Controlled flows of maps from the circle to compact Riemannian manifolds

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Organizers: Mohammad Akil, Emmanuelle Crépeau, Olivier Goubet, Serge Nicaise, Lionel Rosier

This talk is based on the following articles.

1) (J. Krieger, S. Xiang) Semi-global controllability of a semilinear wave equation, 2022.

2) (JMC, J. Krieger, S. Xiang) Global controllability and stabilization of the wave maps equation from a circle to a sphere, 2023.

3) (JMC, S. Xiang) Global controllability to harmonic maps of the heat map flow from a circle to a sphere, 2024.

4) (JMC, J. Krieger, S. Xiang) Global controllability of the wave maps equation from a circle to a Riemannian manifold, 2025.

Outline

1. Heat and wave map flows; two main results
2. Some ideas of the proofs for the controlled wave map flow
3. Some ideas of the proofs for the controlled heat map flow

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   - Heat map flow
   - Wave map flow
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Heat map flow

Let $(\mathcal{M}, g)$ and $\mathcal{N} \subset \mathbb{R}^l$ be two compact Riemannian manifolds. A solution to the heat map flow is a map u from $[0, +\infty) \times \mathcal{M}$ to $\mathcal{N}$.

(1) $$(\partial_t - \Delta)u \perp T_u\mathcal{N}.$$

For the Cauchy problem, one prescribes $u(0, \cdot) = u^0$. Note that (1) can be written in the following form in local coordinates

(2) $$(\partial_t - \Delta)u^i - S^i_{jk}(u)g^{\alpha,\beta}(x)\partial_\alpha u^j \partial_\beta u^k = 0 \,\forall i \in \{1, \dots, l\}.$$

where $(g^{\alpha,\beta}) = (g_{\alpha,\beta})^{-1}$, $S^i_{jk}(u)$ are the coefficients of the second fundamental form at $u \in \mathcal{N}$. For example, in the case $\mathcal{N} = \mathbb{S}^k := \{u \in \mathbb{R}^{k+1} : \, |u| = 1\}$ with $k \in \mathbb{N} \setminus \{0\}$, (2) is just

(3) $$(\partial_t - \Delta)u - u|\nabla u|^2 = 0.$$

Harmonic maps: stationary solutions; $-\Delta u = u|\nabla u|^2$ if $\mathcal{N} = \mathbb{S}^k$.

Two results and a problem

- Well-posedness of the Cauchy problem if the dimension of $\mathcal{M}$ is 1;
- Formation of singularities in finite time in general if the dimension of $\mathcal{M}$ is at least 2 (JMC and J.-M. Ghidaglia (1989), K.-C. Chang, W.-Y. Ding, and R. Ye (1992)). However if $\mathcal{N}$ has non positive sectional curvature the Cauchy problem is well-posed;
- Evolution with extra (control) forces?

Control forces

We assume that $\mathcal{M} = \mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z} \simeq \mathbb{S}^1$ and $\mathcal{N} = \mathbb{S}^k$. Then the heat map equation with extra (=control) forces (also called the controlled heat map equation) takes the form

(1) $$u_t - u_{xx} = |u_x|^2 u + \mathbf{1}_\omega f^{u^\perp},$$

where ω is some given nonempty open subset of $\mathcal{M}$ and where $f^{u^\perp}$ represents the projection of f onto the tangent space $T_u\mathbb{S}^k \subset \mathbb{R}^{k+1}$ and $\mathbf{1}_\omega$ is the characteristic function of ω:

(2) $\mathbf{1}_\omega(x) = 1$ if $x \in \omega$ and $\mathbf{1}_\omega(x) = 0$ if $x \in \mathbb{T}^1 \setminus \omega$.

Roughly speaking one can modify u as we want for $x \in \omega$ provided that it always take values in $\mathbb{S}^k$. Similar equations hold if $\mathcal{N} \neq \mathbb{S}^k$. The extra force f is called the control and we can choose it as we want.

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Wave maps flow

We assume that $\mathcal{M} = \mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$ (with the usual metric).

(1) $$u_t - u_{xx} \perp T_u\mathcal{N},$$

is now replaced by

(2) $$u_{tt} - u_{xx} \perp T_u\mathcal{N}.$$

It is equivalent to

(3) $$u^i_{tt} - u^i_{xx} - S^i_{jk}(u)\partial_\alpha u^j \partial^\alpha u^k = 0,\ i = 1,\ 2, \ldots,\ l.$$

with

(4) $$\partial_0, \partial_1 = \partial_t, \partial_x,$$

(5) $$\partial^0, \partial^1 = -\partial_t, \partial_x.$$

The initial condition is now

(6) $$(u(0,\cdot), u_t(0,\cdot)) = (u^0, u^1) :\ \mathcal{M} \to T\mathcal{N} \subset \mathbb{R}^l \times \mathbb{R}^l.$$

Special case $\mathcal{M} = \mathbb{T}^1$ and $\mathcal{N} = \mathbb{S}^k$

We assume again that $\mathcal{M} = \mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathcal{N} = \mathbb{S}^k$. Then the wave map equation takes the form

(1) $$u_{tt} - u_{xx} = -\left(|u_t|^2 - |u_x|^2\right) u$$

and the controlled equation is

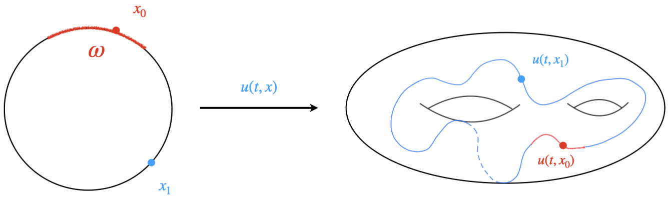
(2) $$u_{tt} - u_{xx} = -\left(|u_t|^2 - |u_x|^2\right) u + \mathbf{1}_\omega f^{u^\perp}$$

where $f^{u^\perp}$ is the orthogonal projection of f onto the tangent space $T_u\mathbb{S}^k$ of $\mathbb{S}^k$ at u:

(3) $$f^{u^\perp} := f - \langle f, u\rangle u \in T_u\mathbb{S}^k.$$

Again, roughly speaking one can modify u as we want for $x \in \omega$ provided that it always takes values in $\mathbb{S}^k$. Note however that now the state, which is now (u, u_t) cannot be deformed arbitrarily: u_t is the time derivative of u.

Two classical problems in control theory



- Controllability: Move from a given state to a desired target state;
- Stabilization: Find control to make energy dissipate exponentially.

Remark

Stabilization is a useful tool to get global controllability results. See, in particular, L. Rosier (1997) for a KdV control system. We also use this tool in this talk.

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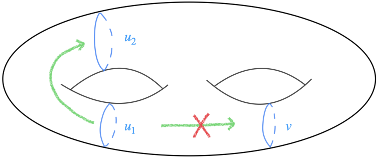
Main results: 1. Global controllability in large time

From now on we always assume that $\mathcal{M} = \mathbb{T}^1$.
Homotopy class



Of course is the initial state and the target state are not in the same homotopy class one cannot reach the target state from the initial state. The next theorem shows that this is the only obstruction.

Main results: 1. Global controllability in large time

Theorem (J. Krieger–S. Xiang (2022), JMC–J. Krieger–S. Xiang 2023, JMC–S. Xiang 2024, JMC–Krieger–Xiang 2025)

For the controlled wave map flow, and the controlled heat map flow if the target state is harmonic,

$$\textit{Homotopic} \Leftrightarrow \textit{Global controllability.}$$

Main results: 2. Global steady state controllability in small time for the controlled heat map flow

Theorem (JMC–S. Xiang (2024))

If the initial state and the target state are harmonic maps and in the same homotopy class one can move, for the controlled heat map flow, from the initial state to the target state in arbitrary small time.

Global stabilization

Let $\mathcal{N} = \mathbb{S}^k$. Our control system is

$$u_{tt} - u_{xx} + (|u_t|^2 - |u_x|^2)u = \mathbf{1}_\omega f^{u^\perp} \tag{1}$$

Question: how to stabilize this system?

Global stabilization

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Question: how to stabilize this system?

Non-trivial steady states $(\bar{\phi}, 0)$: harmonic maps

(2) $$\bar{\phi}_{xx} + |\bar{\phi}_x|^2 \bar{\phi} = 0.$$

Up to isometries, there are given by

(3) $$\bar{\phi}(x) = (\cos(nx), \sin(nx), 0, \ldots, 0),$$

where $n \in \mathbb{Z}$. One has

(4) $$\int_{\mathbb{T}^1} |u_x|^2 dx = 2\pi n^2.$$

Define "ε-approximate harmonic maps":

(5)

$$\mathcal{Q}_\varepsilon := \bigcup_{\bar{\phi}(x):\ \text{a harmonic map}} \left\{ (u, u_t) \in T\mathbb{S}^k : \|(u, u_t) - (\bar{\phi}, 0)\|_{H^1_x \times L^2_x} \leq \varepsilon \right\}$$

Damping and stabilization towards harmonic maps

Damping method

(1) $$u_{tt} - u_{xx} + \left(|u_t|^2 - |u_x|^2\right) u = -a(x)u_t,$$

with $a \geq 0$, support of $a \subset \omega$ and $a \neq 0$. The energy of $[u] := (u, u_t) : \mathbb{T}^1 \to T\mathbb{S}^k$ is

(2) $$E([u]) := \int_{\mathbb{T}^1} \left(|u_t|^2 + |u_x|^2\right) dx.$$

For $\lambda \in [0, +\infty)$, let

(3) $$H(\lambda) := \{[u] = (u, u_t) : \mathbb{T}^1 \to T\mathbb{S}^k;\ E([u]) \leq \lambda\}.$$

Damping and stabilization towards harmonic maps

Note that

(1) $$\frac{d}{dt}E([u]) = -2\int_{\mathbb{T}^1} a(x)|u_t|^2 \leq 0.$$

- (J. Krieger–S. Xiang (2022)) exponential stabilization below 2π energy level set
  Let $\nu > 0$. There exists explicit $c > 0$ and $C > 0$

  (2) $$E(t) \leq Ce^{-ct}E(0)\ \forall [u](0) \in \mathbf{H}(2\pi - \nu)$$

  This is sharp[1] as harmonic maps are steady states.

