$\Gamma_h^0 = \{(a, kh_2) \,|\, 0 \leq k \leq K+1\} \cup \{(jh_1, b) \,|\, 0 \leq j \leq J+1\}$.

$$\begin{cases} u''_{pr}(t) - \frac{u_{j+1k}(t) - 2u_{jk}(t) + u_{j-1k}(t)}{h_1^2} - \frac{u_{jk+1}(t) - 2u_{jk}(t) + u_{jk-1}(t)}{h_2^2} = 0 \\ \qquad\qquad t > 0,\ 1 \leq j \leq J,\ 1 \leq k \leq K \\ u_{jk}(t) = 0 \qquad t \in (0,T),\ (j,k) \in \Gamma_h^1 \\ u_{J+1,k}(t) = v_k^1(t) \qquad t \in (0,T),\ (j,k) \in \Gamma_h^0 \\ u_{j,K+1}(t) = v_j^2(t) \qquad t \in (0,T),\ (j,k) \in \Gamma_h^0 \\ u_{jk}(0) = u_{jk}^0, \quad u'_{jk}(0) = u_{jk}^1 \qquad 1 \leq j \leq J,\ 1 \leq k \leq K. \end{cases} \tag{10}$$

Discrete controllability problem: Given $T > 0$ and $\begin{pmatrix} U^0 \\ U^1 \end{pmatrix} = (u_{jk}^0, u_{jk}^1)_{1 \leq j \leq J,\, 1 \leq k \leq K} \in \mathbb{C}^{2JK}$, there exists a control function $v_h = (v^1, v^2) \in L^2\left(0, T; \mathbb{C}^{J+K}\right)$ such that the corresponding solution $(u_{jk})_{1 \leq j \leq J,\, 1 \leq k \leq K}$ of (10) verifies

$$u_{jk}(T) = u'_{jk}(T) = 0 \qquad (1 \leq j \leq J,\, 1 \leq k \leq K) \tag{11}$$

R. Glowinski and J.L. Lions: *Exact and approximate controllability for distributed parameter systems*, Acta Numerica, 4 (1995), pp 159–328.

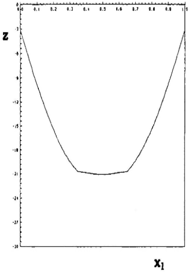


Fig. 45. $z^0(x_1, .5)$.

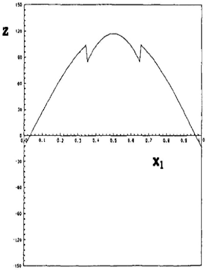


Fig. 46. $z^1(x_1, .5)$.

Figure: Initial data

Numerical Approximations

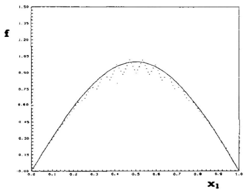


Fig. 48. Variations of $f^0(x_1, .5)$ (——) and $f_*^0(x_1, .5)$ (······) ($h = 1/32$).

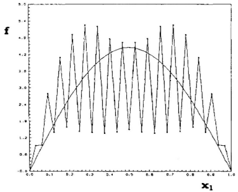


Fig. 49. Variations of $f^1(x_1, .5)$ (——) and $f_*^1(x_1, .5)$ (······) ($h = 1/32$).

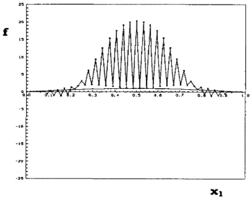


Fig. 50. Variations of $f^0(x_1, .5)$ (——) and $f_*^0(x_1, .5)$ (······) ($h = 1/64$).

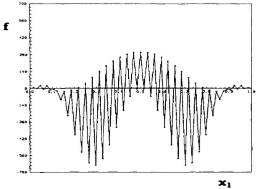


Fig. 51. Variations of $f^1(x_1, .5)$ (——) and $f_*^1(x_1, .5)$ (······) ($h = 1/64$).

Figure: Approximations of the control with $h = \frac{1}{32}$ and $\frac{1}{64}$.

