New results on mixed operators

Genni Fragnelli

University of Siena

Control of PDEs Hauts-de-France, 2ND Edition
Calais, 15th-17th September 2025

joint paper with Dimitri Mugnai (Tuscia),
Analysis and Applications 2025

Motivations

- Nanoparticles
- Hybrid composite materials
- Synthetic or natural polymers combined with metal, ceramic or carbon nanosctructures
- High–performance materials

(Picture from esub.com)

Motivations

Reinforced concrete and buildings

Part of the floor is outside the wall and part of the floor is inside the wall

Motivations

Insurgence of nonlocality in composites was shown by Silling in 2000, immediately studied by Bellido, Cueto, Mora-Corral, Ortega.

Problems have been investigated with the aim of describing the elastic behavior of complex structures composed by two or more different phases having extremely efficient mechanical features, to employ in a wide range of fields as civil engineering and architecture.

Silling: the equilibrium equation is written by a nonlocal operator L

$$L(u(x)) = \int_{\text{some set}} f(u(y) - u(x), y - x)dy.$$

Motivations

Starting from

$$L(u(x)) = \int_{\text{some set}} f(u(y) - u(x), y - x) dy,$$

if *"some set"* $= \mathbb{R}^N$ and

$$f(U, Z) := \frac{U}{|Z|^{N+2s}}, \quad s \in (0, 1),$$

then

$$L(u(x)) = (-\Delta)^s u(x)$$

where (up to a constant)

$$(-\Delta)^s u(x) := \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dx.$$

Question

What happens if we consider both operators in Ω? For instance:

$$\mathcal{L}(u(x)) = (-\Delta)^s u + \Delta^2 u.$$

It is a *mixed operator*.

The mixed local-nonlocal operator

$$\text{Prototype:} \quad -\Delta + (-\Delta)^s, \qquad s \in (0,1)$$

$+$ is important for maximum principles.

The mixed local-nonlocal operator

Prototype: $-\Delta + (-\Delta)^s, \quad s \in (0,1)$

$+$ is important for maximum principles.

They are a natural object to model animal foraging (like shark's hunting behaviour).

Not new in literature

$A : C_0^2(\mathbb{R}^N) \to C(\mathbb{R}^N)$ linear

$$\begin{cases} u \in C_0^2(\mathbb{R}^N), u \geq 0 \\ u(x) = \inf_{\mathbb{R}^N} u = 0 \end{cases} \implies Au(x) \geq 0$$

if and only if A is local + nonlocal (but *right* combination)

- Courrége (1964) and Bony-Courrége-Priouret (1968)
- ...
- Barles et al. (2008, 2012, 2014) - viscosity solutions

And quite studied recently: 2022-

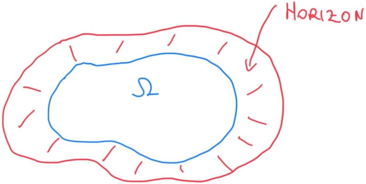
- Regularity: De Filippis-Mingione, Biagi-Dipierro-Valdinoci-Vecchi, Garain-Kinnnen, Garain-Lindgren;
- Qualitative properties: Biagi-Dipierro-Valdinoci-Vecchi, Dipierro-Proietti Lippi-Valdinoci, Mugnai-Proietti Lippi, Biagi-Mugnai-Vecchi;
- Singular problems: Garain-Ukhlov;
- Shape optimization: Biagi-Dipierro-Valdinoci-Vecchi;
- Overdetermined problems: Biswas-Modasyia-Sen;
- Existence of solutions: Maione-Mugnai-Vecchi, Mugnai-Proietti Lippi, Giovannardi-Mugnai-Vecchi;
- Parabolic case: Dipierro-Proietti Lippi-Valdinoci, Mugnai-Proietti Lippi;
- ...

Horizon

For buildings we need a different approach!

Horizon

For buildings we need a different approach!
In 2000 Silling introduced the concept of extended boundary or horizon, i.e. a region around the domain in which the function is identically 0



Peridynamics

The starting points for our work are two recent papers of *Cluni-Gusella-Mugnai-Proietti Lippi-Pucci* (Math. Eng. 2023 and AIMS Math 2025):

$$\begin{cases} -c\Delta u + k(-\Delta)^s_\Omega u = f(x,u) & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1, \end{cases} \qquad (P_{\mathrm{dyn}})$$

where c, $k > 0$ are physical coefficients and f is a perturbation. The operator

$$(-\Delta)^s_\Omega \varphi(x) = \int_\Omega \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}}\, dy \quad \text{for all } \varphi \in C^\infty_c(\Omega),$$

is the *regional fractional Laplacian in* Ω, not in Ω_0 where the equation holds.

Peridynamics

$\Omega \subset \mathbb{R}^N$ is a bounded domain. $\Omega = \Omega_0 \cup \Omega_1$, where Ω_0 is an open set with smooth boundary $\partial\Omega_0$ and $\Omega_0 \cap \Omega_1 = \emptyset$. Furthermore, $\Omega_\delta = \Omega_0 + B_\delta \subset \Omega$ for a suitable radius $\delta > 0$. In this way, the remaining set $\Omega_1 \supset \partial\Omega_0$ and Ω_1 can be seen as the nonlocal boundary of Ω.

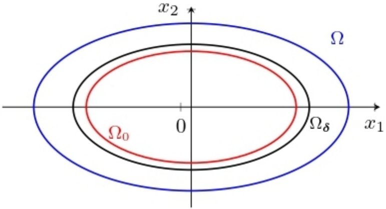


Figure 1: The sets Ω, Ω_0 and Ω_δ

New mixed beam operator

$$Au(x) := a\Delta^2 u(x) + C(-\Delta)^s_\Omega u(x) + V(x)u, \quad x \text{ in } \Omega_0,$$

- a, $C > 0$ and $s \in (0,1)$
- $u = 0$ in Ω_1
- Ω, Ω_0, Ω_1 as above
- V is a bounded, non-negative, continuous function.