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A flux method

Proposition (Krieger–Xiang. 2022, JMC–Krieger–Xiang 2023, JMC–Krieger–Xiang 2025)

There exist some effectively computable $q \geq 1$ *and* $C > 0$ *such that, for any* $\varepsilon \in (0,1)$,

$$\int_0^{32\pi} \int_{\mathbb{T}^1} a(x)|u_t|^2 dxdt \leq C\varepsilon^q \tag{1}$$

implies

$$\exists \bar{t} \in [0, 32\pi] \; s.t. \; [u](\bar{t}) \in \mathcal{Q}_\varepsilon \tag{2}$$

Obstruction to global stabilization

A feedback law is an application
$f : [u] \in H^1L^2(\mathbb{T}^1, T\mathbb{S}^k) \mapsto f([u]) \in L^2(\mathbb{T}^1, \mathbb{R}^{k+1})$. For a feedback law $f : [u] \mapsto f([u])$ the closed-loop wave map equation is

(1) $$u_{tt} - u_{xx} = -\left(|u_t|^2 - |u_x|^2\right) u + \mathbf{1}_\omega f([u])^{u^\perp}.$$

One has the following theorem, where f is assumed to be smooth enough (for example locally Lipschitz).

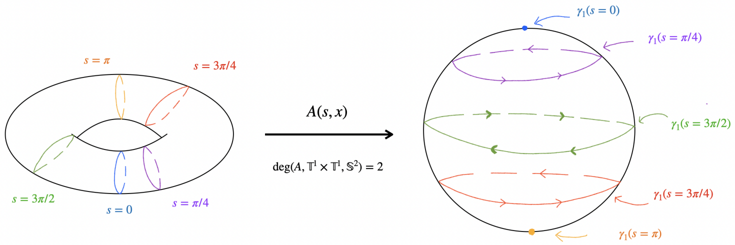
Theorem (JMC–J. Krieger–S. Xiang (2023))

Consider the wave maps flow from $\mathbb{T}^1$ to $\mathbb{S}^k$, there is no feedback control such that, for the closed-loop map equation,

(2) $$E(u(t)) \leq p(t)E(u(0)) \ \forall [u](0) \in \mathbf{H}(2\pi)$$

for some p satisfying $p(+\infty) = 0$.

Topological obstruction ($\mathcal{N} = \mathbb{S}^2$)



Define $A : \mathbb{T}^1_s \times \mathbb{T}^1_x \to \mathbb{S}^2$ as

(1) $$A(s, x) := \begin{cases} (\sin s \cos x, \sin s \sin x, \cos s)^T \ \forall s \in [0, \pi], \\ (-\sin s \cos x, \sin s \sin x, \cos s)^T \ \forall s \in (\pi, 2\pi). \end{cases}$$

Note that $(A(s, \cdot), 0)$ has an energy $\leq 2\pi$:

(2) $$E((A(s, \cdot), 0)) = 2\pi(\sin s)^2 \leq 2\pi.$$

Lemma. The degree of A is 2.

Topological obstruction

- If uniform asymptotic stabilization holds,

(1) $$E(u(t)) \leq p(t)E(u(0)) \ \forall [u](0) \in \mathbf{H}(2\pi)$$

then, denoting by Φ the first component of the closed-loop wave map flow, there exists $T > 0$,

(2) $$|\Phi(T;(A(s,\cdot),0))(x) - \Phi(T;(A(s,\cdot),0))(0)| \leq 1/2 \ \ \forall s \in \mathbb{T}^1_s \ \forall x \in \mathbb{T}^1_x.$$

Hence

(3) $$\text{degree } \left((s,x) \in \mathbb{T}^1 \times \mathbb{T}^1 \mapsto \Phi(T;(A(s,\cdot),0))(x) \in \mathbb{S}^2\right) = 0.$$

- This contradicts the fact that

(4) $$\text{degree } \left((s,x) \in \mathbb{T}^1 \times \mathbb{T}^1 \mapsto \Phi(0;(A(s,\cdot),0)(x) \in \mathbb{S}^2\right) = \text{ degree } A = 2.$$

How to get through harmonic maps for $\mathcal{N} = \mathbb{S}^k$

In the case $\mathcal{N} = \mathbb{S}^k$, the simplest non-trivial harmonic map is given by

(1) $$\varphi(x) := (\cos x, \sin x, 0, \ldots, 0)^T \ \forall x \in \mathbb{T}^1.$$

More generally, every harmonic map can be explicitly written as

(2) $$u(x) = A\varphi(nx) \text{ with } A \in O(k+1), n \in \mathbb{Z}.$$

These non-trivial steady states have energy $2\pi n^2$.
Let us assume we want to arrive to a constant map. With the damping procedure we arrive (close) to an harmonic map of energy $2\pi n^2$. However the damping does not allow to move to a map of energy $< 2\pi n^2$. This cannot be done through damping since in order to do this motion we have to pass through maps which have energy larger than $2\pi n^2$.

Let us explain how to perform such a motion in the case where $\omega = \mathbb{T}^1$, $\mathcal{N} = \mathbb{S}^2$ and the harmonic map is (hence $n = 1$)

$$\phi(x) = (\cos(x), \sin(x), 0)^T. \tag{1}$$

So we consider solution of the controlled wave map equation such that $u(0,x) = (\cos(x), \sin(x), 0)^T$ and $u_t(0,x) = (0,0,0)^T$. It is natural to look for u in the form

$$u(t,x) = (\cos\theta(t)\cos x, \cos\theta(t)\sin x, \sin\theta(t))^T \tag{2}$$

with $\theta(0) = 0$, $\theta'(0) = 0$, $t \in [0,T]$ and $T > 0$. Then one can check that the corresponding control f has to be

$$\begin{aligned} f(t,x) = (-\theta_{tt} + \sin\theta\cos\theta) \\ (-w\sin\theta(t)\cos x, -\sin\theta(t)\sin x, \cos\theta(t))^T. \end{aligned} \tag{3}$$

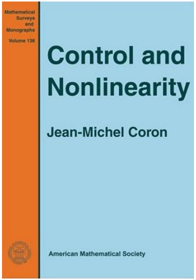
In order to get $E[u](T) < 2\pi$ as we want, it suffises to impose that $\theta'(T) = 0$ and $\theta(T) \in (0, \pi)$. Note that

(1) $$E([u](t)) = 2\pi \left(\theta_t^2 + (\cos\theta)^2\right)(t)$$

and one can check that if $\theta(0) = 0$ and $(E([u](t)) \leq 2\pi$ for every $t \in [0, T]$, then we must have $\theta(t) = 0$ for every $t \in [0, T]$. In particular $E([u](T)) = 2\pi$ which is in agreement that in order to go to a state of energy less than 2π we must pass through states having an energy larger than 2π.

The case where $\omega \neq \mathbb{T}^1$

Power series expansion: JMC-Crépeau, 2004



Power series expansion

Consider

(1) $$\bar{u}_{tt} - \bar{u}_{xx} = (-|u_t|^2 + |\bar{u}_x|^2)\bar{u} + \mathbf{1}_\omega f^\perp$$

and decompose

$$\bar{u} := \bar{u}_0 + \varepsilon \bar{u}_1 + \varepsilon^2 \bar{u}_2 + \dots$$
$$f := \varepsilon f_1 + \varepsilon^2 f_2 + \dots$$

This leads to a cascade of nonlinear systems

(2) $$\bar{u}_{0tt} - \bar{u}_{0xx} = (-|\bar{u}_{0t}|^2| + \bar{u}_{0x}|^2)\bar{u}_0,$$

(3) $$\bar{u}_{1tt} - \bar{u}_{1xx} = 2(-\bar{u}_{0t} \cdot \bar{u}_{1t} + \bar{u}_{0x} \cdot \bar{u}_{1x})\bar{u}_0 + (-|\bar{u}_{0t}|^2 + |\bar{u}_{0x}|^2)\bar{u}_1 + f_1 - (f_1 \cdot \bar{u}_0)\bar{u}_0,$$

(4) $$\bar{u}_{2tt} - \bar{u}_{2xx} = \dots$$

for which, $\bar{u}_0$ is stationary, and $\bar{u}_1$ can be controlled... One succeeds to get, if T is large enough,

(5) $$E(T) - E(0) = -2\pi\varepsilon^2 + O(\varepsilon^3) \text{ as } \varepsilon \to 0^+.$$

The last steps for $\mathcal{N} = \mathbb{S}^k$

In order to get the global controllability for the wave maps in the case $\mathcal{N} = \mathbb{S}^k$ it only remains to

1. Prove the local controllability of the controlled wave equation near $(p, 0)$, $p \in \mathbb{S}^k$ (J. Krieger–S. Xiang (2022)). Note that the linearized control system around $(p, 0)$ are controllable (classical result). It is a difficult task to deduce from these controllabilities the local controllability of the nonlinear system. However it is strongly encouraging.
2. Note that this property implies than one can move from any $(p^1, 0)$ to any $(p^2, 0)$.