Control of the projection of the solution

E. Zuazua, J. Math. Pures et Appl., 1999

- The constants on the boundary observability inequality blow-up as the mesh-size tends to zero.
  This is do to the largest eigenvalues of the corresponding adjoint system which are very different from the continuous ones (numerical spurious high eigenfrequencies).
- We recuperate the uniform observability inequality if we consider only the projections of the solutions over the space generated by the low frequencies:

$$\lambda \max\{h_1, h_2\} \leq 2\delta, \quad \delta \in (0,1).$$

  This is equivalent to the uniform controllability of the projection of the solution over this space.

The aim of this work is to show that we can guarantee the uniform controllability of the entire solution by filtering the high frequencies of the initial data only.

U. Biccari, A. Marica, E. Zuazua: : *Propagation of One- and Two-Dimensional Discrete Waves Under Finite Difference Approximation*, Foundations of Computational Mathematics, 20 (2020), pp. 1401–1438.

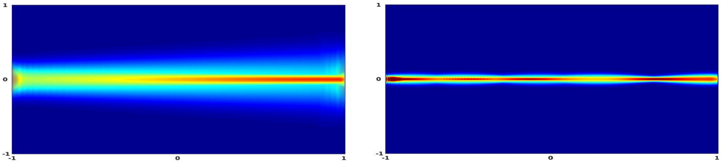


Fig. 16 Numerical solutions with initial datum (3.14) and parameters $(x_0, y_0, \xi_0, \eta_0) = (1, 0, \pi/2, \pi)$. The discretization is done on a uniform mesh (left) and on a non-uniform one obtained through the mesh function **g** (right). The time horizon is $T = 10$ s in both cases (Color figure online)

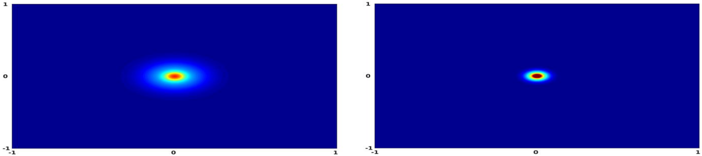


Fig. 17 Numerical solutions with initial datum (3.14) and parameters $(x_0, y_0, \xi_0, \eta_0) = (0, 0, \pi, \pi)$. The discretization is done on a uniform mesh (left) and on a non-uniform one obtained through the mesh function **g** (right). The time horizon is $T = 10$ s in both cases (Color figure online)

The eigenvalues are given by the family $(i\lambda^{\pm}_{mn}(h_1,h_2))_{\substack{1\leq m\leq J\\ 1\leq n\leq K}}$, where

$$\lambda^{\pm}_{mn}(h_1,h_2) = \pm\sqrt{\frac{4}{h_1^2}\sin^2\left(\frac{m\pi h_1}{2}\right)+\frac{4}{h_2^2}\sin^2\left(\frac{n\pi h_2}{2}\right)}, \qquad (12)$$

and the corresponding eigenvectors are

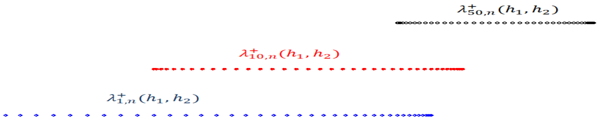
$$\Phi^{\pm}_{mn}(h_1,h_2) = \sqrt{2}\begin{pmatrix} \frac{1}{i\lambda^{\pm}_{mn}(h_1,h_2)} \\ -1 \end{pmatrix}\left(\sin\left(m\pi p h_1\right)\sin\left(n\pi r h_2\right)\right)_{\substack{1\leq p\leq J\\ 1\leq r\leq K}} \qquad (13)$$

Moreover, the vectors $(\Phi^{\pm}_{mn}(h_1,h_2))_{\substack{1\leq p\leq J\\ 1\leq r\leq K}}$ form an orthonormal basis in $\mathbb{C}^{2JK}$.

