Hyperbolic Conservation Laws and Control

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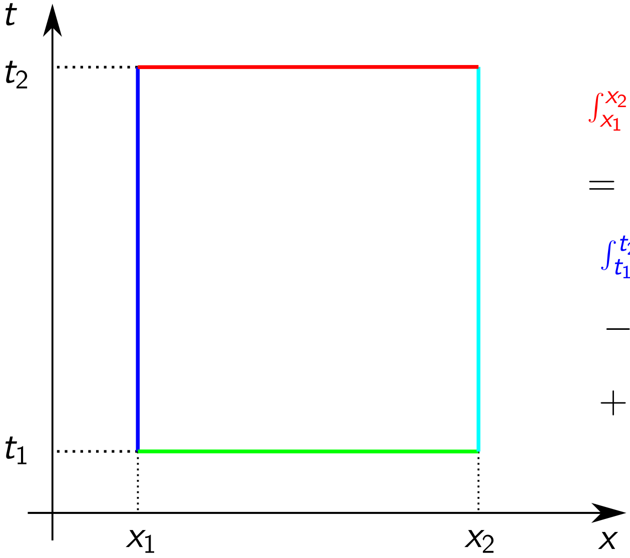
Control of PDEs in Hauts-de-France 2025

Hyperbolic Conservation Laws

Inverse Design

And now for some Algebra



$$\int_{x_1}^{x_2} u(t_2, x)dx$$

$$=$$

$$\int_{t_1}^{t_2} F(t, x_1)dt$$

$$- \int_{t_1}^{t_2} F(t, x_2)dt$$

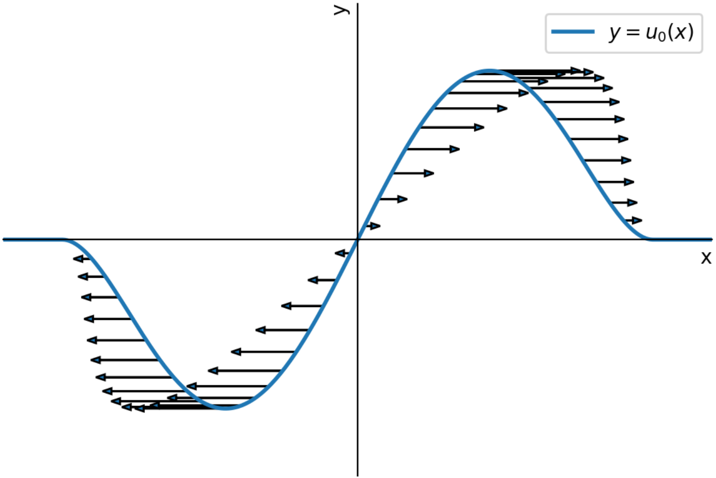
$$+ \int_{x_1}^{x_2} u(t_1, x)dx$$

- Integral Form: u, F just L^1_{loc}
- Differential Form: u, F $\mathcal{C}^1$

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} \partial_t u(t,x)dtdx = -\int_{t_1}^{t_2}\int_{x_1}^{x_2} \partial_x F(t,x)dxdt$$

$$\implies \frac{\partial u}{\partial t}(t,x) + \frac{\partial F}{\partial x}(t,x) = 0$$

- Heat:
$$\partial_t u + \partial_x(-\kappa \partial_x u) = 0$$
- Burgers:
$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0$$
- LWR:
$$\partial_t u + \partial_x \left(u(t,x) v_{\max} \left(1 - \frac{u(t,x)}{u_{\max}} \right) \right) = 0$$


y
$y = u_0(x)$
x

$f(u) = \frac{u^2}{2}$, $\forall u_0 \in \mathcal{C}_c^\infty(\mathbb{R}) \setminus \{0\}$

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & t > 0, \quad x \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

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$$\exists T > 0, \quad \exists X \in \mathbb{R}, \qquad \partial_x u(t, X) \underset{t \to T^-}{\to} -\infty \tag{2}$$

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$$||u(t,\cdot)||_{L^\infty(\mathbb{R})} \leq ||u_0||_{L^\infty(\mathbb{R})} \tag{3}$$

⚠ No linearization techniques!