How to move from $(p^1, 0)$ to $(p^2, 0)$

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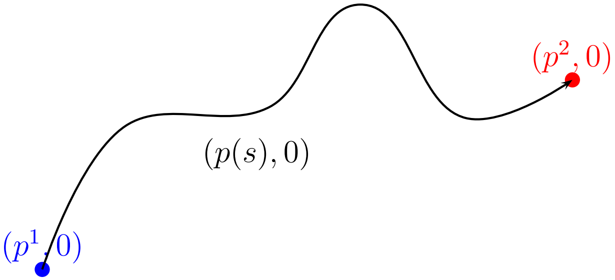

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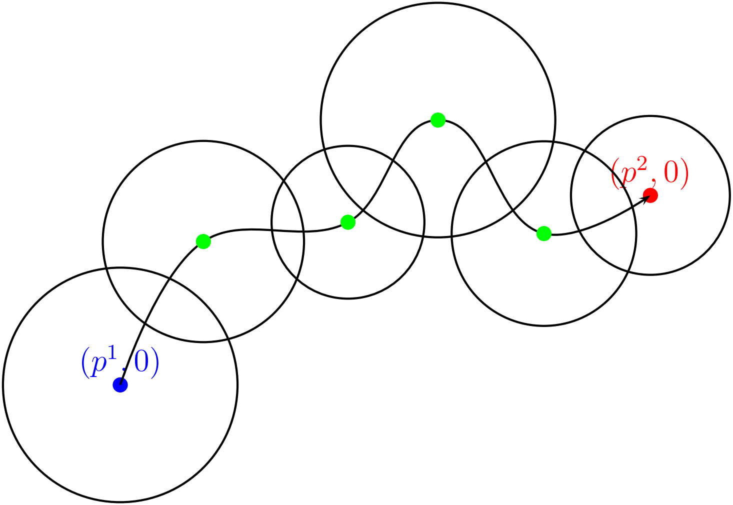

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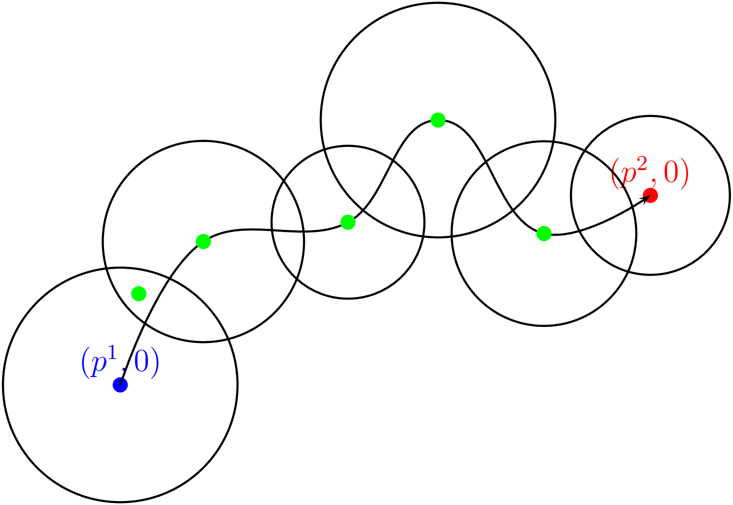

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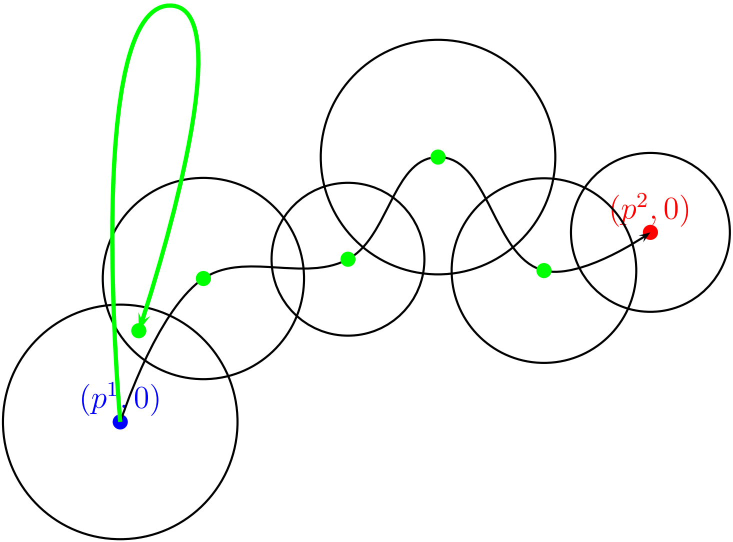

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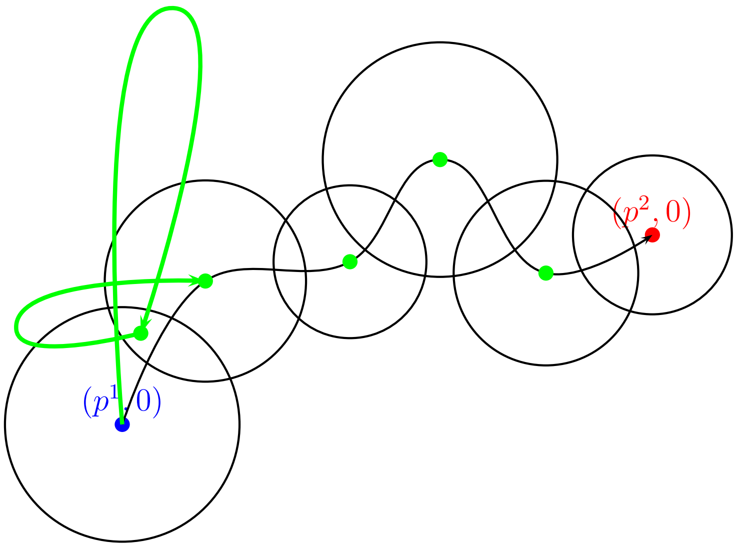

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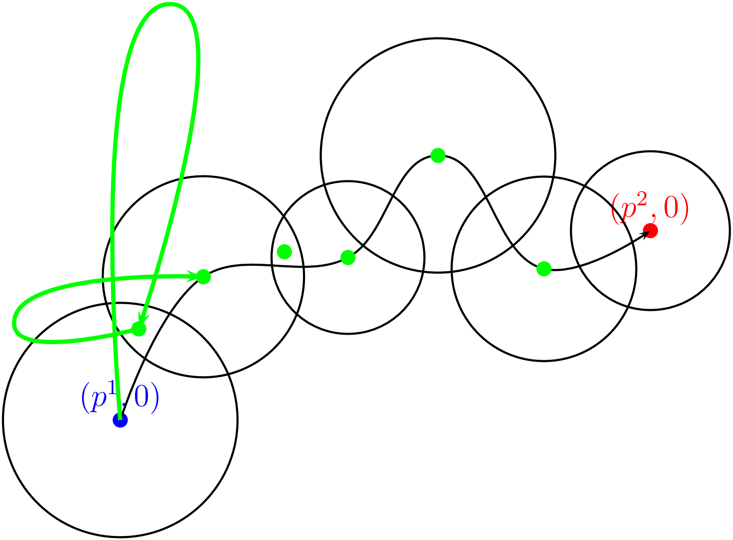

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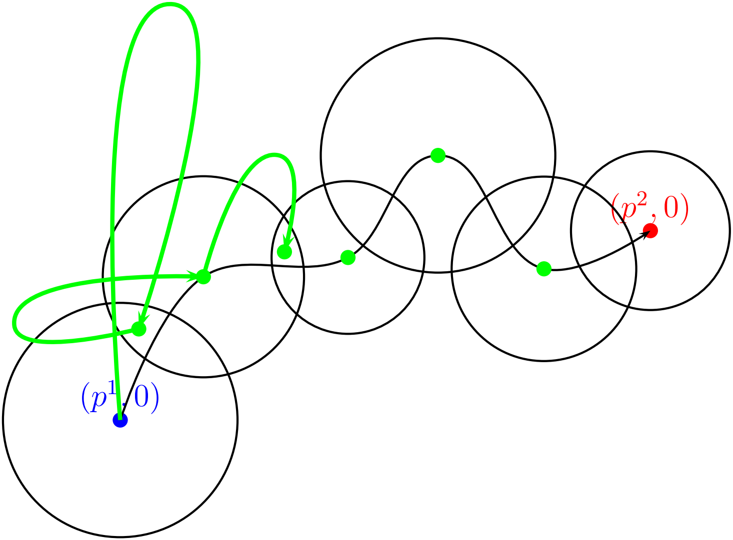

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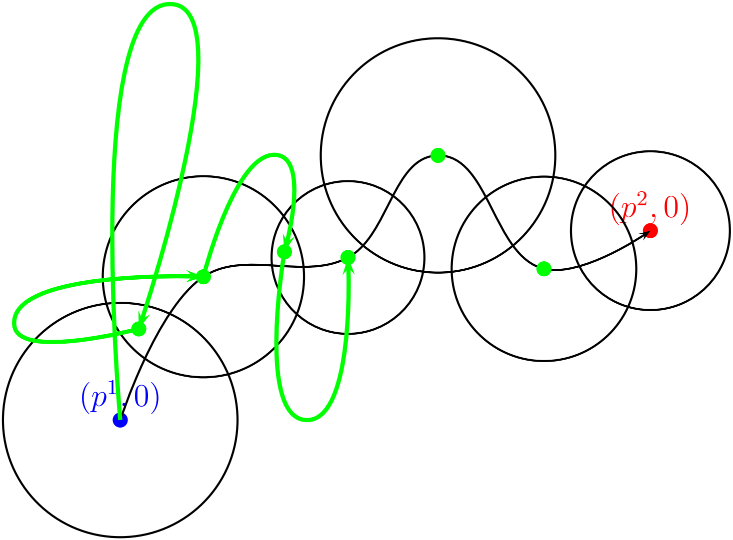

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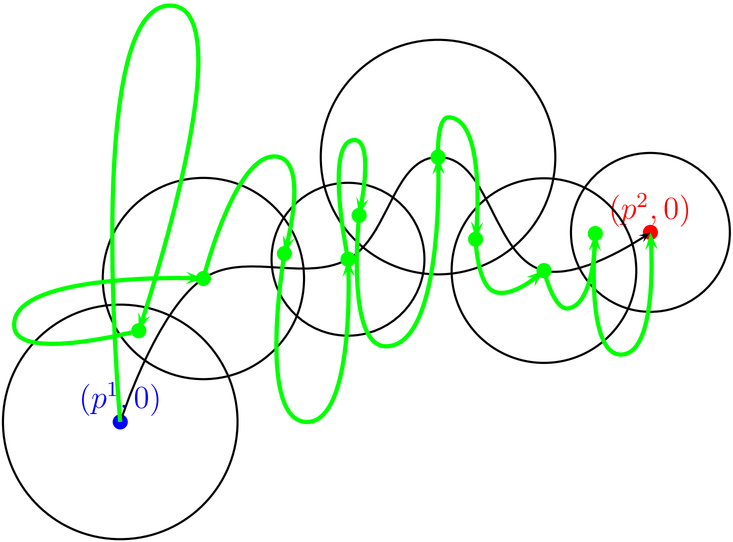

The last steps for $\mathcal{N} = \mathbb{S}^k$

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2. Note that this property implies than one can move from any $(p^1, 0)$ to any $(p^2, 0)$.
3. Observe that if $(u(t, \cdot), u_t(t, \cdot))$ is a solution of the controlled wave maps equation on the time interval $[0, T]$ for the force $f(t, \cdot)$, then $(u(T - t, \cdot), -u_t(T - t, \cdot))$ is a solution of the controlled wave maps equation on the time interval $[0, T]$ for the force $f(T - t, \cdot)$. Let us call this obvious property the “time reversal property”.