We denote by

$$\psi_m(h) = \left(\sin\left(m\pi p h\right)\right)_{1\leq p\leq J}.$$

A few comments



- The family $(\lambda^{\pm}_{mn}(h_1,h_2))_{1\leq n\leq J}$ does not have uniform gap with respect to h_1: the last eigenvalues are at distance h_1.

Moment problem for the discrete problem

Theorem

Given $T > 0$, system (10) is null-controllable in time T if, and only if, for any initial data $\begin{pmatrix} U^0 \\ U^1 \end{pmatrix} \in \mathbb{C}^{2JK}$ *of the form*

$$\begin{pmatrix} U^0 \\ U^1 \end{pmatrix} = \sum_{\substack{1\le m\le J \\ 1\le n\le K}} \alpha_{mn}^{\pm}(h_1,h_2)\Phi_{mn}^{\pm}(h_1,h_2), \tag{14}$$

there exists $v_h = \begin{pmatrix} v_h^1 \\ v_h^2 \end{pmatrix} \in L^2\left(0,T;\mathbb{C}^{J+K}\right)$*, such that, for each* $1 \le m \le J$, $1 \le n \le K$*, we have*

$$\int_0^T \left(\langle v_h^1(t), \psi_n(h_2)\rangle + \langle v_h^2(t), \psi_m(h_1)\rangle\right) e^{i\lambda_{mn}^{\pm} t} dt = \widetilde{\alpha}_{mn}^{\pm}(h_1,h_2). \tag{15}$$

Let $v_h^1(t) = \sum_{n=1}^{J} v_n^1(t)\psi_n(h_2)$, $v_h^2(t) = \sum_{m=1}^{K} v_m^2(t)\psi_m(h_1)$ and $\delta \in (0,1)$.

If $\left(\theta_{mn}^{1\pm}\right)_{1\leq n\leq K}$ is $(1,m)$-biorthogonal to the family $\left(e^{i\lambda_{mn}^{\pm}t}\right)_{1\leq n\leq K}$ and $\left(\theta_{mn}^{2\pm}\right)_{1\leq m\leq J}$ is $(2,n)$-biorthogonal to the family $\left(e^{i\lambda_{mn}^{\pm}t}\right)_{1\leq m\leq J}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ then a "formal" solution is:

$$v_n^1(t) = \begin{cases} \sum\limits_{\frac{a\sigma}{b}n<m<\delta(J+1)} \widetilde{\alpha}_{mn}^{\pm}\theta_{mn}^{2\pm}\left(\frac{T}{2}-t\right) & \text{if } n<\delta(J+1) \\ 0 \quad \text{if } n\geq\delta(J+1), \end{cases}$$

$$v_m^2(t) = \begin{cases} \sum\limits_{\frac{b}{a\sigma}m\leq n<\delta(K+1)} \widetilde{\alpha}_{mn}^{\pm}\theta_{mn}^{1\pm}\left(\frac{T}{2}-t\right) & \text{if } m<\delta(K+1) \\ 0 \quad \text{if } m\geq\delta(K+1). \end{cases}$$

We need to construct and evaluate the biorthogonal sequences!

The biorthogonal sequences

Theorem

Let $\eta, \delta \in (0,1)$ *and* $T_1 > \pi^2 e\sigma + \frac{4b\sqrt{\sigma^2+1}}{(1-\eta)\cos^2\frac{\delta\pi}{2}}$. *There exist* $h_1^*, h_2^* > 0$ *and two constants* C, $\rho_1 > 0$ *such that, for every* $h_1 \in (0, h_1^*)$, $h_2 \in (0, h_2^*)$ *and* $1 \leq m \leq J$, *a* $(1,m)$*-biorthogonal sequence* $\left(\theta_{mn}^{1\pm}\right)_{1\leq n\leq K}$ *to the family of exponential functions* $\left(e^{i\,\lambda_{mq}^{\pm} t}\right)_{1\leq q\leq K}$ *in* $L^2\left(-\frac{T_1}{2}, \frac{T_1}{2}\right)$ *can be constructed with the property that for any sequence of scalars* $(a_n^{\pm})_{1\leq n\leq K} \subset \mathbb{C}$ *the following inequality holds:*

