

Relaxation enhancement by controlled incompressible fluid flows

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1 Introduction

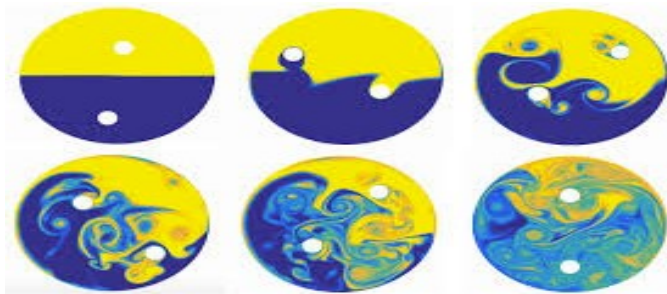
2 Main results

3 Proof of Theorem 6

4 Proof of Theorem 7

5 Conclusions and open questions

Mixing and enhanced dissipation by fluid flow



Mixing: disperse one material or field in another medium.

Enhanced dissipation (diffusion): Fasten the natural diffusion of one material or field in another medium.

PDE description of enhanced dissipation (I): the context

$$\frac{\partial \phi_a}{\partial t}(t, x) + a \tilde{v}(x) \cdot \nabla \phi_a(t, x) - \Delta \phi_a(t, x) = 0 \quad (t \geq 0, x \in M),$$

$$\phi_a(0, x) = f(x) \quad (x \in M),$$

where M is a manifold without boundary, $a > 0$ and $\tilde{v} = \tilde{v}(x)$ is free divergence. Then

$$\|\phi_a(t, \cdot) - \overline{\phi_a}\|_{L^2(M)} \leq \exp(-\lambda t) \|f - \overline{f}\|_{L^2(M)} \quad (a \geq 0, t \geq 0),$$

where λ is the smallest positive eigenvalue of the operator $-\Delta$, where Δ is the Laplace–Beltrami operator on M .

Can we choose \tilde{v} to “significantly” improve this decay rate when $a \gg 1$?

Definition 1

The free divergence vector field \tilde{v} on M is said *relaxation enhancing* if for every $\tau, \delta > 0$ there exists $a^*(\tau, \delta)$ such that for any $a \geq a^*(\tau, \delta)$ and every $f \in L^2(M)$, with $\|f\|_{L^2(M)} \leq 1$ and $\int_M f(x) dx = 0$, such that

$$\|\phi_a(\tau, \cdot)\|_{L^2(M)} < \delta. \quad (1)$$

Theorem 2

A continuous free divergence field $\tilde{v} : M \rightarrow \mathbb{R}^3$ is relaxation-enhancing if and only if the operator $\varphi \mapsto \tilde{v} \cdot \nabla \varphi$ has no eigenvectors in the Sobolev space $H^1(M)$, other than constant functions.

Remark 1

Constructing relaxation-enhancing field is far of being obvious, even if $M = \mathbb{T}^2$, see, for instance, Fayad [Fay02].

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An abstract framework

- X is a Hilbert space.
- $\Gamma : \mathcal{D}(\Gamma) \rightarrow X$ is a strictly positive operator.
- $L : \mathcal{D}(L) \rightarrow X$ symmetric on X , with $L \in \mathcal{L}(\mathcal{D}(\Gamma^{\frac{1}{2}}); X)$.

Theorem 3

The following two statements are equivalent:

- *For every $\tau, \delta > 0$ there exists $a(\tau, \delta)$ such that for any $a > a(\tau, \delta)$ and any f with $\|f\|_X = 1$, the solution ϕ_a of $\dot{\phi}_a = iaL\phi_a - \Gamma\phi_a, \phi_a(0) = f$ satisfies $\|\phi_a(\tau)\|_X < \delta$.*
- *The operator L has no eigenvectors lying in $\mathcal{D}(\Gamma^{\frac{1}{2}})$.*

Main ingredients of the proof in Wei [Wei21])

Theorem 4 (Gearhart-Prüss type)

Let A be an m -dissipative operator in a Hilbert space X and let \mathbb{T} be the contraction semigroup generated by A . Then we have

$$\|\mathbb{T}_t\|_{\mathcal{L}(X)} \leq \exp\left(-t\psi(A) + \frac{\pi}{2}\right) \quad (t \geq 0),$$

where $\psi(A) = \frac{1}{\sup_{\omega \in \mathbb{R}} \|(i\omega\mathbb{I} - A)^{-1}\|_{\mathcal{L}(X)}}$.