Indeed....

We want to move from $(u_1, v_1) : \mathbb{T}^1 \to T\mathbb{S}^k$ to $(u_2, v_2) : \mathbb{T}^1 \to T\mathbb{S}^k$. Using damping + the passing through harmonic maps we arrive close to $(p_1, 0)$ with $p^1 \in \mathbb{S}^k$. Then, using the local controllability around $(p_1, 0)$ we finally arrive to $(p_1, 0)$ exactly. This uses the time interval $[0, T_1]$. We do the same starting from $(u_2, -v_2)$. Using the time interval $[0, T_2]$, we arrive at $(p_2, 0)$ with $p^2 \in \mathbb{S}^k$. Let us denote by $f_2(t, x)$ the control forced that is used for that. As mentioned in the previous slide, we know that it is possible to move from $(p_1, 0)$ to $(p_2, 0)$. This can be done during some time interval $[0, T_3]$. So we know than we can move from (u_1, v_1) to $(p_2, 0)$ during the time interval $[0, T_1 + T_3]$. Starting at time $T_1 + T_3$ from $(p_2, 0)$ and using the control force $f_2(T_1 + T_3 + T_2 - t, x)$, the time reversal property gives us that we arrive at (u_2, v_2) at time $T_1 + T_3 + T_2$.

$\square$

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   - The case of a general $\mathcal{N}$
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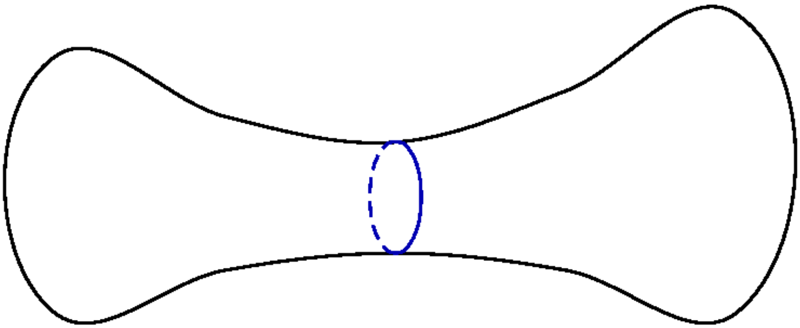
General $\mathcal{N}$

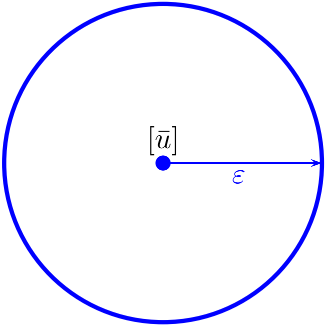


Main new difficulty: How to move from the blue geodesic to a point. Surely one cannot do as above: one needs a large motion when we start from the blue harmonic map (with 0 speed) to arrive at a state having an energy less than the energy of the blue harmonic map (with 0 speed).

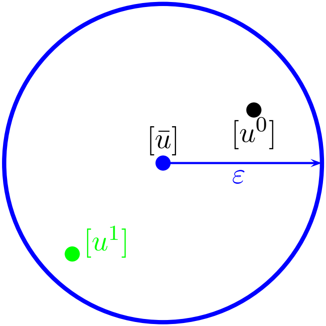
Local controllability

$[\bar{u}]$

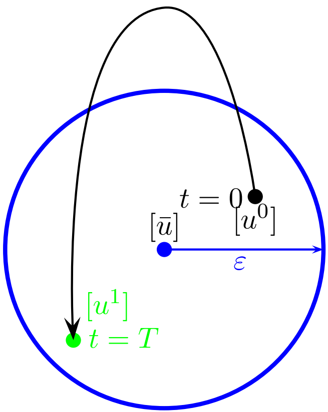
Local controllability



Local controllability



Local controllability



The return method (JMC (1992))

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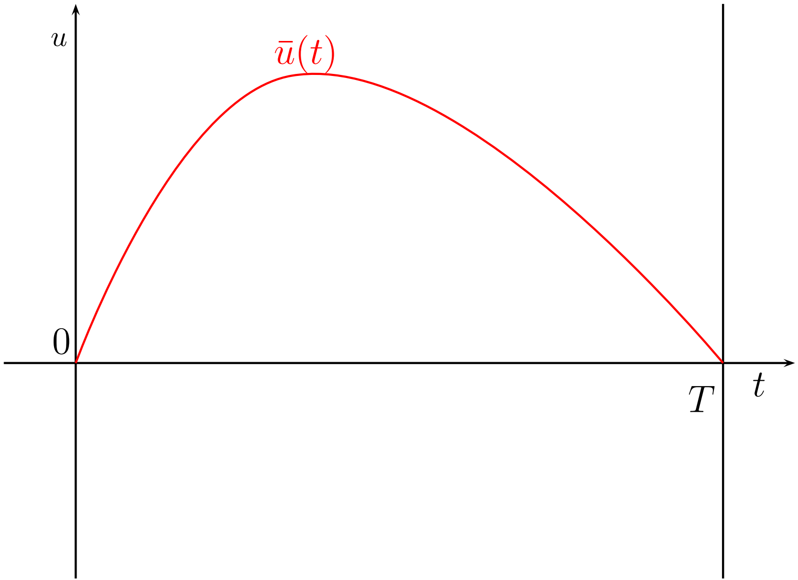
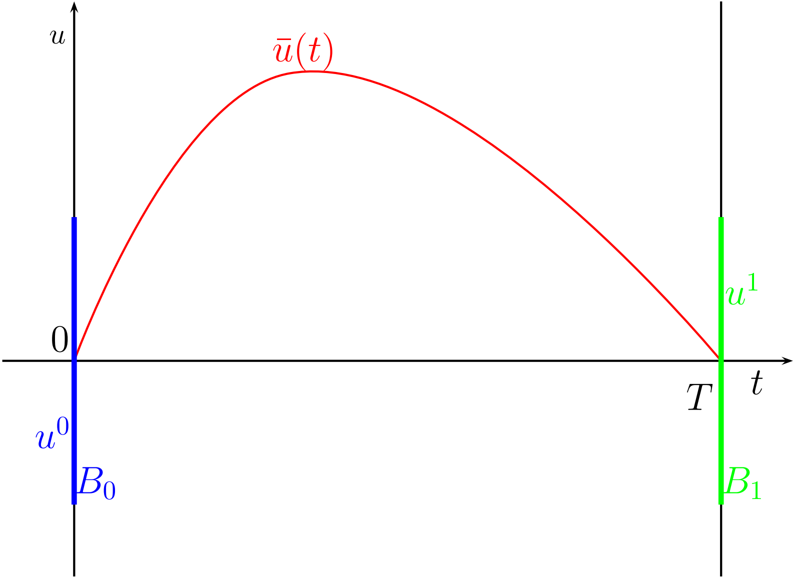
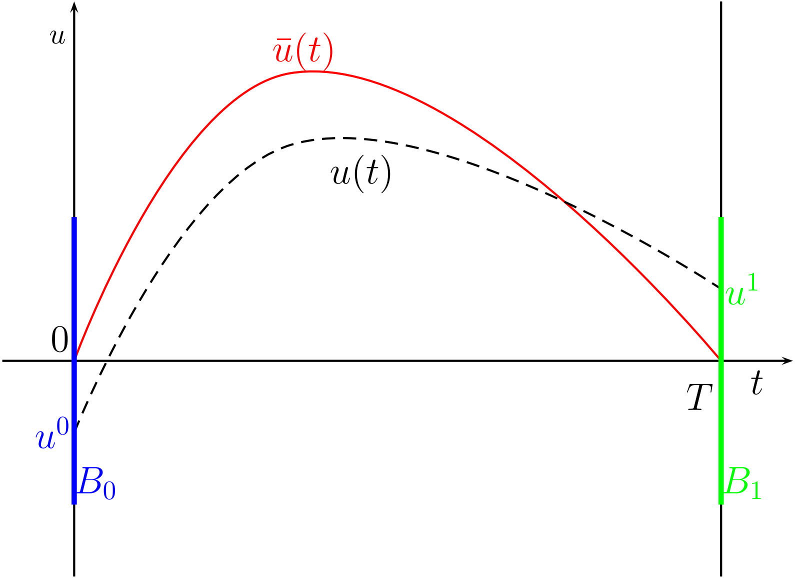


The return method (JMC (1992))

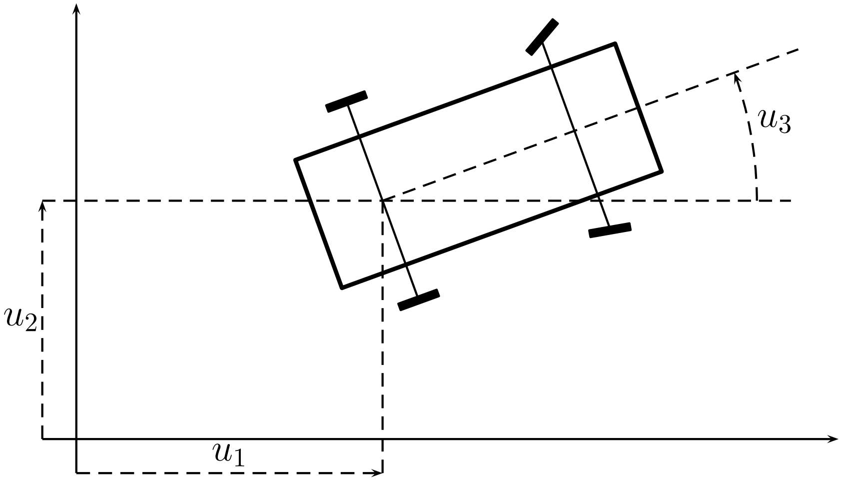


The return method (JMC (1992))