Hopf 1950:

$$\begin{cases} \partial_t u_\epsilon + \partial_x (u_\epsilon^2/2) = \epsilon \partial_{xx}^2 u_\epsilon \\ u_\epsilon(0,x) = u_0(x) \end{cases}$$

- Distributional Solutions ⚠
- Motivation: Navier-Stokes $\to$ Euler
- Singular limit
- Cole-Hopf " $\Longrightarrow$ " heat equation

Kruzkov 1970:

- $\forall k \in \mathbb{R}$, $\forall \phi \geq 0$ in $\mathcal{C}^1$

$$\int_{\mathbb{R}} |u_0 - k|\phi(0,\cdot)\mathrm{d}x + \int_{\mathbb{R}^+}\int_{\mathbb{R}} |u - k|\partial_t \phi$$
$$+ \operatorname{sign}(u-k)(f(u) - f(k))\partial_x \phi \mathrm{d}x\mathrm{d}t \geq 0 \quad (4)$$

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- $\forall u_0 \in \mathrm{L}^\infty$, $\exists! u$ maximal (global)

$$(t,x) \mapsto (S_t u_0)(x) := u(t,x), \quad (5)$$

$$(S_t)_{t\geq 0} : \mathrm{L}^1_{\mathrm{loc}} \text{ contracting} \quad (6)$$

Colombo-Goatin 2005
Andreianov-Karlsen-Risebro 2011:

- Toll gate at $\bar{x}$:

$$\begin{cases} \partial_t u + \partial_x f_{\mathrm{LWR}}(u) = 0 \\ f_{\mathrm{LWR}}(u(\cdot, \bar{x})) \leq M. \end{cases}$$

- Alternative Monotone Semigroup of Distributional Solutions
- Crandall-Tartar $\implies$ L^1 contractive
- Vanishing Viscosity *NOT* universal!

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And now for some Algebra

- Entropy solutions $\implies$ $(S_t^{CL})_{t\geq 0}$ on $L^\infty(\mathbb{R})$
- Reachable states: $T > 0$ determine

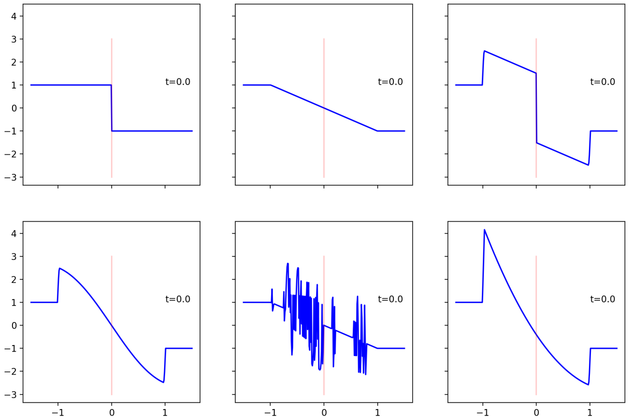
$$\{w \in L^\infty(\mathbb{R}) \, : \, \exists u_0 \in \mathrm{L}^\infty(\mathbb{R}) \, S_T^{CL} u_0 = w\}$$

- Inverse Design: $T > 0$, $w \in \mathrm{L}^\infty(\mathbb{R})$ determine

$$I_T(w) := \{u_0 \in \mathrm{L}^\infty(\mathbb{R}) \, : \, S_T^{CL} u_0 = w\}$$

1. Entropy Solutions Irreversible dynamics
2. Entropy Semigroup Compactifying
   De Lellis-Golse 2005
   Ancona-Glass-Nguyen 2015, 2019, 2020
3. Sonic boom minimization
   Gosse-Zuazua 2017
4. Traffic Flow Data Assimilation
   tollgate estimates, accident localization
5. Control theory
   Russell's extension method
   Ancona-Marson 1998, Horsin 1998...