Second step. Prove, by contradiction, that $\lim_{a \rightarrow \infty} \psi(iaL - \Gamma) = \infty$.

Basic idea (I)

Instead imposing a prescribed velocity field \tilde{v} at each point of the fluid, obtaining a velocity field $v(t, x)$ by controlling an incompressible fluid flow.

Consider the incompressible Euler equations on \mathbb{T}^2 , i.e.,

$$\frac{\partial v}{\partial t}(t, x) + (v \cdot \nabla)v(t, x) + \nabla p(t, x) = (Bu(t, \cdot))(x) \quad (t \geq 0, x \in \mathbb{T}^2)$$

$$(\operatorname{div} v)(t, x) = 0 \quad (t \geq 0, x \in \mathbb{T}^2),$$

$$v(0, x) = 0 \quad (x \in \mathbb{T}^2),$$

where $B \in \mathcal{L}(U, L^2(\mathbb{T}^2))$ is the **control operator**.

Definition 5

The control system is said *relaxation enhancing* if for every $\tau, \delta > 0$ there exists $u \in L^2([0, \tau]; L^2(\mathbb{T}^2))$ s. t. the solution φ_v of

$$\frac{\partial \varphi_v}{\partial t}(t, x) + v(t, x) \cdot \nabla \varphi_v(t, x) - \Delta \varphi_v(t, x) = 0 \quad (t \geq 0, x \in \mathbb{T}^2),$$

$$\varphi_v(0, x) = f(x) \quad (x \in \mathbb{T}^2).$$

satisfies $\|\varphi_v(\tau, \cdot)\|_{L^2(\mathbb{T}^2)} < \delta$ for all f with $\|f\|_{L^2(\mathbb{T}^2)} \leq 1$ and $\int_{\mathbb{T}^2} f(x) dx = 0$.

Theorem 6

Let $\theta_1(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin x_2$, $\theta_2(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos x_2$,
 $\theta_3(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(x_1 - x_2)$, and $\theta_4(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(x_1 - x_2)$, where
 $x = (x_1, x_2)$. Then the Euler control system with $U = \mathbb{R}^4$ and

$$Bu = \sum_{j=1}^4 u_j(t)\theta_j(x),$$

is relaxation enhancing.

Second main result

Theorem 7

Let $\omega \subset \mathbb{T}^2$ be open and such that $\mathbb{T}^2 \setminus \omega$ is simply-connected. Let $B \in \mathcal{L}(L^2(\mathbb{T}^2))$ be defined by $Bu = \mathbb{I}_\omega u$, where \mathbb{I}_ω is the characteristic function of ω . Moreover let

$$X = \left\{ g \in L^2(\mathbb{T}^2) \mid \int_{\mathbb{T}} g(x_1, x_2) dx_1 = 0 \text{ for almost every } x_2 \in \mathbb{T} \right\},$$

and let P_X be the orthogonal projector from $L^2(\mathbb{T}^2)$ onto X . Then for every $\tau, \delta > 0$ there exists $u \in L^2([0, \tau]; L^2(\mathbb{T}^2))$ such that

$$\|P_X \varphi_v(\tau, \cdot)\|_{L^2(\mathbb{T}^2)} < \delta,$$

for every $f \in L^2(\mathbb{T}^2)$ with $\|f\|_{L^2(\mathbb{T}^2)} \leq 1$.