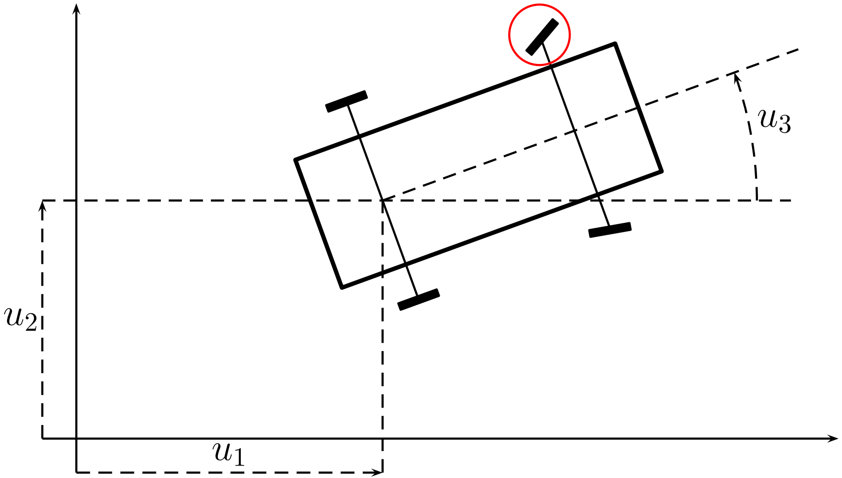


Example in finite dimension: The baby stroller

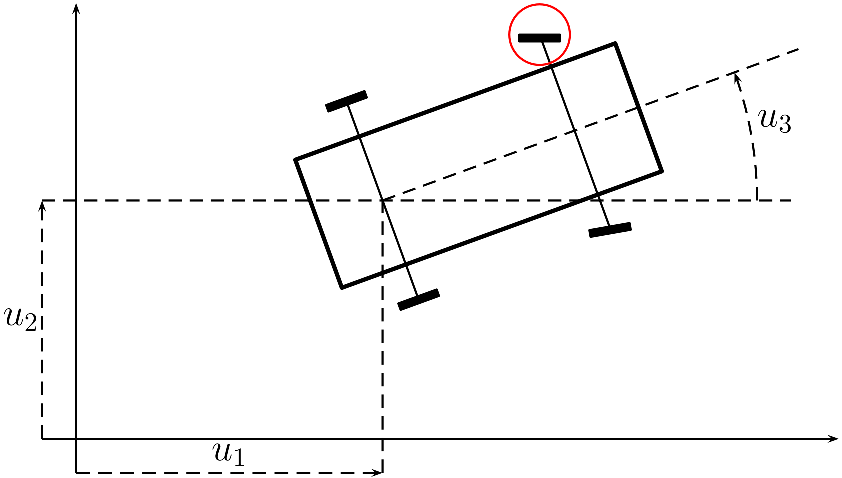
The baby stroller: The dynamic equations of motion



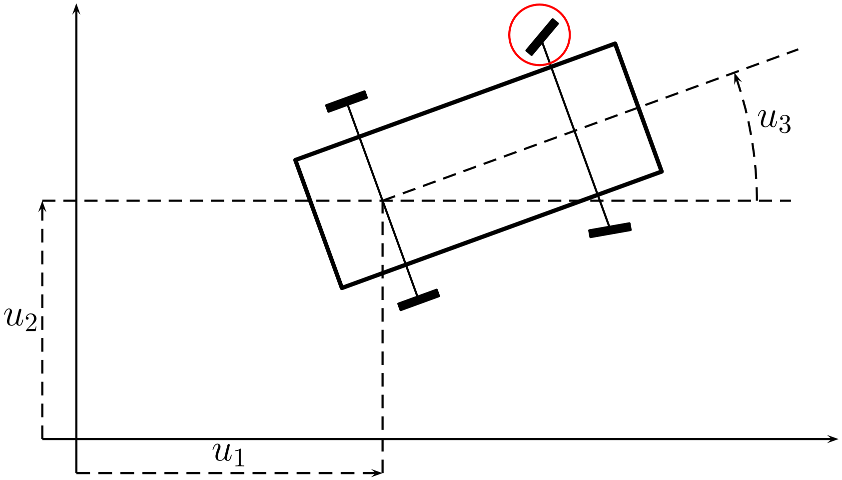
The baby stroller: The dynamic equations of motion



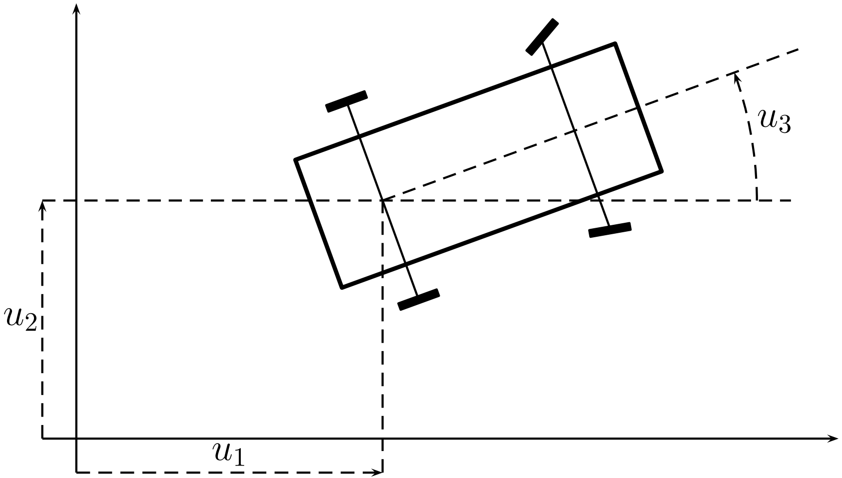
The baby stroller: The dynamic equations of motion



The baby stroller: The dynamic equations of motion



The baby stroller: The dynamic equations of motion



$$\dot{u}_1 = f_1 \cos u_3,\ \dot{u}_2 = f_1 \sin u_3, \dot{u}_3 = f_2. \tag{1}$$

How to come back to the initial state.

We go back to the baby stroller control system

(1) $$\dot{u}_1 = f_1 \cos u_3,\ \dot{u}_2 = f_1 \sin u_3,\ \dot{u}_3 = f_2.$$

For every $\bar{f} : [0,T] \to \mathbb{R}^2$ such that, for every t in $[0,T]$, $\bar{f}(T-t) = -\bar{f}(t)$, every solution $\bar{u} : [0,T] \to \mathbb{R}^3$ of

(2) $$\dot{\bar{u}}_1 = \bar{f}_1 \cos \bar{u}_3,\ \dot{\bar{u}}_2 = \bar{f}_1 \sin \bar{u}_3, \dot{\bar{u}}_3 = \bar{f}_2,$$

satisfies $\bar{u}(0) = \bar{u}(T)$. The linearized control system around $(\bar{u}, \bar{f})$ is

(3) $$\left\{ \begin{array}{l} \dot{u}_1 = -\bar{f}_1 u_3 \sin \bar{u}_3 + f_1 \cos \bar{u}_3,\ \dot{u}_2 = \bar{f}_1 u_3 \cos \bar{u}_3 + f_1 \sin \bar{u}_3, \\ \dot{u}_3 = f_2, \end{array} \right.$$

which is controllable if (and only if) $\bar{f} \not\equiv 0$. We have got the controllability of the baby stroller system without using Lie brackets. We have only used controllability results for **linear** control systems.

Return method: References

- Stabilization of driftless systems in finite dimension: JMC (1992),
- Euler equations of incompressible fluids: JMC (1993,1996), O. Glass (1997,2000), O. Glass-Th. Horsin (2010, 2012, 2016),
- Control of driftless systems in finite dimension: E.D. Sontag (1995),
- Navier-Stokes equations: JMC (1996), JMC and A. Fursikov (1996), A. Fursikov and O. Imanuvilov (1999), S. Guerrero, O. Imanuvilov and J.-P. Puel (2006), JMC and S. Guerrero (2009), M. Chapouly (2009), JMC and P. Lissy (2014),
- Saint-Venant equations: JMC (2002),
- Vlasov Poisson: O. Glass (2003),

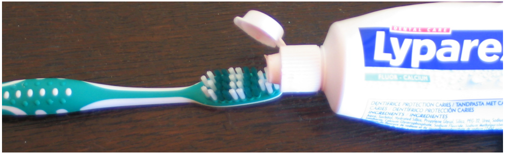
Return method: References (continued)

- Isentropic Euler equations: O. Glass (2006),
- Schrödinger equation: K. Beauchard (2005), K. Beauchard and JMC (2006),
- Hyperbolic/wave equations: JMC, O. Glass and Z. Wang (2009), F. Alabau, JMC and G. Olive (2017), C. Zhang (2017),
- Ensemble controllability of Bloch equations: K. Beauchard, JMC and P. Rouchon (2010),
- Parabolic systems: JMC, S. Guerrero and L. Rosier (2010), JMC and J.-Ph. Guilleron (2017),
- Uniform controllability of scalar conservation laws in the vanishing viscosity limit: M. Léautaud (2010).

Main difficulty for the return method: Return to the initial state

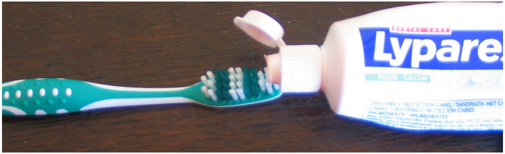
Main difficulty for the return method: Return to the initial state

Initial state

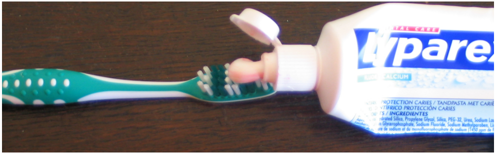


Main difficulty for the return method: Return to the initial state

Initial state

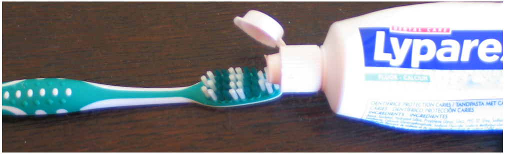


Intermediate state

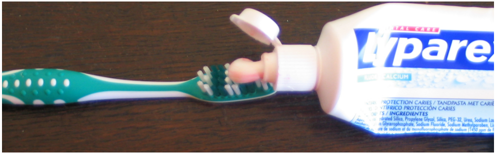


Main difficulty for the return method: Return to the initial state

Initial state



Intermediate state

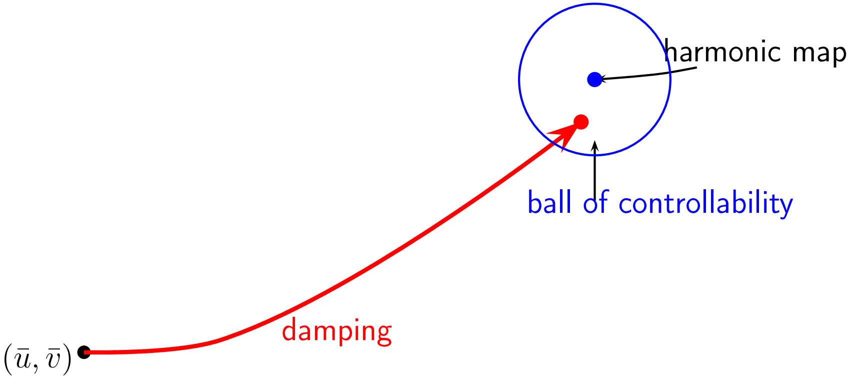


It is difficult to return to the initial state. How to do it for the controlled wave map flow?