The abstract setting

The natural solution space associated the operator is

$$\mathbb{H}_0^{2,s}(\Omega_0,\Omega) = \left\{ u \in H_0^2(\Omega_0) \cap H_0^s(\Omega) \,:\, u = 0 \text{ a.e. in } \Omega_1 \right\}$$

where

- $H_0^2(\Omega_0)$ is the usual Sobolev space of order 2, defined as the completion of $C_c^\infty(\Omega_0)$ with respect to the norm

$$\|u\|_{H_0^2(\Omega_0)} := \|\Delta u\|_{L^2(\Omega_0)}, \quad u \in H_0^2(\Omega_0),$$

- $H_0^s(\Omega)$ is the completion of $C_c^\infty(\Omega)$, with respect to the Gagliardo seminorm

$$[u]_{2,\Omega} = \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy \right)^{1/2}.$$

The abstract setting

$$Au(x) := a\Delta^2 u(x) + C(-\Delta)^s_{\Omega} u(x) + V(x)u, \quad x \text{ in } \Omega_0,$$

On $\mathbb{H}^{2,s}_0(\Omega_0, \Omega)$ we consider the norm

$$\|u\| := \left(a\|\Delta u\|^2_{L^2(\Omega_0)} + C[u]^2_{2,\Omega} + \int_{\Omega_0} V(x)u^2 dx \right)^{1/2}.$$

The abstract setting

$$Au(x) := a\Delta^2 u(x) + C(-\Delta)^s_\Omega u(x) + V(x)u, \quad x \text{ in } \Omega_0,$$

On $\mathbb{H}^{2,s}_0(\Omega_0, \Omega)$ we consider the norm

$$\|u\| := \left(a\|\Delta u\|^2_{L^2(\Omega_0)} + C[u]^2_{2,\Omega} + \int_{\Omega_0} V(x)u^2 dx \right)^{1/2}.$$

Proposition

- *If $N > 4$, $\mathbb{H}^{2,s}_0(\Omega_0, \Omega)$ embeds compactly in $H^1_0(\Omega_0)$ and in $L^p(\Omega_0)$ for any $p \in [1, 2^*)$, and continuously in $L^{2^*}(\Omega)$, where $2^* = \dfrac{2N}{N-4}$.*
- *If $N = 4$, then $\mathbb{H}^{2,s}_0(\Omega_0, \Omega)$ embeds compactly in $H^1_0(\Omega_0)$ and in $L^p(\Omega_0)$ for all $p \geq 1$.*
- *If $N \leq 3$, then $\mathbb{H}^{2,s}_0(\Omega_0, \Omega)$ embeds compactly in $H^1_0(\Omega_0)$ and in $C^0(\bar{\Omega})$.*

Eigenvalue problem

$$\begin{cases} a\Delta^2 u + C(-\Delta)^s_\Omega u + V(x)u = \lambda u & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1. \end{cases} \qquad (P_\lambda)$$

Proposition

(a) $\lambda_1 > 0$, $\lambda_1 = \min_{u \in \mathbb{H}^{2,s}_0(\Omega_0,\Omega)\setminus\{0\}} \frac{\|u\|^2}{\int_{\Omega_0} u^2 dx}$ *with associated eigenfunction* $e_1 \in \mathbb{H}^{2,s}_0(\Omega_0,\Omega)$ *which realizes the minimum.*

(b) *The set of the eigenvalues of problem is a sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$ *s.t.* $0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_k \le \lambda_{k+1} \le \dots$ *and* $\lambda_k \to \infty$ *as* $k \to \infty$. *Moreover, for any* $k \in \mathbb{N}$, $\lambda_{k+1} = \min_{u \in \mathbb{P}_{k+1}\setminus\{0\}} \frac{\|u\|^2}{\int_{\Omega_0} u^2 dx}$, *where* $\mathbb{P}_{k+1} = \left\{ u \in \mathbb{H}^{2,s}_0(\Omega_0,\Omega) \, : \, \langle u, e_j \rangle = 0 \text{ for any } j = 1,\dots,k \right\}$, *and there exists a function* $e_{k+1} \in \mathbb{P}_{k+1}$, *which is an eigenfunction associated to* λ_{k+1} *and realizes the minimum.*

Eigenvalue problem

$$\begin{cases} a\Delta^2 u + C(-\Delta)^s_\Omega u + V(x)u = \lambda u & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1. \end{cases} \tag{P_λ}$$

Proposition

(c) *The sequence* $\{e_k\}_{k\in\mathbb{N}}$ *is an orthogonal basis of* $\mathbb{H}^{2,s}_0(\Omega_0,\Omega)$ *and orthonormal basis of* $L^2(\Omega_0)$.

(d) *Each eigenvalue* λ_k *has finite multiplicity. More precisely, if* λ_k *is such that*

$$\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+h} < \lambda_{k+h+1} \tag{1}$$

for some $h \in \mathbb{N} \cup \{0\}$, *then the set of eigenfunctions associated to* λ_k *coincides with*

$$\mathrm{span}\{e_k, \ldots, e_{k+h}\}.$$

Eigenvalue problem

Remark

What is natural in these types of problems is that λ_1 is simple and e_1 is positive, in our case meaning e_1 positive in Ω_0.

Eigenvalue problem

Remark

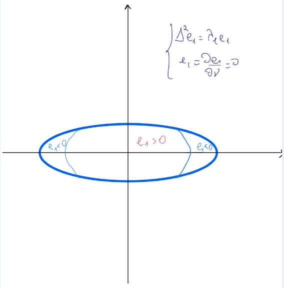
What is natural in these types of problems is that λ_1 is simple and e_1 is positive, in our case meaning e_1 positive in Ω_0.

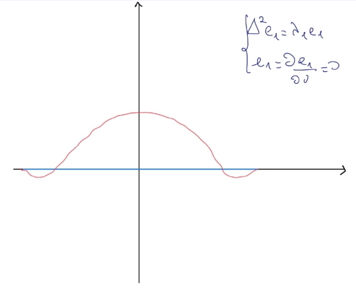
However, it is natural that in our case a similar result cannot hold in general due to the presence of the biharmonic operator with peridynamical Dirichlet conditions in Ω_1 which imply classical Dirichlet conditions in Ω_0...

Eigenvalue problem

Remark

Indeed, for Δ^2 with Dirichlet conditions in Ω_0 such a result is false: Garabedian (1951), Grunau-Sweers (1996, 1999), when Ω_0 is a long ellipsis:

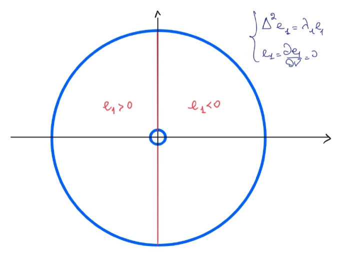


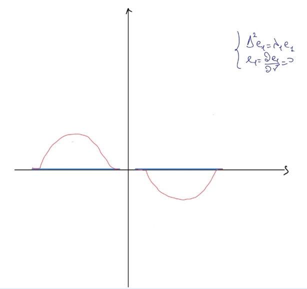


Eigenvalue problem

Remark

Indeed, for Δ^2 with Dirichlet conditions in Ω_0 such a result is false: Duffin-Shaffer (1952), Coffman-Duffin-Shaffer (1978) for annuli with small inner hole:





Eigenvalue problem

Open problem:

is there Ω_0 s.t. $e_1 > 0$ in Ω_0 with λ_1 simple?