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$\mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$, $J_T = [0, T]$. Moreover, for every $l \in \mathbb{N}$ we set

$$H^l := \left\{ \varphi \in H^l(\mathbb{T}^d; \mathbb{R}), \int_{\mathbb{T}^d} \varphi(x) \, dx = 0 \right\},$$

$$H^0 := \left\{ \varphi \in L^2(\mathbb{T}^d; \mathbb{R}), \int_{\mathbb{T}^d} \varphi(x) \, dx = 0 \right\},$$

$$\mathcal{H} := \left\{ v \in L^2(\mathbb{T}^d; \mathbb{R}^d) : \operatorname{div} v = 0, \int_{\mathbb{T}^d} v(x) \, dx = 0 \right\},$$

$$H_\sigma^k := H_\sigma^k(\mathbb{T}^d; \mathbb{R}^d) := H^k(\mathbb{T}^d; \mathbb{R}^d) \cap \mathcal{H} \quad (k \in \mathbb{N}).$$

Main ingredient for Thm 6: an approx. controllability result

$$\frac{\partial v}{\partial t}(t, x) + (v \cdot \nabla)v(t, x) + \nabla p(t, x) = \sum_{j=1}^4 u_j(t) \theta_j(x) \quad (t \geq 0, x \in \mathbb{T}^2),$$

$$(\operatorname{div} v)(t, x) = 0 \quad (t \geq 0, x \in \mathbb{T}^2),$$

$$v(0, x) = v_0(x) \quad (x \in \mathbb{T}^2),$$

Theorem 8

For every $\varepsilon > 0$, $\psi \in W^{1,1}(J_T; H_\sigma^{k+2}) \cap L^1(J_T; H_\sigma^{k+3})$, and $R > 0$, there is a control $u \in C^\infty(J_T; \mathbb{R}^m)$ such that $\|v - \psi\|_{T,k} < \varepsilon$,

where $v(0) = \psi(0)$ and $\|h\|_{T,X} := \sup_{t \in J_T} \left\| \int_0^t h(s) ds \right\|_X$.

Second ingredient: a continuous dependence result

$$\frac{\partial \varphi_v}{\partial t}(t, x) + v(t, x) \cdot \nabla \varphi_v(t, x) - \Delta \varphi_v(t, x) = 0 \quad (t \geq 0, x \in \mathbb{T}^2),$$

$$\varphi_v(0, x) = f(x) \quad (x \in \mathbb{T}^2).$$

Theorem 9

Let $T > 0$ and let $k \in \mathbb{N}$, $k > \frac{d}{2}$. For every $f \in H^{k+2}$ and $v_1, v_2 \in C(J_T; H_\sigma^{k+2})$ let f, v_1, v_2 be such that

$$\|f\|_{k+2} + \|v_1\|_{C(J_T; H_\sigma^{k+2})} + \|v_2\|_{C(J_T; H_\sigma^{k+2})} \leq R$$

for some $R > 0$. Then there exists $C := C(R, T, k, d) > 0$ such that $\|\varphi_{v_1} - \varphi_{v_2}\|_{C(J_T; H^k)} \leq C \|v_1 - v_2\|_{T, k}^{1/2}$.

Main steps of the proof

- 1 Let $\tilde{v} \in C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ relaxation-enhancing and let $a > 0$ be such that if $\|f\|_{L^2(\mathbb{T}^2)} \leq 1$ and $\int_{\mathbb{T}^2} f(x) dx = 0$ then

$$\|\phi_a(\tau, \cdot)\|_{L^2(\mathbb{T}^2)} < \frac{\delta}{3}.$$

- 2 Use Theorem 8 to obtain a control $u \in C^\infty(J_T; \mathbb{R}^4)$ such that $v = v(u)$ satisfies $\|v - a\tilde{v}\|_{T,k} \ll 1$.
- 3 Use Theorem 9 to prove that $\|\phi_v(\tau)\|_{L^2(\mathbb{T}^2)} < \delta$.

The notion of saturating subspace (Agrachev et al)

Consider the system $\dot{x} = Ax + N(x) + Bu$, with state space X , with N quadratic. Let $E := \text{Ran } B \subset X$ be a finite dimensional subspace. We denote by $\mathcal{F}(E)$ the largest vector space $F \subset X$ such that any $\eta_1 \in F$ can be represented in the form

$$\eta_1 = \eta - \sum_{i=1}^p N(\zeta^i) \quad \text{with} \quad \eta, \zeta^1, \dots, \zeta^p \in E.$$

We set:

$$E_0 = E, \quad E_j = \mathcal{F}(E_{j-1}) \quad \text{for } j \in \mathbb{N}, \quad E_\infty = \bigcup_{j=1}^{\infty} E_j. \quad (2)$$

Definition 10

The space E is said to be *saturating* if E_∞ is dense in X .