Burgers Slowdown: 10x



Oleinik 56, f convex:

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1. $w \in S_T(L^\infty(\mathbb{R})) \implies w \in \mathrm{BV}(\mathbb{R})$
2. ⚠ r_w^T Cantor's staircase

Colombo-Perrollaz 2020:

$I_T(w) \neq \emptyset \implies$

1. $I_T(w)$ convex cone
2. $I_T(w)$ $L^1_{loc} - F_\sigma$

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2. Else unbounded L^∞, L^∞ closed L^1_{loc}
3. No facet of finite dimension ($\neq$ vertex)

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Complete characterization of $I_T(w)$

1. LWR: varying speed limit, varying number of lanes, turns

$$\partial_t u(t,x) + \partial_x\left(u(t,x) v_{\max}(x)\left(1 - \frac{u(t,x)}{u_{\max}(x)}\right)\right) = 0$$

2. Non-homogeneous conservation laws
   2.1 Flux $(x,u) \mapsto H(x,u)$
   2.2 Richer geometry! Characteristics equation:

$$\begin{cases} \dot{q}(t) = \partial_2 H(q(t), p(t)) \\ \dot{p}(t) = -\partial_1 H(q(t), p(t)) \end{cases}$$

   2.3 ⚠ Literature ⚠

Colombo-Perrollaz-Sylla 2023:

$$T > 0,\ w \in L^\infty(\mathbb{R}),\ \pi_w^T(x) := q(0)$$

$$\begin{cases} \dot q(t) = \partial_2 H(q(t), p(t)) & q(T) = x \\ \dot p(t) = -\partial_1 H(q(t), p(t)) & p(T) = w(x) \end{cases}$$

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$I_T(w)$ not singleton $\implies$

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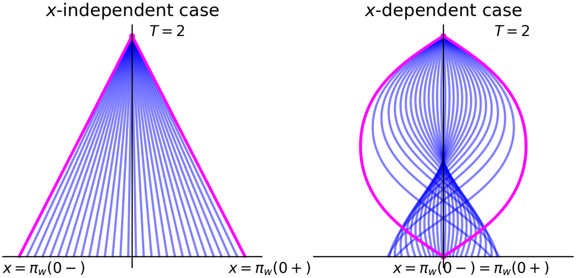
. . . exactly like the homogeneous case?

OOPS



x-independent case
$T=2$
$x=\pi_w(0-)$
$x=\pi_w(0+)$
x-dependent case
$T=2$
$x=\pi_w(0-)=\pi_w(0+)$

Colombo-Perrollaz-Sylla 2023: $\exists T > 0$, H, w

- $I_T(w) \neq \emptyset$,
- $\exists (E, Q)$ entropy-entropy flux pair such that $\forall u_0 \in I_T(w)$,

$$\partial_t E(u) + \partial_x \left(Q(x, u) \right) + E'(u) \partial_1 H(x, u) - \partial_1 Q(x, u) \neq 0$$

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- Isentropic solutions "=" $\overline{\{\text{classical solutions}\}}$.
- Range of S_T Homogeneous $\sim$ classical solutions.
- Range of S_T Non-Homogeneous larger.

Literature:

- Esteve-Zuazua 2020, 2022, 2023
- Liard-Zuazua 2023
- Coclite-De Nitti-Donadello-Peru 2024: chromatography system
- Ancona-Talamini 2024: discontinuous flux
- Colombo-Perrollaz 2025: localization
- Applications: Traffic Flow, Machine Learning

Limitations:

- Robustness
- General vs Particular
- Sampling
- Numerics
- General fluxes, $\mathbb{R}^d$, systems

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Esteve-Zuazua:

$$w \text{ reachable} \iff S_T^+ \circ S_T^-(w) = w \tag{7}$$

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$$w \text{ reachable} \iff S_T^+ \circ S_T^-(w) = w \qquad (7)$$

Liard-Zuazua: for Burgers

$$S_T^+ \circ S_T^- \quad \mathrm{L}^2 \text{ projection on } S_T^+(\mathrm{L}^\infty) \qquad (8)$$