Eigenvalue problem

Open problem:

is there Ω_0 s.t. $e_1 > 0$ in Ω_0 with λ_1 simple?

Guess: if Ω_0 is a ball, we know for Δ^2 with Dirichlet conditions that $e_1 > 0$ in Ω_0 (Boggio 1905). So...

A *toy-linear* elliptic case

$$(P_L) \quad \begin{cases} a\Delta^2 u + C(-\Delta)^s_\Omega u + V(x)u = \lambda u + f(x) & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1, \end{cases}$$

where a, $C > 0$ and $f \in L^2(\Omega_0)$.

Definition

$u \in \mathbb{H}^{2,s}_0(\Omega_0, \Omega)$ is a (weak) *solution* of problem (P_L) if

$$a \int_{\Omega_0} \Delta u \Delta v \, dx + C \iint_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy$$
$$+ \int_{\Omega_0} V(x)uv \, dx = \int_{\Omega_0} (\lambda u + f(x))v \, dx$$

for every function $v \in \mathbb{H}^{2,s}_0(\Omega_0, \Omega)$.

A *toy-linear* elliptic case

$$(P_L) \quad \begin{cases} a\Delta^2 u + C(-\Delta)^s_{\Omega} u + V(x)u = \lambda u + f(x) & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1, \end{cases}$$

where a, $C > 0$ and $f \in L^2(\Omega_0)$.

Solutions of (P_L) are critical points of $J : \mathbb{H}^{2,s}_0(\Omega_0, \Omega) \to \mathbb{R}$, defined as

$$J(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_{\Omega_0} u^2 - \int_{\Omega_0} f(x)u\,dx.$$

A *toy-linear* elliptic case

$$(P_L) \quad \begin{cases} a\Delta^2 u + C(-\Delta)^s_\Omega u + V(x)u = \lambda u + f(x) & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1, \end{cases}$$

where a, $C > 0$ and $f \in L^2(\Omega_0)$.

Solutions of (P_L) are critical points of $J : \mathbb{H}^{2,s}_0(\Omega_0, \Omega) \to \mathbb{R}$, defined as

$$J(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_{\Omega_0} u^2 - \int_{\Omega_0} f(x)u\,dx.$$

Proposition

Let $f \in L^2(\Omega_0)$ and $\lambda \neq \lambda_k$ for all $k \in \mathbb{N}$. Then (P_L) admits a unique solution $u \in \mathbb{H}^{2,s}_0(\Omega_0, \Omega)$. If f is nontrivial, then also the solution is nontrivial.

An idea of the proof

- $\lambda < \lambda_1$: J is coercive, weakly lower semicontinuous and strictly convex, thus $\exists!$ $\min J$

An idea of the proof

- $\lambda < \lambda_1$: J is coercive, weakly lower semicontinuous and strictly convex, thus $\exists!\ \min J$
- $\lambda > \lambda_1$: Saddle Theorem of Rabinowitz.

In any case, J has a nontrivial critical point u which is a nontrivial solution of (P_L).

The *real nonlinear* elliptic case with *linear* growth

The real result holds if $\lambda u + f(x)$ is replaced by $g(x,u)$ where g is a Carathéodory function satisfying the following assumption: either

$$\limsup_{|s|\to+\infty} \frac{g(x,s)}{s} < \lambda_1$$

or $\exists\ \lambda_i$ s.t.

$$\lambda_i < \liminf_{|s|\to+\infty} \frac{g(x,s)}{s} \le \limsup_{|s|\to+\infty} \frac{g(x,s)}{s} < \lambda_{i+1}.$$

A *nonlinear* elliptic case

$$(P_N) \quad \begin{cases} a\Delta^2 u + C(-\Delta)^s_\Omega u + V(x)u = \lambda u + f(x,u) & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1, \end{cases}$$

$f : \Omega_0 \times \mathbb{R} \to \mathbb{R}$ *Carathéodory function such that* $f(\cdot, 0) = 0$ *a.e. in* Ω_0.
Recall the critical Sobolev exponent

$$2^* = \begin{cases} \dfrac{2N}{N-4} & \text{if } N > 4, \\ \infty & \text{if } N \leq 4, \end{cases}$$

and its Hölder conjugate

$$(2^*)' = \begin{cases} \dfrac{2N}{N+4} & \text{if } N > 4, \\ 1 & \text{if } N \leq 4. \end{cases}$$

A *nonlinear* elliptic case

$$(P_N) \quad \begin{cases} a\Delta^2 u + C(-\Delta)^s_\Omega u + V(x)u = \lambda u + f(x,u) & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1, \end{cases}$$

Definition

$u \in \mathbb{H}^{2,s}_0(\Omega_0, \Omega)$ is a (weak) *solution* of problem (P_N) if

$$a \int_{\Omega_0} \Delta u \Delta v \, dx + C \iint_{\Omega\times\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \, dxdy$$
$$+ \int_{\Omega_0} V(x) uv \, dx = \int_{\Omega_0} (\lambda u + f(x,u)) v \, dx$$

for every function $v \in \mathbb{H}^{2,s}_0(\Omega_0, \Omega)$.

A *nonlinear* elliptic case

$$(P_N) \quad \begin{cases} a\Delta^2 u + C(-\Delta)^s_\Omega u + V(x)u = \lambda u + f(x,u) & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1, \end{cases}$$

Solutions of (P_N) are critical points of $I : \mathbb{H}^{2,s}_0(\Omega_0, \Omega) \to \mathbb{R}$, defined as

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_{\Omega_0} u^2 - \int_{\Omega_0} F(x,u)\,dx,$$

where $F(x,t) := \int_0^t f(x,\tau)d\tau$.

A *nonlinear* elliptic case

$$(P_N) \quad \begin{cases} a\Delta^2 u + C(-\Delta)^s_\Omega u + V(x)u = \lambda u + f(x,u) & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1, \end{cases}$$

Solutions of (P_N) are critical points of $I : \mathbb{H}^{2,s}_0(\Omega_0, \Omega) \to \mathbb{R}$, defined as

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_{\Omega_0} u^2 - \int_{\Omega_0} F(x,u)\,dx,$$

where $F(x,t) := \int_0^t f(x,\tau)d\tau$.
At least, formally.