A toy model of Theorem 8

$$\dot{x} + Ax + f(x) = Bu, \quad x(0) = x_0 \in \mathbb{R}^3,$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad f(x_1, x_2, x_3) = \begin{bmatrix} x_1 x_2 \\ x_3^2 - x_1^2 \\ -x_3 x_2 \end{bmatrix}.$$

Here $\text{Ran } B$ is saturating in \mathbb{R}^3 .

Theorem 11

Given any $\psi \in W^{1,2}((0, T); \mathbb{R}^3)$ and $\varepsilon > 0$, there exists a control $u \in C^\infty([0, T]; \mathbb{R}^2)$ such that the corresponding solution x with $x_0 = \psi(0)$ satisfies

$$|x(T) - \psi(T)| + \|x - \psi\|_T < \varepsilon.$$

Remark 2

The above tracking controllability does not hold for $f = 0$.

Main ingredient of the proof of Theorem 7

For $\omega \subset \mathbb{T}^2$, we consider the system

$$\partial_t v + (v \cdot \nabla)v + \nabla p = \mathbb{I}_\omega u, \quad \operatorname{div}(v) = 0, \quad v(\cdot, 0) = v_0.$$

Theorem 12

Given $T > 0$, $v_0, v_T \in H^2(\mathbb{T}^2; \mathbb{R}^2)$, and a force $h \in L^2((0, T); H^2(\mathbb{T}^2; \mathbb{R}^2))$, there exists a control

$$u \in C^0([0, T]; L^2(\mathbb{T}^2; \mathbb{R}^2)) \cap L^\infty([0, T]; H^1(\mathbb{T}^2; \mathbb{R}^2))$$

obeys $v(T, \cdot) = v_T$.

Second ingredient: using shear flows

Proposition 1

Let the **shear flow** \tilde{v} be defined by

$$\tilde{v}(x_1, x_2) = \begin{bmatrix} \alpha(x_2) \\ 0 \end{bmatrix} \quad \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{T}^2 \right).$$

Assume that $\alpha \in C^0(\mathbb{T}; \mathbb{R})$ and suppose that there is no open subset of \mathbb{T}^2 on which α is constant. Then for every $\tau, \delta > 0$ there exists $a^*(\tau, \delta)$ such that for every $f \in L^2(\mathbb{T}^2)$, with $\|f\|_{L^2(\mathbb{T}^2)} \leq 1$ we have

$$\|P_X \phi_a(\tau, \cdot)\|_{L^2(\mathbb{T}^2)} < \delta,$$

where P_X is the orthogonal projector from $L^2(\mathbb{T}^2)$ onto X .

Main steps of the proof

- 1 Apply Theorem 12 to obtain $v\left(\frac{T}{2}, \cdot\right) = \tilde{v}$.
- 2 Remarking that \tilde{v} is a stationary solution of the Euler equations, take $u = 0$ for $t > \frac{T}{2}$.
- 3 Apply Theorem 3 with

$$\mathcal{D}(\Gamma) = H^2(\mathbb{T}^2) \cap X, \quad \mathcal{D}(L) = \mathcal{D}(\Gamma^{\frac{1}{2}}),$$

$$\Gamma g = -P_X \Delta g \quad (g \in \mathcal{D}(\Gamma)),$$

$$Lg = i(\tilde{v} \cdot \nabla g) \quad \left(g \in \mathcal{D}(\Gamma^{\frac{1}{2}})\right).$$

Executive summary

- We have obtained relaxation enhancing flows by controlling an incompressible fluid.
- The controls are “localized” either in the frequency or in the physical space
- The results are limited to flows on \mathbb{T}^2 .

Some open questions

- More general geometries.
- “Full” enhanced dissipation by controls localized in space.
- Taylor-Couette type flows.