$$\left\|\sum_{n=1}^{K} a_n^{\pm}\theta_{mn}^{1\pm}\right\|^2_{L^2\left(-\frac{T_1}{2},\frac{T_1}{2}\right)} \leq C\left[e^{\frac{b}{a}\pi^2 m}\sum_{1\leq n<\frac{b}{a\sigma}m}|a_n^{\pm}|^2 \right. \tag{16}$$

$$\left. + \sum_{\frac{b}{a\sigma}m\leq n<\delta(K+1)}|a_n^{\pm}|^2 + e^{2\rho_1 K}\sum_{\delta(K+1)\leq n\leq K}|a_n^{\pm}|^2\right].$$

Theorem

Let $\eta, \delta \in (0,1)$ *and* $T_2 > \pi^2 e\sigma + \frac{4b\sqrt{\sigma^2+1}}{\sigma(1-\eta)\cos^2\frac{\delta\pi}{2}}$. *There exist* $h_1^{**}, h_2^{**} > 0$ *and two constants* C, $\rho_2 > 0$ *such that for every* $h_1 \in (0, h_1^{**})$, $h_2 \in (0, h_2^{**})$ *and* $1 \leq n \leq K$ *a* $(2,n)$*-biorthogonal sequence* $\left(\theta_{mn}^{2\pm}\right)_{1\leq m\leq J}$ *to the family of exponential functions* $\left(e^{i\lambda_{pn}^{\pm}t}\right)_{1\leq p\leq J}$ *in* $L^2\left(-\frac{T_2}{2}, \frac{T_2}{2}\right)$ *can be constructed with the property that for any sequence of scalars* $(a_m^{\pm})_{1\leq m\leq J} \subset \mathbb{C}$ *the following inequality holds:*

$$\left\| \sum_{m=1}^{J} a_m^{\pm}\theta_{mn}^{2\pm} \right\|^2_{L^2\left(-\frac{T_2}{2}, \frac{T_2}{2}\right)} \leq C \left[e^{\frac{a}{b}\pi^2 n} \sum_{1\leq m\leq \frac{a\sigma}{b}n} |a_m^{\pm}|^2 \right. \tag{17}$$

$$\left. + \sum_{\frac{a\sigma}{b}n<m<\delta(J+1)} |a_m^{\pm}|^2 + e^{2\rho_2 J} \sum_{\delta(J+1)\leq m\leq J} |a_m^{\pm}|^2 \right].$$

In particular we have that

$$\left\|\theta_{mn}^{1\pm}\right\|_{L^2\left(-\frac{T_1}{2},\frac{T_1}{2}\right)} \leq C \qquad \left(\frac{b}{a\sigma}m \leq n < \delta(K+1)\right), \tag{18}$$

$$\left\|\theta_{mn}^{2\pm}\right\|_{L^2\left(-\frac{T_2}{2},\frac{T_2}{2}\right)} \leq C \qquad \left(\frac{a\sigma}{b}n < m < \delta(J+1)\right). \tag{19}$$

Recall that we construct controls with precisely these biorthogonal terms:

$$v_n^1(t) = \begin{cases} \displaystyle\sum_{\frac{a\sigma}{b}n<m<\delta(J+1)} \widetilde{\alpha}_{mn}^{\pm}\theta_{mn}^{2\pm}\left(\frac{T}{2}-t\right) & \text{if } \ n < \delta(J+1) \\ 0 & \text{if } \ n \geq \delta(J+1), \end{cases}$$

$$v_m^2(t) = \begin{cases} \displaystyle\sum_{\frac{b}{a\sigma}m\leq n<\delta(K+1)} \widetilde{\alpha}_{mn}^{\pm}\theta_{mn}^{1\pm}\left(\frac{T}{2}-t\right) & \text{if } \ m < \delta(K+1) \\ 0 & \text{if } \ m \geq \delta(K+1). \end{cases}$$