Return to the initial state for the controlled wave maps

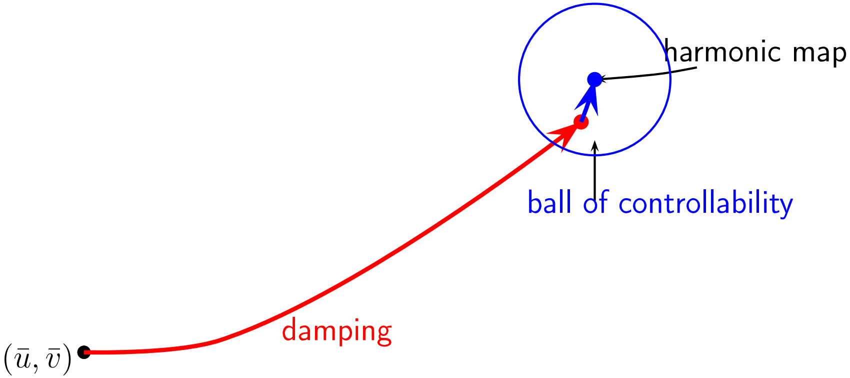
$(\bar{u}, \bar{v})$ •

Return to the initial state for the controlled wave maps

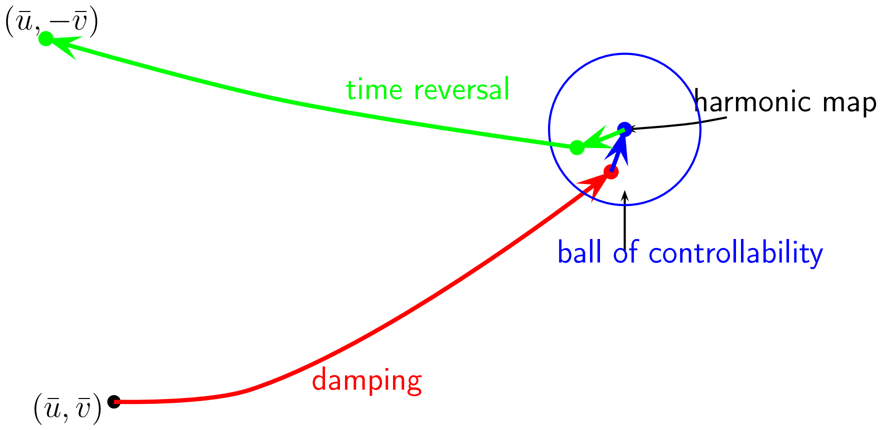


One arrives at the harmonic map (the blue point) at time τ.

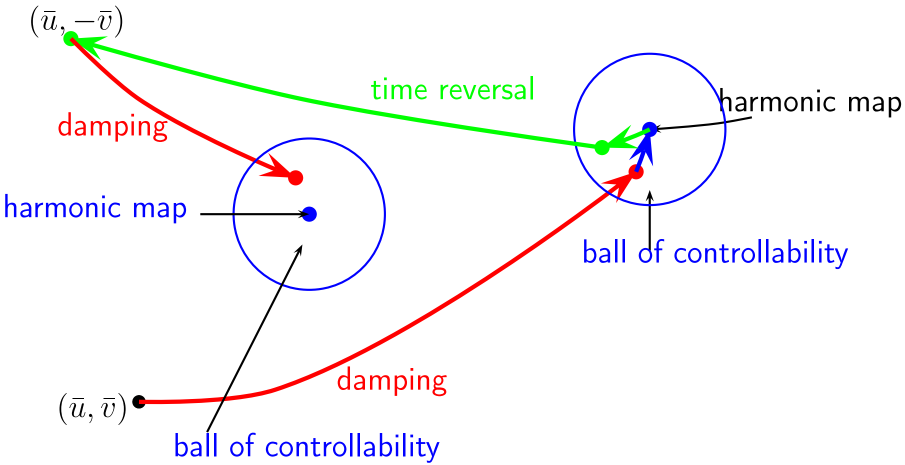
Return to the initial state for the controlled wave maps



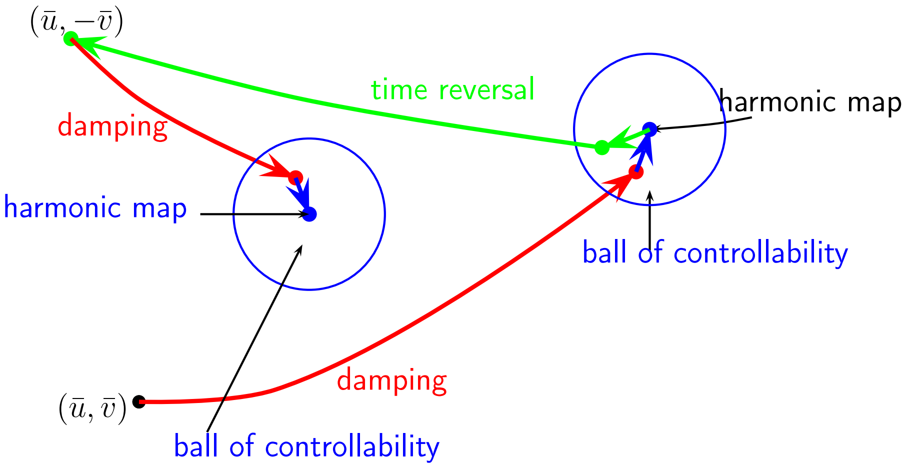
Return to the initial state for the controlled wave maps



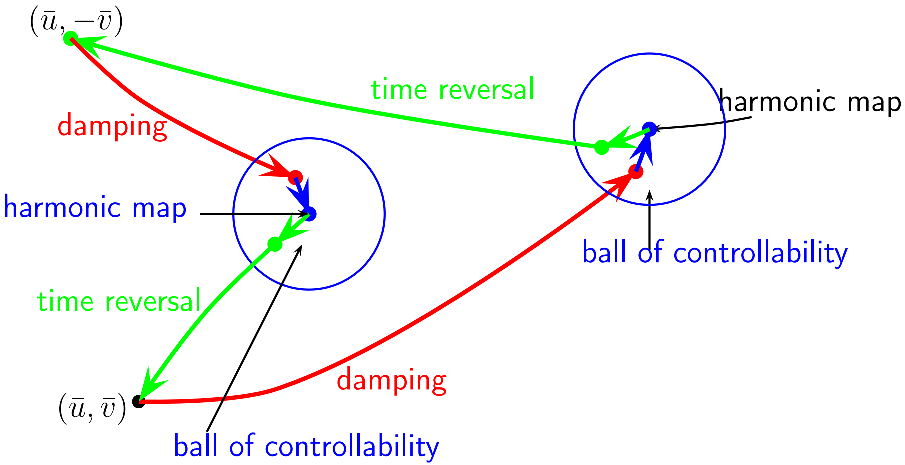
Return to the initial state for the controlled wave maps

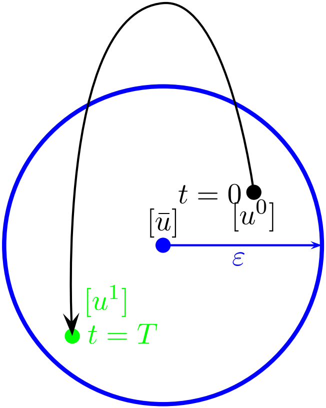


Return to the initial state for the controlled wave maps



Return to the initial state for the controlled wave maps





A key point is to get uniform estimates on ε and T: $\varepsilon \geq \varepsilon(M) > 0$ and $T \leqslant T(M) < +\infty$ if $E([\bar{u}]) \leqslant M$. Such estimates can be proved. With these uniform estimates, one can deduce the global controllability from the local controllability.

1. Heat and wave map flows; two main results
2. Some ideas of the proofs for the controlled wave map flow
3. Some ideas of the proofs for the controlled heat map flow
   - Global controllability to harmonic maps in large time
   - Controllability in small time between two harmonic maps

1. Heat and wave map flows; two main results
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   - Global controllability to harmonic maps in large time
   - Controllability in small time between two harmonic maps

Theorem (JMC–Xiang 2024)

For the heat flow $\mathbb{T}^1 \to \mathbb{S}^n$, global controllability to homotopic harmonic maps.

Main ideas:

- Part 1: global convergence towards harmonic maps (omitted),
- Part 2: local controllability around harmonic maps
- Part 3: Controllability in small time between two harmonic maps

Part 2 using quantitative rapid stabilization

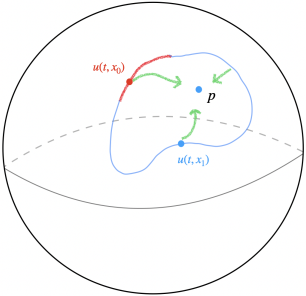
A local control problem with geometric constraint

Stereographic projection:

(1) $$\partial_t u - \Delta u = |\partial_x u|^2 u + \mathbf{1}_\omega f^\perp$$

becomes

(2) $$\partial_t v - \Delta v + \frac{2v \cdot v_x}{4 + |v^2|} v_x - \frac{2|v_x|^2}{4 + |v|^2} v = \mathbf{1}_\omega g$$



- first used by Y. Liu (2018) for local control of heat flow
- a nonlinear control problem without constraint
- nonlinear heat: Liu–Takahashi–Tucsnak (2013)
- we use quantitative rapid stabilization approach

Quantitative rapid stabilization

Notation: P_λ low-frequency projection: projection on the eigenfunctions with eigenvalues less than λ.