- A, B partially ordered sets $A \underset{\alpha}{\overset{\beta}{\rightleftarrows}} B$
- (β, α) Galois Connexion:

$$\forall (a, b) \in A \times B,\ \beta(a) \leq_B b \iff a \leq_A \alpha(b) \tag{9}$$

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- Then

$$\begin{cases} \alpha \circ \beta \geq \mathrm{Id} \geq \beta \circ \alpha & \alpha, \beta \nearrow \\ \alpha \circ \beta \circ \alpha = \alpha & \beta \circ \alpha \circ \beta = \beta \\ (\alpha \circ \beta)^2 = \alpha \circ \beta & (\beta \circ \alpha)^2 = \beta \circ \alpha \end{cases} \tag{10}$$

- Simple on $\mathbb{R}$: $j : \mathbb{Z} \to \mathbb{R}$ embedding

$$\begin{cases} j(n) \le x \iff n \le \lfloor x \rfloor \\ x \le j(n) \iff \lceil x \rceil \le n \end{cases} \tag{11}$$

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- Original: $K \subset L \subset M$ fields

$$\mathrm{Fix}(H) \supset L \iff H \subset \mathrm{Gal}(M/L) \tag{12}$$

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- Algebraic Geometry: $V \subset \mathbb{K}^n$, $I \subset \mathbb{K}[X_1, \dots, X_n]$

$$\{P \in \mathbb{K}[X_1, \dots, X_n] : \forall x \in V,\ P(x) = 0\} \supset I$$
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- Convex Analysis:

$$\forall (f, g) \in \mathcal{F}(E, \overline{\mathbb{R}}) \quad f^* \le g \iff f \ge g^* \tag{14}$$

- Entropy solutions: $(S_t^{CL})_{t\geq 0}$ on $L^\infty(\mathbb{R})$

$$\partial_t u + \partial_x H(x,u) = 0, \quad t > 0, \quad x \in \mathbb{R} \tag{NHCL}$$

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- Viscosity solutions : $(S_t^{HJ})_{t\geq 0}$ on $\mathrm{Lip}(\mathbb{R})$

$$\partial_t U + H(x,\partial_x U) = 0, \quad t>0, \quad x\in\mathbb{R} \tag{HJB}$$

- Correspondence:

$$\begin{array}{ccc} U_o & \longrightarrow & S_t^{HJ} U_o \\ \partial_x \downarrow & & \downarrow \partial_x \\ u_o & \longrightarrow & S_t^{CL} u_o \end{array}$$

- ⚠ **Not really** $(S_t^{HJ})_{t\geq 0} \stackrel{\partial_x}{\to} (S_t^{CL})_{t\geq 0}$!

$$\begin{cases} \partial_t U + H(\partial_x U) = 0 \\ U(0) = U_0 \end{cases} \quad \overset{\text{viscosity}}{\Longrightarrow} \quad U(t) = (S_t^+)_{t\geq 0} U_0 \tag{15}$$

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$$

Galois Connexion : $\forall U_0, V_1 \in \mathrm{Lip}(\mathbb{R})$

$$
\begin{aligned}
& V_1 \leq S_T^+ U_0 \\
\iff & V_1(x) \leq \inf_{y \in \mathbb{R}} TH^*((x-y)/T) + U_0(y), \quad \forall x \\
\iff & V_1(x) \leq TH^*((x-y)/T) + U_0(y), \quad \forall x, y \\
\iff & V_1(x) - TH^*((x-y)/T) \leq U_0(y), \quad \forall x, y \\
\iff & \sup_{x \in \mathbb{R}} V_1(x) - TH^*((x-y)/T) \leq U_0(y), \quad \forall y \\
\iff & S_T^- V_1 \leq U_0
\end{aligned}
$$

Colombo-Perrollaz 2025:

- Relaxation interpretation of $S^+ \circ S^-$ and $S^- \circ S^+$
- Boundary value problems
- Discontinuous Flux/Hamiltonian
- Networks
- Multi dimensional HJB
- "Optimal Control Problem" $\implies$ Galois Connexion!

- CL-HJB correspondence?
- Lattices?
- Riesz Space?
- Semigroup?
- Numerical schemes and Galois Connexion?
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Thank You