A *nonlinear* elliptic case

Set $F(x,t){:=}\int_0^t f(x,\tau)d\tau$ and $\sigma(x,t){:=}\ f(x,t)t{-}2F(x,t)$ in $\Omega_0\times\mathbb{R}$. Assume:

(f_1) $\exists\ \alpha\in L^q(\Omega_0), \beta\in L^\infty(\Omega_0),\ \alpha\geq 0,\ \beta>0, q\in((2^*)',2), r\in(2,2^*)$:

$$|f(x,t)|\leq \alpha(x)+\beta(x)|t|^{r-1} \quad \textit{for a.e. } x\in\Omega_0 \textit{ and for all } t\in\mathbb{R};$$

(f_2) $\lim\limits_{t\to\pm\infty}\dfrac{F(x,t)}{|t|^2}=+\infty$ *uniformly for a.e.* $x\in\Omega_0$;

(f_3) $\lim\limits_{t\to 0}\dfrac{f(x,t)}{|t|}=0$ *uniformly for a.e.* $x\in\Omega_0$;

(f_4) $\exists\ \theta\geq 1,\ \beta\in L^1(\Omega_0),\ \beta\geq 0,$ *s.t.*

$$\sigma(x,t_1)\leq\theta\sigma(x,t_2){+}\beta(x) \quad \textit{a.e. } x\in\Omega_0, \textit{ all } 0\leq t_1\leq t_2 \textit{ or } t_2\leq t_1\leq 0.$$

Some comments on (f_4)

(f_4) $\exists\, \theta \geq 1$, $\beta \in L^1(\Omega_0)$, $\beta \geq 0$, s.t.

$$\sigma(x,t_1) \leq \theta\sigma(x,t_2)+\beta(x) \quad \text{a.e. } x \in \Omega_0, \text{ all } 0 \leq t_1 \leq t_2 \text{ or } t_2 \leq t_1 \leq 0.$$

(f_4) from Mugnai-Proietti Lippi (NA 2019, $p-$Laplacian), improvement of those in Mugnai-Papageorgiou (TAMS 2014, $p-q-$Laplacian) and Jeanjean (PRSE-A 1999, Laplacian).

Some comments on (f_4)

(f_4) $\exists\, \theta \geq 1$, $\beta \in L^1(\Omega_0)$, $\beta \geq 0$, s.t.

$$\sigma(x,t_1) \leq \theta\sigma(x,t_2)+\beta(x) \quad \text{a.e. } x \in \Omega_0, \text{ all } 0 \leq t_1 \leq t_2 \text{ or } t_2 \leq t_1 \leq 0.$$

(f_4) from Mugnai-Proietti Lippi (NA 2019, $p-$Laplacian), improvement of those in Mugnai-Papageorgiou (TAMS 2014, $p-q-$Laplacian) and Jeanjean (PRSE-A 1999, Laplacian).

Example: $f(x,t) = \beta t \log(1+|t|), \qquad \beta > 0.$

Some comments on (f_4)

Ambrosetti-Rabinowitz-condition 1973:
$f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\exists\, p > 2, R > 0$ so that $0 < pF(s, x) \leq f(x, s)s$ for all $x \in \bar{\Omega}$ and for all s s.t. $|s| > R$.
Clearly this condition implies that $F(x, s) \geq \frac{|s|^p}{p} - d$ for a suitable constant $d > 0$.

Definition

$I \in C^1(H, \mathbb{R})$, where H is a Hilbert space satisfies the Palais-Smale condition, (PS), if any sequence $(u_n)_n \subseteq H$ such that $(I(u_n))_n$ is bounded and $I'(u_n) \to 0$ in H as $n \to \infty$ admits a convergent subsequence in H.

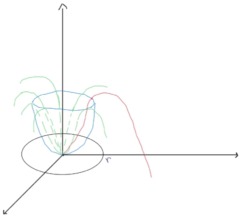
Mountain Pass Theorem

Theorem (Ambrosetti-Rabinowitz, 1973)

Assume $I \in C^1(H, \mathbb{R})$, where H is a Hilbert space, such that

- ⋆ *I satisfies the (PS) condition,*
- ⋆ *$I(0) = 0$,*
- ⋆ *$\exists\, r$ and a such that $I(u) \geq a$ if $\|u\| = r$*
- ⋆ *$\exists\, v \in H$ with $\|v\| > r$ s.t. $I(v) \leq 0$.*

If we define $\Gamma := \{g \in C([0,1]; H) : g(0) = 0, g(1) = v\}$ and $c := \inf_{g\in\Gamma} \max_{0\leq t\leq 1} I(g(t))$, then c is a critical value for I.



Some comments on (f_4)

However, under (f_4) we are not able to prove the (PS) condition.

Some comments on (f_4)

However, under (f_4) we are not able to prove the (PS) condition.

But we can prove the Cerami condition:

Definition

I satisfies the Cerami condition, (C), if any sequence $(u_n)_n$ in $\mathbb{H}_0^{2,s}(\Omega_0,\Omega)$ s.t. $(I(u_n))_n$ is bounded and $(1+\|u_n\|)I'(u_n)\to 0$ as $n\to\infty$ admits a convergent subsequence.

Recall (f_4): $\exists\,\theta\geq 1$, $\beta\in L^1(\Omega_0)$, $\beta\geq 0$, s.t.

$$\sigma(x,t_1)\leq\theta\sigma(x,t_2)+\beta(x)\quad \text{a.e. } x\in\Omega_0,\text{ all } 0\leq t_1\leq t_2 \text{ or } t_2\leq t_1\leq 0$$

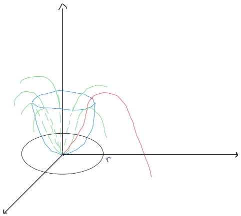
Mountain Pass Theorem

Theorem

Assume $I \in C^1(H, \mathbb{R})$, where H is a Hilbert space, such that

- ⋆ *I satisfies the (C) condition,*
- ⋆ $I(0) = 0$,
- ⋆ *$\exists\, r$ and a such that $I(u) \geq a$ if $\|u\| = r$*
- ⋆ *$\exists\, v \in H$ with $\|v\| > r$ s.t. $I(v) \leq 0$.*

If we define $\Gamma := \{g \in C([0,1]; H) : g(0) = 0, g(1) = v\}$ and $c := \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} I(g(t))$, then c is a critical value for I.