- **A Weierstrass Product:**

$$P^{1,\pm}_{mn}(z)=\prod_{\substack{1\leq q\leq K\\ q\neq n}}\frac{z-\lambda^{\pm}_{mq}}{\lambda^{\pm}_{mn}-\lambda^{\pm}_{mq}}\prod_{1\leq q\leq K}\frac{z-\lambda^{\mp}_{mq}}{\lambda^{\pm}_{mn}-\lambda^{\mp}_{mq}}\qquad (1\leq n\leq K),$$

$$|P^{1,\pm}_{mn}(x)|\leq C\begin{cases}\exp\left(\frac{1}{2}(\frac{h_2}{h_1}+\pi)\varphi_1(x)\right) & \left(|x|\leq\frac{2}{h_1}\sin\frac{m\pi h_1}{2}\right)\\ 1 & \left(\frac{2}{h_1}\sin\frac{m\pi h_1}{2}<|x|<\sqrt{\frac{4}{h_1^2}\sin^2\frac{m\pi h_1}{2}+\frac{4}{h_2^2}}\right)\\ \exp\left(\varphi_2(x)\right) & \left(|x|\geq\sqrt{\frac{4}{h_1^2}\sin^2\frac{m\pi h_1}{2}+\frac{4}{h_2^2}}\right),\end{cases}$$

where

$$\varphi_1(x)=\sqrt{\frac{4}{h_1^2}\sin^2\frac{m\pi h_1}{2}-x^2},$$

$$\varphi_2(x)=\frac{2}{h_2}\ln\left(\frac{h_2}{2}\sqrt{x^2-\frac{4}{h_1^2}\sin^2\frac{m\pi h_1}{2}}+\sqrt{x^2\frac{h_2^2}{4}-\frac{h_2^2}{h_1^2}\sin^2\frac{m\pi h_1}{2}-1}\right).$$

The construction of the biorthogonal sequences

- **The multipliers:**

P. Lissy, I. Roventa, Math. Comp., 2019

$$|M^{\pm}_{mn}(x-\lambda^{\pm}_{mn})| \leq \begin{cases} \exp\left(-\varphi_2(x)\right) & \left(|x| \geq \sqrt{\frac{4}{h_1^2}\sin^2\frac{m\pi h_1}{2}+\frac{4}{h_2^2}}\right) \\ C & \left(|x| < \sqrt{\frac{4}{h_1^2}\sin^2\frac{m\pi h_1}{2}+\frac{4}{h_2^2}}\right) \end{cases}$$

S. Micu, L. Teresa, Asymptotic Analysis, 2010

$$|G^{\pm}_{mn}(x-\lambda^{\pm}_{mn})| \leq \begin{cases} \exp\left(-\frac{1}{2}(\frac{h_2}{h_1}+\pi)\varphi_1(x)\right) & \left(|x| \leq \frac{2}{h_1}\sin\frac{m\pi h_1}{2}\right) \\ C & \left(|x| > \frac{2}{h_1}\sin\frac{m\pi h_1}{2}\right) \end{cases}$$

The construction of the biorthogonal sequences

- **The entire function**

$$\Psi_{mn}^{1,\pm}(z) := P_{mn}^{1,\pm}(z)M_{mn}^{\pm}(z-\lambda_{mn}^{\pm})G_{mn}^{\pm}(z-\lambda_{mn}^{\pm})\frac{\sin\epsilon(z-\lambda_{mn}^{\pm})}{\epsilon(z-\lambda_{mn}^{\pm})}.$$

- **Th. Paley-Wienner** $\Rightarrow$ $(\theta_{mn}^{1,\pm})_n = (\widehat{\Psi}_{mn}^{1,\pm})_n$ **biorthogonal**