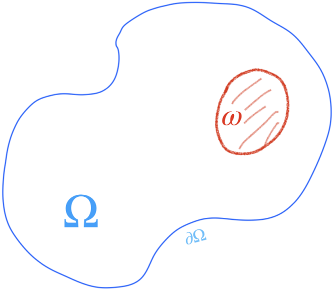
Theorem (S. Xiang (2020))

For $\lambda > 0$ the heat equation

$$y_t - \Delta y = -\gamma_\lambda 1_\omega P_\lambda y$$

satisfies $\|y(t)\|_{L^2} \leq Ce^{C\sqrt{\lambda}} e^{-\lambda t} \|y(0)\|_{L^2}$

(Relies on G. Lebeau and L. Robbiano (1995))



Constructive physical feedback + *Theoretical microlocal analysis*

Frequency Lyapunov (Xiang)

Stabilization

Quantitative rapid stabilization: Navier–Stokes

Theorem (Xiang, 2023)

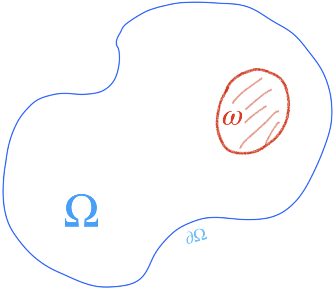
There exists explicit $C > 0$ s.t. for any $\lambda > 0$ the 2D incompressible Navier-Stokes equation

$$u_t - \Delta u + (u.\nabla)u + \nabla p = -\gamma_\lambda 1_\omega P_\lambda u$$

satisfies

$$\|u(t)\|_{L^2} \leq Ce^{C\sqrt{\lambda}} e^{-\lambda t} \|u(0)\|_{L^2}$$

provided $\|u(0)\|_{L^2} \leq (Ce^{C\sqrt{\lambda}})^{-1}$



- This method can be easily adapted to nonlinear models
- Finite time stabilization and null controllability
- Optimal costs

Quantitative rapid stabilization: heat flow

Theorem (JMC–Xiang, 2024)

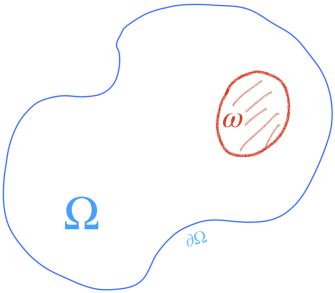
There exists explicit $C > 0$ *s.t. for any* $\lambda > 0$

$$\partial_t v - \Delta v + \frac{2v \cdot v_x}{4 + |v^2|} v_x - \frac{2|v_x|^2}{4 + |v|^2} v = -\lambda e^{C_0 \sqrt{\lambda}} \mathbf{1}_\omega P_\lambda v$$

satisfies

$$\|u(t)\|_{H^1} \leq C e^{C\sqrt{\lambda}} e^{-\lambda t} \|u(0)\|_{H^1}$$

provided $\|u(0)\|_{H^1} \leq (C e^{C\sqrt{\lambda}})^{-1}$



- This method can be easily adapted to nonlinear models
- Finite time stabilization and null controllability
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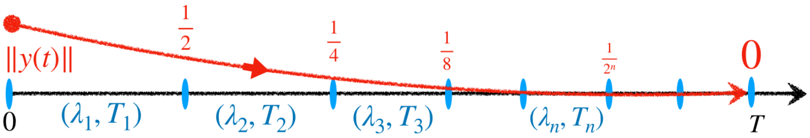
Finite time stabilization

Finite time stabilization: find a sequence $\{(\lambda_k, T_k)\}$, on each interval we perform quantitative rapid stabilization



Theorem (JMC–Xiang, 2024)

There exists $C > 1$ such that for any $T \in (0,1)$, for any initial state $u_0 \in H^1(\mathbb{T}^1)$ and any point $p \in \mathbb{S}^k$ satisfying

(1) $$\|u_0 - p\|_{H^1(\mathbb{T}^1)} \leq e^{-\frac{C}{T}}$$

there is a control f satisfying

(2) $$\|f\|_{L^\infty(0,T;L^2)} \leq e^{\frac{C}{T}} \|u_0 - p\|_{H^1}$$

such that the solution of the heat flow satisfies $u(T, \cdot) = p$.

Part 3: Controllability in small time between two homotopic harmonic maps

Open problem (A longstanding open problem)

Consider the nonlinear controlled heat equation

(1) $$u_t - u_{xx} - u^3 = \mathbf{1}_\omega f,\ x \in \mathbb{T}^1.$$

Does the small-time global controllability between steady states holds?

Part 3: Controllability in small time between two homotopic harmonic maps

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(1) $$u_t - u_{xx} - u^3 = \mathbf{1}_\omega f,\ x \in \mathbb{T}^1.$$

Does the small-time global controllability between steady states holds?

Theorem (JMC-Xiang 2024: a positive answer!)

Consider controlled heat map flow.

(2) $$\partial_t u^i - \Delta u^i + S^i_{jk}(u)\partial^\alpha u^j \partial_\alpha u^k = \mathbf{1}_\omega f$$

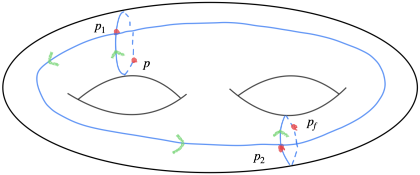
For any homotopic harmonic maps u_0 and u_1, one can move from u_0 to u_1 in arbitrary small positive time.

Part 3: Controllability in small time between two homotopic harmonic maps

(1)
$$\partial_t u^i - \Delta u^i + S^i_{jk}(u)\partial^\alpha u^j \partial_\alpha u^k = \mathbf{1}_\omega f^\perp$$

Let $\Gamma \subset \mathcal{N}$ be a geodesic (complete, closed or not).

(2) $$\Gamma = \{\bar{u}(x) : x \in \mathbb{T}^1\}$$

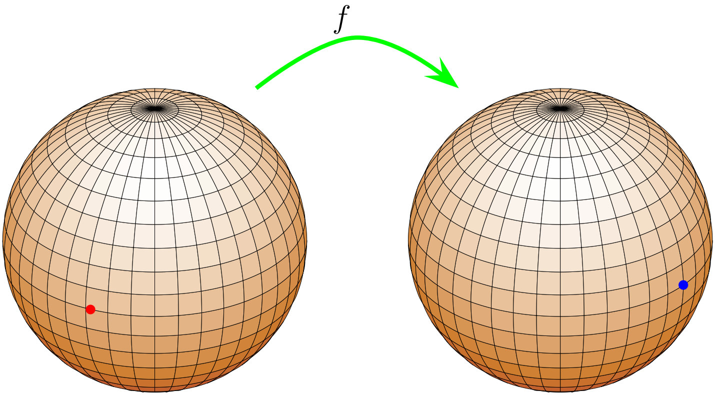


- If initial state $u_0 \in \Gamma$ takes its value in Γ and $f \in T\Gamma$, then u takes its values in Γ
- Let $u(t,x) = \bar{u}(\varphi(t,x))$. Then, if φ satisfies

  (3) $$\partial_t \varphi - \Delta \varphi = \mathbf{1}_\omega g,$$

  $u(t,x)$ is the solution to the heat map equation with $f = g\bar{u}'(\varphi)$.

How to move from a point to another point



Initial state: $x \in \mathbb{T}^1 \mapsto (\cos(\varphi^0), \sin(\varphi^0), 0)^T$, target state: $x \in \mathbb{T}^1 \mapsto (\cos(\varphi^1), \sin(\varphi^1), 0)^T$.

We take u of the form

(1) $$u(t,x) = (\cos(\varphi(t,x)), \sin(\varphi(t,x)), 0)^T,$$

with

(2) $$\varphi_t - \varphi_{xx} = \chi_\omega g,\ \varphi(t,0) = \varphi(t,2\pi),\ \varphi_x(t,0) = \varphi_x(t,2\pi).$$

Then, if $f = g(t,x)(-\sin(\varphi(t,x)), \cos(\varphi(t,x)), 0)^T$, u is a solution of the controlled heat maps flow. We have

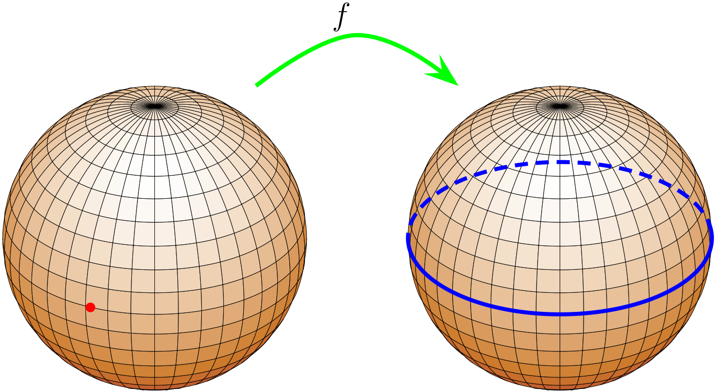
(3) $$\varphi(0,x) = \varphi^0$$

and we want to have, for a given $T > 0$,

(4) $$\varphi(T,x) = \varphi^1.$$

It is known since Hector Fattorini and David Russell (1971) that there are g's allowing to do this motion.

How to move from a point to a non constant harmonic map



Initial state $x \in \mathbb{T}^1 \mapsto (\cos(\varphi^0), \sin(\varphi^0), 0)^T$, target state $x \in \mathbb{T}^1 \mapsto (\cos(x), \sin(x), 0)^T$.