A *nonlinear* elliptic case

$$(P_N) \quad \begin{cases} a\Delta^2 u + C(-\Delta)^s_\Omega u + V(x)u = \lambda u + f(x,u) & \text{in } \Omega_0, \\ u = 0 & \text{in } \Omega_1, \end{cases}$$

Solutions of (P_N) are critical points of $I : \mathbb{H}^{2,s}_0(\Omega_0, \Omega) \to \mathbb{R}$, defined as

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_{\Omega_0} u^2 - \int_{\Omega_0} F(x,u)\,dx,$$

where we recall $F(x,t) := \int_0^t f(x,\tau)d\tau$.

Theorem

Let f as before and $\lambda \in \mathbb{R}$. Then (P_N) admits at least one nontrivial solution.

A second order peridynamical beam-type problem

$$(\mathscr{P}_{\rm lin}) \quad \begin{cases} u_{tt} + a\Delta^2 u + C(-\Delta)^s_\Omega u + h(t)t^\alpha u_t + V(t,x)u = 0 & \text{in } I \times \Omega_0, \\ u(t,x) = 0 & \text{on } I \times \Omega_1, \end{cases}$$

- $I = [1, \infty)$,
- $\alpha \in \mathbb{R}$,
- h is a continuous function satisfying $1/C \leq h(t) \leq C$ for all $t \in I$ and for some $C > 0$,
- V is bounded continuous in $I \times \Omega_0$ and V_t continuous in $I \times \Omega_0$.

Problem

Asymptotic stability of solutions (AS):

Given an initial condition (u_0, u_1), we look for conditions s.t. if $u(t,x)$ solves $(\mathscr{P}_{\mathrm{lin}})$, then

$$\text{``}\lim_{t\to\infty} E_u(t) = 0\text{''},$$

where E_u is the energy associated to the solution u of the problem.

Beam equation: The non-degenerate classical case

If we consider a non-degenerate beam equation

$$(NB) \quad u_{tt}(t,x) + a(x)u_{xxxx}(t,x) = 0$$

where, for example, $a(x) := \frac{EI}{\rho(x)}$, EI is the flexural rigidity coefficient and ρ is the mass density, (AS) is studied by

- ⋆ Chen, Krantz, Ma, Wayne, West 1987
- ⋆ Chen, Delfour, Krall, Payre, 1987
- ⋆ Chen, Zhou 1990
- ⋆ F., Mugnai 2008 . . .

with a degenerate damping:

- ⋆ Cavalcanti, Domingos Cavalcanti, Jorge Silva, Narciso, 2021
- ⋆ Cong, Chunyou 2023
- ⋆ Narciso, Ekinci, Piskin, 2023. . .

Beam equation. The degenerate case

If there is a degenerate leading operator

Camasta, F., SICON 2024 and DCDS 2024,
Camasta, F., Pignotti NORWA 2025,

Beam equation. The degenerate case

$$(DP) \quad \begin{cases} u_{tt} + a(x)u_{xxxx} = 0, & (t,x) \in Q, \\ u(t,0) = 0, \;\; u_x(t,0) = 0, & t > 0, \\ \beta u(t,1) - u_{xxx}(t,1) + u_t(t,1) = 0, & t > 0, \\ \gamma u_x(t,1) + u_{xx}(t,1) + u_{tx}(t,1) = 0, & t > 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), & x \in (0,1) \end{cases}$$

- ⋆ $Q := (0, +\infty) \times (0,1)$
- ⋆ a degenerates at 0.
- ⋆ β, γ are non negative constants

Types of degeneracy

We will consider two types of degeneracy:

Definition

Weakly degenerate function (WD): a is (WD) at 0 if $a \in C[0,1] \cap C^1(0,1]$, $a(0)=0$, $a>0$ on $(0,1]$ and $\exists\, K_a \in (0,1)$ s.t. $\boxed{xa'(x) \le K_a a(x)}$ a.e. $x \in [0,1]$.

Types of degeneracy

We will consider two types of degeneracy:

Definition

Weakly degenerate function (WD): a is (WD) at 0 if $a \in C[0,1] \cap C^1(0,1]$, $a(0) = 0$, $a > 0$ on $(0,1]$ and $\exists\ K_a \in (0,1)$ s.t. $\boxed{xa'(x) \leq K_a a(x)}$ a.e. $x \in [0,1]$.

Prototype: $a(x) = x^\alpha$, $\alpha \in (0,1)$.

Types of degeneracy

We will consider two types of degeneracy:

Definition

Weakly degenerate function (WD): a is (WD) at 0 if $a \in C[0,1] \cap C^1(0,1]$, $a(0) = 0$, $a > 0$ on $(0,1]$ and $\exists\ K_a \in (0,1)$ s.t. $\boxed{xa'(x) \le K_a a(x)}$ a.e. $x \in [0,1]$.

Prototype: $a(x) = x^\alpha$, $\alpha \in (0,1)$.

Definition

Strongly degenerate function (SD): a is (SD) at $x_0 \in [0,1]$ if $a \in C^1[0,1]$, $a(0) = 0$, $a > 0$ on $(0,1]$ and $\exists\ K_a \in [1,2)$ s.t. $\boxed{xa'(x) \le K_a a(x)}$ a.e. $x \in [0,1]$.

Types of degeneracy

We will consider two types of degeneracy:

Definition

Weakly degenerate function (WD): a is (WD) at 0 if
$a \in C[0,1] \cap C^1(0,1]$, $a(0) = 0$, $a > 0$ on $(0,1]$ and $\exists\ K_a \in (0,1)$ s.t.
$\boxed{xa'(x) \leq K_a a(x)}$ a.e. $x \in [0,1]$.

Prototype: $a(x) = x^\alpha$, $\alpha \in (0,1)$.

Definition

Strongly degenerate function (SD): a is (SD) at $x_0 \in [0,1]$ if
$a \in C^1[0,1]$, $a(0) = 0$, $a > 0$ on $(0,1]$ and $\exists\ K_a \in [1,2)$ s.t.
$\boxed{xa'(x) \leq K_a a(x)}$ a.e. $x \in [0,1]$.

Prototype: $a(x) = x^\alpha$, $\alpha \in [1,2)$.