Uniform boundedness of the sequence of controls

Theorem

Let $\delta, \eta \in (0,1)$, $h_1^0 = \min\{h_1^*, h_1^{**}\}$, $h_2^0 = \min\{h_2^*, h_2^{**}\}$ *and* $T = \max\{T_1, T_2\}$, *where* (h_1^*, h_2^*, T_1) *and* $(h_1^{**}, h_2^{**}, T_2)$ *are given by Theorem 3 and Theorem 4, respectively. There exists a constant* $C > 0$ *such that for any* $h_1 \in (0, h_1^0)$, $h_2 \in (0, h_2^0)$ *and each initial data* $\begin{pmatrix} U^0 \\ U^1 \end{pmatrix} \in \mathbb{C}^{2JK}$ *of the form*

$$\begin{pmatrix} U^0 \\ U^1 \end{pmatrix} = \sum_{\substack{1 \leq m \leq \delta(J+1) \\ 1 \leq n \leq \delta(K+1)}} \alpha_{mn}^{\pm}(h_1, h_2) \Phi_{mn}^{\pm}(h_1, h_2), \tag{20}$$

there exists a control $v_h \in L^2\left(0, T; \mathbb{C}^{J+K}\right)$ *of* (10) *such that*

$$\|v_h\|_{L^2(0,T;\mathbb{C}^{J+K})} \leq C \left\| \begin{pmatrix} U^0 \\ U^1 \end{pmatrix} \right\|_{0,-1}. \tag{21}$$

Theorem

Let $\delta, \eta \in (0,1)$, $h_1^0 = \min\{h_1^*, h_1^{**}\}$, $h_2^0 = \min\{h_2^*, h_2^{**}\}$ *and* $T = \max\{T_1, T_2\}$, *where* (h_1^*, h_2^*, T_1) *and* $(h_1^{**}, h_2^{**}, T_2)$ *are given by Theorem 3 and Theorem 4, respectively. There exists a constant* $C > 0$ *such that for any* $h_1 \in (0, h_1^0)$, $h_2 \in (0, h_2^0)$ *and each initial data* $\begin{pmatrix} U^0 \\ U^1 \end{pmatrix} \in \mathbb{C}^{2JK}$ *given by* (14) *where the Fourier coefficients verify*

$$\sum_{\substack{1\leq m\leq J \\ 1\leq n\leq K}} \left(e^{\frac{2\rho_1}{\delta}n} + e^{\frac{2\rho_2}{\delta}m}\right) \left|\alpha_{mn}^{\pm}(h_1,h_2)\right|^2 < C, \tag{22}$$

there exists a control $v_h \in L^2\left(0,T;\mathbb{C}^{J+K}\right)$ *of* (10) *such that*

$$\|v_h\|_{L^2(0,T;\mathbb{C}^{J+K})} \leq C. \tag{23}$$

About the filtration

Let us consider the discretization by points of the initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1)$:

$$\begin{pmatrix} U^0 \\ U^1 \end{pmatrix} = \begin{pmatrix} u^0(jh)_{1\leq j\leq N} \\ u^1(jh)_{1\leq j\leq N} \end{pmatrix}, \tag{24}$$

For $i \in \{0,1\}$ and some $\tau > 0$, let $\widetilde{U}^i(t)$ be the solution of the system

$$\begin{cases} (\widetilde{U}^i)'(t) + hL_h\widetilde{U}^i(t) = 0 \qquad (t \in (0,\tau)) \\ \widetilde{U}^i(0) = U^i, \end{cases} \tag{25}$$

where L_h is the finite differences discrete Laplace operator.

We can use instead of $\begin{pmatrix} U^0 \\ U^1 \end{pmatrix}$ the initial data $\begin{pmatrix} \widetilde{U}_h^0(\tau) \\ \widetilde{U}_h^1(\tau) \end{pmatrix}$, which verifies the conditions from the previous theorem. Indeed, the new data have almost the same low modes as the initial ones, but the high modes have negative exponential weights.