Note that the method used for the wave maps (reverse the time) can no longer be used. We still have

(1) $$\varphi_t - \varphi_{xx} = \chi\omega g,\ \varphi(t,0) = \varphi(t,2\pi),\ \varphi_x(t,0) = \varphi_x(t,2\pi).$$

The initial condition is

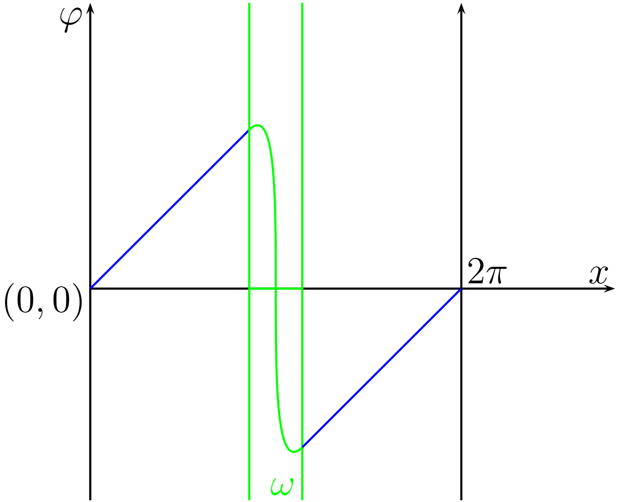
(2) $$\varphi(0,x) = \varphi^0$$

We would like to have

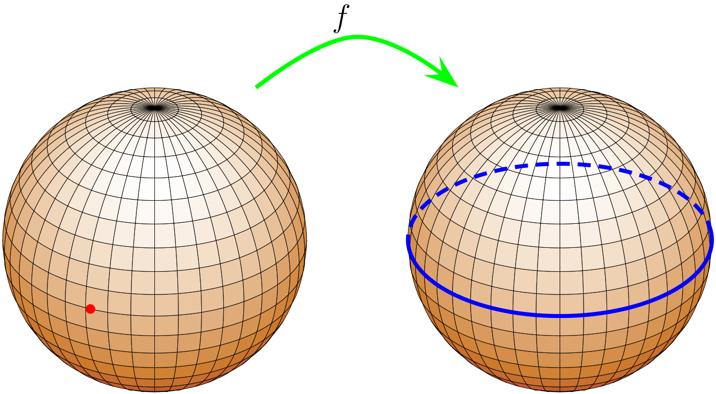
(3) $$\varphi(T,x) \equiv x[2\pi],$$

which is impossible as one easily using the continuity of φ.

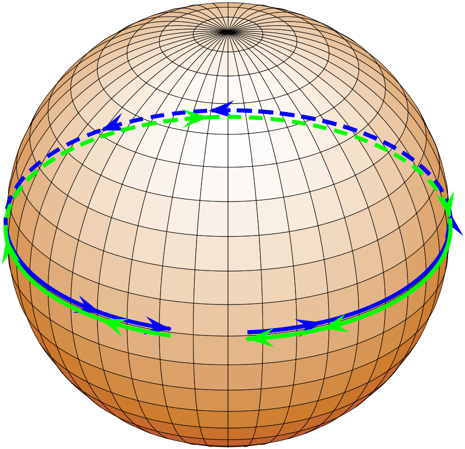
A target that one can reach for φ



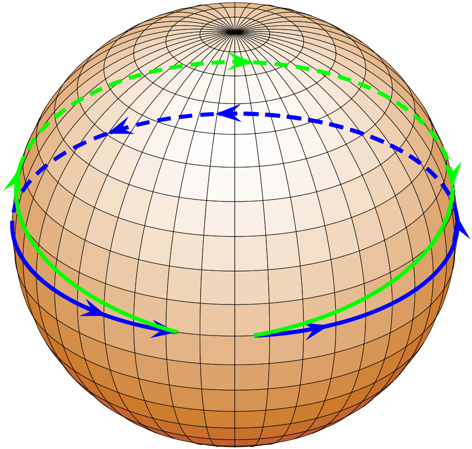
Using the extra dimension



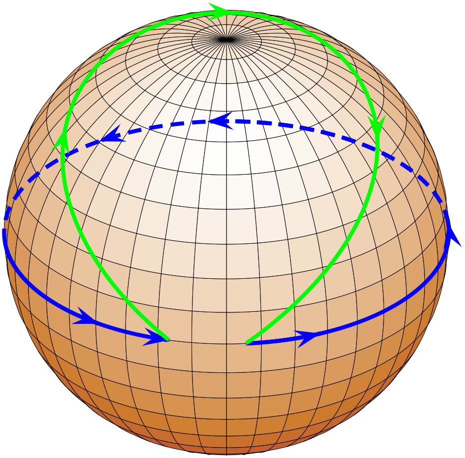
Using the extra dimension



Using the extra dimension



Using the extra dimension



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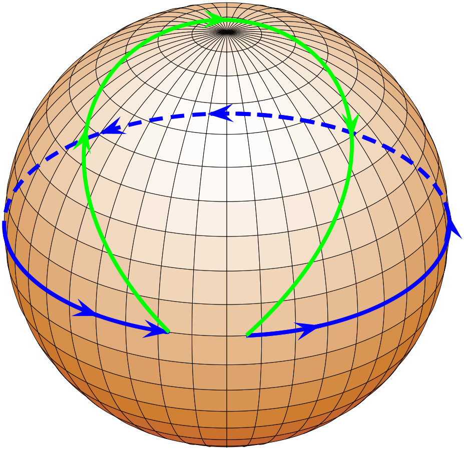
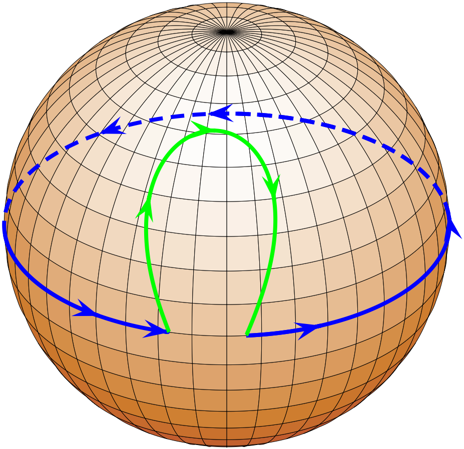
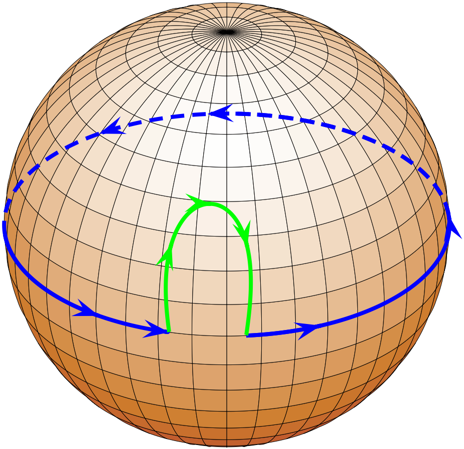


Using the extra dimension



Using the extra dimension



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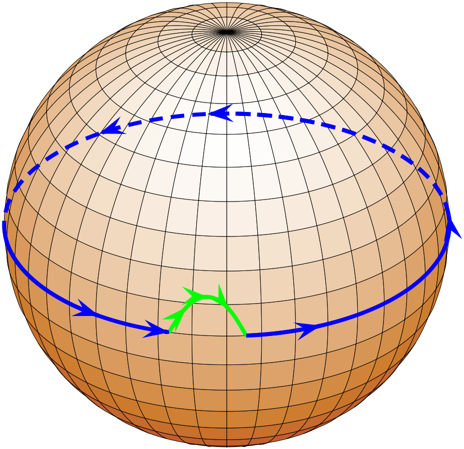
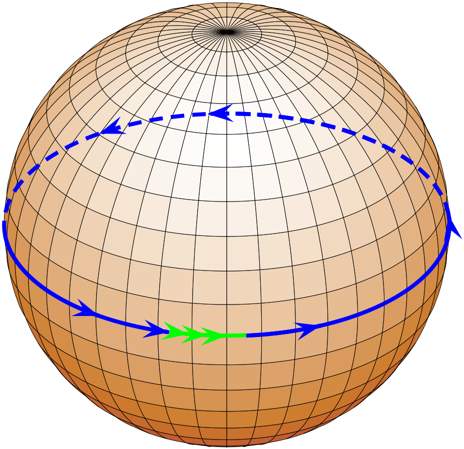
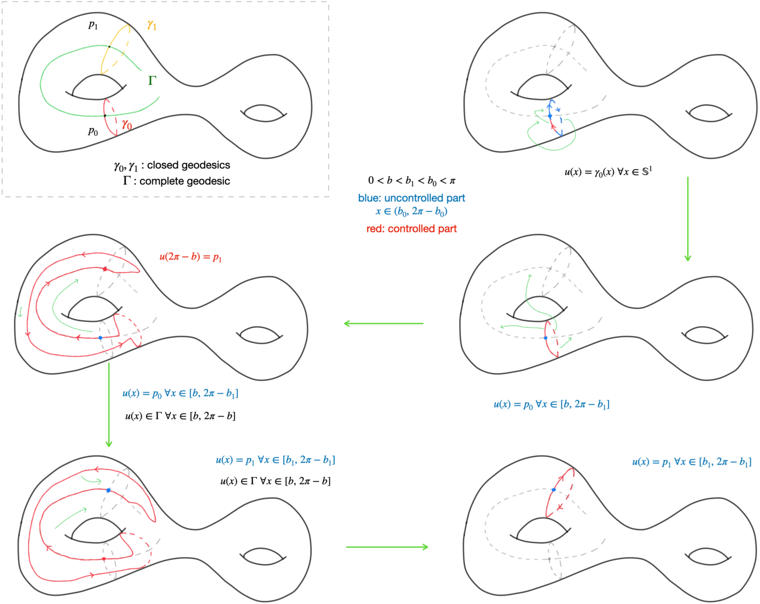


Using the extra dimension



Another case



Open problems

1. Suppose we want to go from a point close to the South Pole to a different point also close to the South Pole. If we want to go quickly from one point to the other, with our construction we need to pass close to the North Pole. Is is possible to propose another motion which avoids passing close the North Pole?
2. Global controllability to homotopic harmonic maps in small time?
3. What happens if one replaces $\mathbb{T}^1$ by $\mathbb{T}^n$ with $n \geq 2$. Note that already with $n = 2$ we can have blow-up in finite time without the control force. Can the force make it possible to avoid the blow-up?