Suitable Hilbert spaces and energy in non divergence form

$$L^2_{\frac{1}{a}}(0,1) := \left\{ u \in L^2(0,1) \;\mid\; \|u\|_{\frac{1}{a}} < \infty \right\},$$

$$H^1_{\frac{1}{a},0}(0,1) := \left\{ u \in L^2_{\frac{1}{a}}(0,1) \cap H^1(0,1) : u(0) = 0 \right\},$$

$$H^2_{\frac{1}{a},0}(0,1) := \left\{ u \in H^1_{\frac{1}{a},0}(0,1) \cap H^2(0,1) : u'(0) = 0 \right\},$$

Definition

$$E_u(t) = \frac{1}{2}\left(\int_0^1 \left(\frac{1}{a} u_t^2(t,x) + u_{xx}^2(t,x) \right) dx + \beta u^2(t,1) + \gamma u_x^2(t,1) \right),$$

$\forall\ t \geq 0$.

Stability

Theorem (Camasta, F., SICON 2024)

Assume (H1).

$$\frac{dE_u(t)}{dt} = -u_t^2(t,1) - u_{tx}^2(t,1), \quad \forall\, t \geq 0.$$

Hence E_u *is nonincreasing. In particular,* $\exists\, C > 0$ *such that*

$$E_u(t) \leq E_u(0)e^{1-\frac{t}{C}}, \quad \forall\, t \in [C, +\infty).$$

Beam equation. The degenerate case in the divergence form

$$(DP_D)\quad \begin{cases} u_{tt}(t,x) + (a(x)u_{xx})_{xx}(t,x) = 0, & (t,x) \in Q, \\ u(t,0) = 0, & t > 0, \\ \begin{cases} u_x(t,0) = 0, & \text{if } a(x) \approx x^\alpha, \alpha \in (0,1), \\ (au_{xx})(t,0) = 0, & \text{if } a(x) \approx x^\alpha, \alpha \in [1,2), \end{cases} & t > 0, \\ Bu(t,1) = 0, & t > 0, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x \in (0,1), \end{cases}$$

where $Bu(t,1) = \begin{cases} \beta u(t,1) - (au_{xx})_x(t,1) + u_t(t,1), \\ \gamma u_x(t,1) + (au_{xx})(t,1) + u_{tx}(t,1). \end{cases}$

Suitable Hilbert spaces in divergence form

We consider:

$$W_a^2(0,1) := \{u \in H^1(0,1) : u' \text{ is abs. continuous in [0,1]}, \\ \sqrt{a}u'' \in L^2(0,1), u(0) = 0\},$$

if $a(x) \approx x^\alpha, \alpha \in (0,1)$ and

$$W_a^2(0,1) := \{u \in H^1(0,1) : u' \text{ is loc. abs. continuous in } (0,1], \\ \sqrt{a}u'' \in L^2(0,1), u(0) = 0\},$$

if $a(x) \approx x^\alpha, \alpha \in [1,2)$. Define, in both cases,

$$\mathcal{Q}(0,1) := \{u \in W_a^2(0,1) : au'' \in H^2(0,1)\},$$

Beam equation. The degenerate case in the divergence form

Assume a is (WD) or (SD)

⋆ The energy
$E_u(t) := \frac{1}{2}\int_0^1 (u_t^2(t,x) + a(x)u_{xx}^2(t,x))dx + \frac{\beta}{2}u^2(t,1) + \frac{\gamma}{2}u_x^2(t,1)$, $t > 0$,
associated to the original problem is nonincreasing

⋆ If $\beta > 0$ and $\gamma > 0$ then there exists $M > 0$ such that
$E_u(t) \le E_u(0)e^{1-\frac{t}{M}}$, for all $t \in [M, +\infty)$.

Actually the previous stability result holds also if β and/or γ are 0 in the weakly degenerate case since in this case we are able to estimate $u_x(t,1)$

(Camasta, F. DCDS 2024)

Stability

⋆ Camasta, F., Pignotti NORWA 2025:

$$u_{tt} + a(x)u_{xxxx} + k(t)\chi_{\mathcal{P}}(x)u_t(t-\tau, x) = f(y)$$

where $k \in L^1_{loc}([0,+\infty)) \cap C([0,+\infty))$ and there exists $\Lambda > 0$ such that

$$\int_{t-\tau}^{t} |k(s)|ds \leq \Lambda, \qquad \forall\, t \geq 0,$$

$\mathcal{P} \subset (0,1)$, $f(y) = |y|^q y$, with $q > 0$, or $f(y) = \left(\int_0^1 |y|^2 dx\right)^{\frac{p}{2}} y$, with $p \geq 1$ (also for operator in div. form or for a degenerate second order operator in div. or in non div. form)

Coming back to our problem

$$(\mathscr{P}_{\mathrm{lin}}) \quad \begin{cases} u_{tt} + a\Delta^2 u + C(-\Delta)^s_\Omega u + h(t)t^\alpha u_t + V(t,x)u = 0 & \text{in } I \times \Omega_0, \\ u(t,x) = 0 & \text{on } I \times \Omega_1, \end{cases}$$

The natural solution space is

$$X = \left\{ \phi \in C(I, \mathbb{H}^{2,s}_0(\Omega_0, \Omega)) \cap C^1(I, L^2(\Omega)) \,:\, E_\phi \text{ is locally bounded on } I \right\},$$

where the energy E_ϕ is defined as

$$E_\phi(t) = \frac{1}{2}\int_{\Omega_0} \phi_t^2(t,x)dx + \frac{1}{2}\|\phi(t,\cdot)\|^2_{\mathbb{H}^{2,s}_0(\Omega_0,\Omega)} + \frac{1}{2}\int_{\Omega_0} V(t,x)\phi^2(t,x)dx.$$

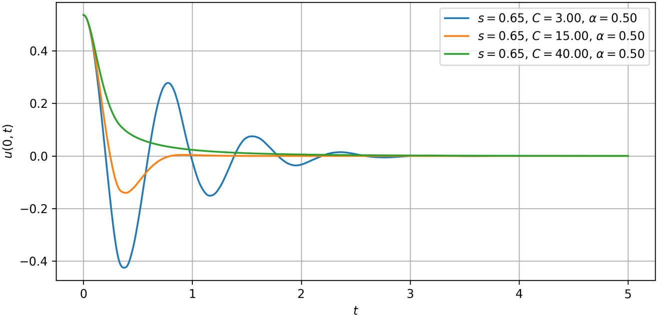
A second order peridynamical beam-type problem

$$(\mathscr{P}_{\mathrm{lin}}) \quad \begin{cases} u_{tt} + a\Delta^2 u + C(-\Delta)^s_\Omega u + h(t)t^\alpha u_t + V(t,x)u = 0 & \text{in } I \times \Omega_0, \\ u(t,x) = 0 & \text{on } I \times \Omega_1, \end{cases}$$