Numerical experiments

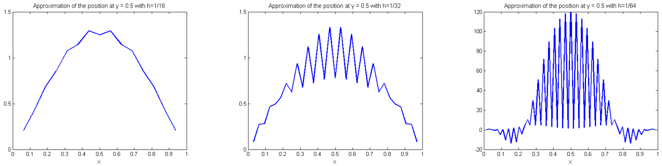


Figure: Approximations of the position with three values of $h \in \left\{\frac{1}{16}, \frac{1}{32}, \frac{1}{64}\right\}$ without filtration.

Numerical experiments

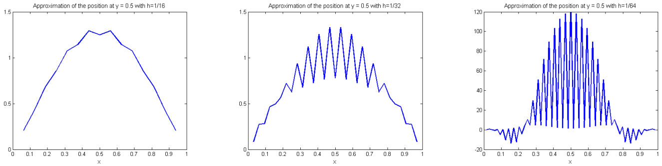


Figure: Approximations of the position with three values of $h \in \left\{\frac{1}{16}, \frac{1}{32}, \frac{1}{64}\right\}$ without filtration.

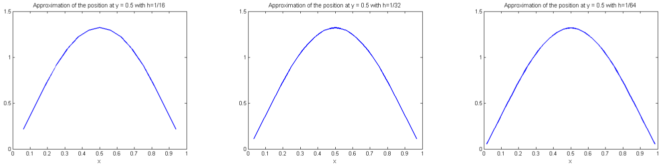


Figure: Approximations of the position with three values of $h \in \left\{\frac{1}{16}, \frac{1}{32}, \frac{1}{64}\right\}$ with filtration $\delta = \frac{1}{2}$.

Numerical experiments

h	1/16	1/32	1/64	1/128	1/256
without f.	40	97	517	Non-convergence	Non-convergence
with f.	36	53	136	12	10

Table: Number of iterations needed for convergence, for different values of h, for the Numerical experiment.

h	1/16	1/32	1/64	1/128	1/256
without f.	8.1003	8.1753	8.5012	Non-convergence	Non-convergence
with f.	8.1025	8.0955	8.0923	8.0920	8.0921

Table: The L^2-norm of the control, for different values of h, for the Numerical experiment.

Other remedies for high frequency pathologies

- **Tychonoff regularization**
  R. Glowinski, C. H. Li, and J.-L. Lions: *A numerical approach to the exact boundary controllability of the wave equation (I). Dirichlet controls: Description of the numerical methods.* Japan J. Appl. Math., 7 (1990), 1–76.
- **Two-grid algorithm**
  R. Glowinski and C. H. Li Glowinski: *On the numerical implementation of the Hilbert uniqueness method for the exact boundary controllability of the wave equation*, C. R. Acad. Sci. Paris Sr. I Math., 311 (1990), 135–142.
  E. Zuazua, L. Ignat: *Convergence of a two-grid algorithm for the control of the wave equation*, Journal of the European Mathematical Society 11 (2009), 351–391.
- **Mixed finite elements**
  C. Castro, S. M., A Munch: *Numerical approximation of the boundary control for the wave equation with mixed finite elements in a square*, IMA Journal of Numerical Analysis 28 (2008), 186–214.

- **Space-time finite elements**
  N. Cîndea, A. Münch: *A mixed formulation for the direct approximation of the control of minimal -norm for linear type wave equations*, Calcolo 52 (3) (2015), 245-288.
  E. Burman, A. Feizmohammadi, A. Münch, L. Oksanen: *Spacetime finite element methods for control problems subject to the wave equation*, ESAIM: Control, Optimisation and Calculus of Variations 29 (2023).
- **Huygens' Principle**
  C. Rosier, L. Rosier: *Numerical control of the wave equation and Huygens' Principle*, MATH. REPORTS 24(74), 1-2 (2022), 319338.
- **Russell's Method**
  N. Cîndea, S Micu, M Tucsnak: *An approximation method for exact controls of vibrating systems*, SICON 49 (3) (2011), 1283-1305.
- HANDBOOK OF NUMERICAL ANALYSIS, VOLUME 24, NUMERICAL CONTROL: PART A, B, Ed. E. Trélat and E. Zuazua, North-Holland, 2023.

THANK YOU VERY MUCH!