Theorem

If $|\alpha| \leq 1$, then all solutions u of $(\mathscr{P}_{\mathrm{lin}})$ are such that

$$\lim_{t\to\infty} E_u(t) = 0 \quad i.e. \quad \lim_{t\to\infty} (\|u_t\|_2 + \|u\|) = 0.$$


$s = 0.65$, $C = 3.00$, $\alpha = 0.50$
$s = 0.65$, $C = 15.00$, $\alpha = 0.50$
$s = 0.65$, $C = 40.00$, $\alpha = 0.50$
$u(0,t)$
0.4
0.2
0.0
−0.2
−0.4
0
1
2
3
4
5
t

An Idea of the Proof: Pucci-Serrin (CPAM 1996) for local operators

Actually Pucci-Serrin consider the nonlinear damped wave system with Dirichlet data

$$(PS) \quad \begin{cases} u_{tt} - \Delta u + Q\left(t, x, u, u_t\right) + f(x, u) = 0, & (t, x) \in J \times \Omega, \\ u(t, x) = 0, & (t, x) \in J \times \partial\Omega, \end{cases}$$

where

⋆ $J = [1, \infty)$ and $\Omega \subset \mathbb{R}^N$.

⋆ $u(t, x) \in \mathbb{R}^k$, $k \geq 1$,

⋆ $Q \in C\left(J \times \Omega \times \mathbb{R}^k \times \mathbb{R}^k; \mathbb{R}^k\right)$ is a nonlinear damping, so that

$$(Q(t, x, u, v), v) \geq 0 \text{ for all arguments } t, x, u, v,$$

where $(\cdot, \cdot)$ is the inner product in $\mathbb{R}^k$.

⋆ $f \in C\left(\Omega \times \mathbb{R}^k; \mathbb{R}^k\right)$ is a restoring force derivable from a potential F, i.e.

$$(f(x, u), u) \geq 0 \quad f(x, u) = \frac{\partial F}{\partial u}(x, u).$$

An idea of the Proof

Theorem (Pucci, Serrin CPAM 1996)

Under some technical assumptions on f and Q, if $\exists\ k \in C(J) \cap BV(J)$, $k \geq 0$ s.t.

$$k \notin L^1(J),$$

$$\frac{\liminf_{t\to\infty} \int_0^t \left(\delta + \sigma^{1-m}\right) k^m ds}{\left(\int_0^t k ds\right)^m} < \infty,$$

then every strong solution u of (PS) is such that

$$\lim_{t\to\infty} E_u(t) = 0$$

An idea of the Proof

Corollary (Pucci, Serrin CPAM 1996)

Suppose

$$\liminf_{t\to\infty} \frac{1}{t^m}\int_0^t \left(\delta + \sigma^{1-m}\right) ds < \infty.$$

Then for every strong solution u of (PS) we have

$$\lim_{t\to\infty} E_u(t) = 0$$

Proof: Take $k = 1$ in the previous theorem.

Stability if $|\alpha| \leq 1$

Hence if $|\alpha| \leq 1$ the solution of

$$(\mathscr{P}_{\text{lin}}) \quad \begin{cases} u_{tt} + a\Delta^2 u + C(-\Delta)^s_\Omega u + h(t)t^\alpha u_t + V(t,x)u = 0 & \text{in } I \times \Omega_0, \\ u(t,x) = 0 & \text{on } I \times \Omega_1, \end{cases}$$

is s.t.

$$\lim_{t \to \infty} E_u(t) = 0.$$

Stability if $|\alpha| \leq 1$

Hence if $|\alpha| \leq 1$ the solution of

$$(\mathscr{P}_{\text{lin}}) \quad \begin{cases} u_{tt} + a\Delta^2 u + C(-\Delta)^s_\Omega u + h(t)t^\alpha u_t + V(t,x)u = 0 & \text{in } I \times \Omega_0, \\ u(t,x) = 0 & \text{on } I \times \Omega_1, \end{cases}$$

is s.t.

$$\lim_{t\to\infty} E_u(t) = 0.$$

Question: what happens if $|\alpha| > 1$?

A second order peridynamical beam-type problem

If $|\alpha| > 1$, we consider the special case

$$(\mathscr{P}_{\mathrm{lin}}) \quad \begin{cases} u_{tt} + a\Delta^2 u + C(-\Delta)^s_\Omega u + h(t)t^\alpha u_t + V(x)u = 0 & \text{in } I \times \Omega_0, \\ u(t,x) = 0 & \text{on } I \times \Omega_1. \end{cases}$$

Definition

$\psi = \psi(x) \in L^2(\Omega_0)$ is *attainable* for $(\mathscr{P}_{\mathrm{lin}})$ if $\exists$ u solution of $(\mathscr{P}_{\mathrm{lin}})$ s.t.

$$\lim_{t \to \infty} \int_{\Omega_0} |u(t,x) - \psi(x)|^2 dx = 0.$$

A second order peridynamical beam-type problem

If $|\alpha| > 1$, we consider the special case

$$(\mathscr{P}_{\mathrm{lin}}) \quad \begin{cases} u_{tt} + a\Delta^2 u + C(-\Delta)^s_\Omega u + h(t)t^\alpha u_t + V(x)u = 0 & \text{in } I \times \Omega, \\ u(t,x) = 0 & \text{on } I \times \Omega_1. \end{cases}$$

Theorem

If $|\alpha| > 1$ then every function $\psi \in Y := \mathrm{span}\,\{e_k\}_{k=1}^\infty$ is attainable for problem $(\mathscr{P}_{\mathrm{lin}})$ and thus the set of attainable functions is dense in $L^2(\Omega_0)$.

A second order peridynamical beam-type problem

Theorem

If $|\alpha| > 1$ then every function $\psi \in Y := \operatorname{span}\{e_k\}_{k=1}^{\infty}$ is attainable for problem $(\mathscr{P}_{\text{lin}})$ and thus the set of attainable functions is dense in $L^2(\Omega_0)$.

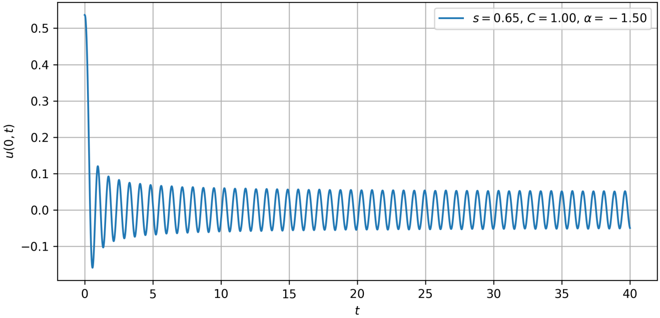
Proof:

- $u_k(t,x) = w_k(t)e_k(x)$ solves $(\mathscr{P}_{\text{lin}}) \Leftrightarrow w_k$ solves $w'' + h(t)t^{\alpha}w' + \lambda_k w = 0$,
- $\lim_{t\to\infty}\|u_k(t,\cdot) - e_k\|_{L^2(\Omega_0)} = 0$.
- $\psi \in Y \Rightarrow \psi = \sum_{i=1}^{j}\beta_i e_{k_i}$. Set

$$u(t,x) = \sum_{i=1}^{j}\beta_i u_{k_i}(t,x) = \sum_{i=1}^{j}\beta_i w_{k_i}(t)e_{k_i}(x).$$

Since $(\mathscr{P}_{\text{lin}})$ is linear, u is a solution and

$$\|u(t,\cdot) - \psi\|_{L^2(\Omega_0)} \to 0 \text{ as } t \to \infty.$$


s = 0.65, C = 1.00, α = − 1.50
u(0, t)
0.5
0.4
0.3
0.2
0.1
0.0
−0.1
0
5
10
15
20
25
30
35
40
t

A peridynamical parabolic problem

$$(\mathscr{P}_{\mathrm{par}}) \quad \begin{cases} u_t + a\Delta^2 u + C(-\Delta)^s_\Omega u + V(x)u = |u|^{p-2}u & \text{in } I \times \Omega_0, \\ u(t,x) = 0 & \text{on } I \times \Omega_1, \end{cases}$$

$p > 2$ and $p \le \frac{2N}{N-4}$ (if $N \ge 5$).
The natural solution space is

$$\mathcal{L}^2(\Omega_0, \Omega) := \{u \in L^2(\Omega) \, : \, u = 0 \text{ in } \Omega_1\},$$

where the energy E_u is defined as

$$E_u(t) := \frac{1}{2}\|u(t,\cdot)\|^2_{\mathbb{H}_0^{2,s}(\Omega_0,\Omega)} - \frac{1}{p}\int_{\Omega_0} |u(t,x)|^p dx, \quad t > 0.$$

A peridynamical parabolic problem

Theorem

Let u the classical solution of $(\mathscr{P}_{\rm par})$ associated to the initial value u_0. If the norm of u_0 is large enough (larger than a well known constant) and the initial energy $E_u(u_0)$ is small enough (bounded from above by a well known constant) then there exists $T > 0$ such that

$$\lim_{t \to T^-} \int_{\Omega_0} u(t,x)^2 dx = +\infty,$$

namely u blows up.

Work in progress

Controllability for

⋆ hyperbolic problems for both operators:

$$Au := a\Delta u + C(-\Delta)^s_\Omega u$$

(in collaboration with S. Dipierro, D. Mugnai and E. Valdinoci) and

$$Au := a\Delta^2 u + C(-\Delta)^s_\Omega u$$

(in collaboration with D. Mugnai)

Work in progress

Controllability for

⋆ hyperbolic problems for both operators:

$$Au := a\Delta u + C(-\Delta)^s_\Omega u$$

(in collaboration with S. Dipierro, D. Mugnai and E. Valdinoci) and

$$Au := a\Delta^2 u + C(-\Delta)^s_\Omega u$$

(in collaboration with D. Mugnai)

⋆ parabolic problems for the previous both operators
(in collaboration with D. Mugnai)

Work in progress

Controllability for

⋆ hyperbolic problems for both operators:

$$Au := a\Delta u + C(-\Delta)^s_\Omega u$$

(in collaboration with S. Dipierro, D. Mugnai and E. Valdinoci) and

$$Au := a\Delta^2 u + C(-\Delta)^s_\Omega u$$

(in collaboration with D. Mugnai)

⋆ parabolic problems for the previous both operators
(in collaboration with D. Mugnai)

Stability for

⋆ hyperbolic problems with more general operators
(in collaboration with S. Dipierro, D. Mugnai and E. Valdinoci)

Future pespectives

Stability for

⋆ parabolic problems for both operators:

$$Au := a\Delta u + C(-\Delta)^s_\Omega u$$

or

$$Au := a\Delta^2 u + C(-\Delta)^s_\Omega u$$

Future pespectives

Stability for

- ⋆ parabolic problems for both operators:

$$Au := a\Delta u + C(-\Delta)^s_\Omega u$$

or

$$Au := a\Delta^2 u + C(-\Delta)^s_\Omega u$$

- ⋆ problems of degenerate mixed operator and/or other dampings

Future pespectives

Stability for

- ⋆ parabolic problems for both operators:

$$Au := a\Delta u + C(-\Delta)^s_\Omega u$$

or

$$Au := a\Delta^2 u + C(-\Delta)^s_\Omega u$$

- ⋆ problems of degenerate mixed operator and/or other dampings

Controllability for

- ⋆ all the previous degenerate problems

Conclusion

Thank you for your attention!

An Idea of the Proof

Assumptions:

(H1): $\exists\, p > 1$ s.t.

$$|f(x,u)| \leq C\left[1+|u|^{p-1}\right],$$

for $C > 0$ and if $N \geq 3$ and $p > \frac{2N}{N-2}$, $\exists\, \kappa > 0$ and $\kappa_1 \geq 0$ for which

$$(f(x,u),u) \geq \kappa|u|^p - \kappa_1|u|.$$

(H2): $\sigma(t)\omega(|v|) \leq |Q(t,x,u,v)| \leq \delta(t)\left[|v|^{m-1}+|v|^{q-1}\right],$

where $\delta \in L^1_{\mathrm{loc}}(J)$, $q = \max\left\{p, \frac{2N}{N-2}\right\}$ ($2 < q < \infty$ if $N = 2$), $1 < m \leq q$, $\omega \in C([0,\infty);\mathbb{R})$ s.t. $\omega(0) = 0$, $\omega(\tau) > 0$ for $0 < \tau < 1$, $\omega(\tau) = \tau$ for $\tau \geq 1$. and σ is s.t. $\sigma \geq 0$ in J, $1/\sigma \in L^{m-1}_{\mathrm{loc}}(J)$.

(H3): Q is tame: $\exists\, \gamma \geq 1$ s.t.

$$|Q(t,x,u,v)|\cdot|v| \leq \gamma(Q(t,x,u,v),v) \qquad \text{(automatic if } N=1\text{)}$$