Control theory and splitting methods

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Our goal is to highlight some of the deep links between numerical splitting methods and control theory. We consider evolution equations of the form $\dot{x} = f_0(x) + f_1(x)$, where f_0 encodes a non-reversible dynamic, so that one is interested in schemes only involving forward flows of f_0. In this context, a splitting method can be interpreted as a trajectory of the control-affine system $\dot{x}(t) = f_0(x(t)) + u(t)f_1(x(t))$, associated with a control u which is a finite sum of Dirac masses. The general goal is then to find a control such that the flow of $f_0 + u(t)f_1$ is as close as possible to the flow of $f_0 + f_1$.

Using this interpretation and classical tools from control theory, we revisit well-known results concerning numerical splitting methods, and we prove a handful of new ones, with an emphasis on splittings with additional positivity conditions on the coefficients. First, we show that there exist numerical schemes of any arbitrary order involving only forward flows of f_0 if one allows complex coefficients for the flows of f_1. Equivalently, for complex-valued controls, we prove that the Lie algebra rank condition is equivalent to the small-time local controllability of a system. Second, for real-valued coefficients, we show that the well-known order restrictions are linked with so-called "bad" Lie brackets from control theory, which are known to yield obstructions to small-time local controllability. We use our recent basis of the free Lie algebra to precisely identify the conditions under which high-order methods exist.

This is a joint work with Adrien Laurent and Frédéric Marbach.

Controlled flows of maps from the circle to compact Riemannian manifolds

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In this presentation, we explore recent developments concerning the evolution of maps from the circle into compact Riemannian manifolds, with a particular focus on their dynamics in relation to the Dirichlet energy. Specifically, we examine two fundamental flow equations: the wave maps equation and the heat maps equation. By introducing a localized control force into these flows, we investigate key questions of controllability and stabilization, shedding light on the extent to which these systems can be guided towards desired states. We shall emphasize the interplay between geometric analysis, partial differential equations, and control theory.

This talk is based on the following articles:

1. Joachim Krieger and Shengquan Xiang, Semi-global controllability of a semilinear wave equation, 2022.
2. JMC, Joachim Krieger, and Shengquan Xiang, Global controllability and stabilization of the wave maps equation from a circle to a sphere, 2023.
3. JMC and Shengquan Xiang, Global controllability to harmonic maps of the heat flow from a circle to a sphere, 2024.
4. JMC, Joachim Krieger, and Shengquan Xiang, Global controllability of the wave maps equation from a circle to a Riemannian manifold, 2025.

On the reachable space for parabolic equations

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In this article, we provide a description of the reachable space for the heat equation with various lower order terms, set in the euclidean ball of $\mathbb{R}^d$ centered at 0 and of radius one and controlled from the whole external boundary. Namely, we consider the case of linear heat equations with lower order terms of order 0 and 1, and the case of a semilinear heat equations. In the linear case, we prove that any function which can be extended as an holomorphic function in a set of the form $\Omega_\alpha = \{z \in \mathbb{C}^d,\, |\Re(z)| + \alpha|\Im(z)| < 1\}$ for some $\alpha \in (0,1)$ and which admits a continuous extension up to $\overline{\Omega}_\alpha$ belongs to the reachable space. In the semilinear case, we prove a similar result for sufficiently small data. Our proofs are based on well-posedness results for the heat equation in a suitable space of holomorphic functions over Ω_α for $\alpha > 1$.

New results on mixed operators

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We study some evolution problems driven by mixed operators of peridynamical type. First we need to study the eigenvalue and some linear and nonlinear elliptic problems. Then, we will show controllability and stability results, for which the obtained spectral properties will be used. As far as we know, these are the first results in this field.

A model of rigid vortex filament in Euler flows

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We consider the evolution of a rigid solid having the shape of a tube in a 3D perfect incompressible fluid and study the limit of the system as the tube shrinks to a curve, keeping a fixed circulation. Under certain assumptions, the system converges a model of a rigid vortex filament coupling the fluid and the position of the curve. This follows a series of papers in the 2D case where the dynamics of small solids in a fluid converges to point vortices models, which I will also describe. I will also talk about the possible connection with control theory.

Dynamic Stabilization of Two-String Systems with Dynamical Interior Mass: Unveiling the Role of Higher-Order Nodal Damping

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We investigate the uniform stabilization of two elastic strings in series, coupled with a dynamic mass at an interior node, under three damping schemes: classical boundary damping, lower-order nodal (tip-velocity) feedback, and a novel higher-order nodal (strain-velocity) feedback. It is shown that when higher-order nodal damping is paired with boundary damping the full system is unconditionally exponentially stable; by contrast, boundary damping alone, or boundary plus lower-order nodal feedback, admits at best the sharp t^{-1} decay first established by [Littman-Taylor'02] and found in the strong stabilization result of [Hansen-Zuazua'95]. Remarkably, even in the absence of any boundary dissipation, higher-order nodal feedback alone enforces exponential decay provided the wave-speed ratio satisfies an explicit arithmetic condition, whereas lower-order nodal feedback remains confined to the t^{-1} rate, refining and completing earlier partial results of [Chen-Coleman-West'87] and [Lee-You'89]. These findings are illustrated by Finite-Difference simulations of solution profiles, eigenvalue spectra, and energy-decay curves across varying damping configurations, speed ratios, and mesh resolutions, which confirm the decisive role of the arithmetic condition in distinguishing exponential, polynomial, or no decay.

Uniqueness and stability in bottom detection through surface measurements of water waves

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This work investigates the geometric inverse problem of recovering the shape of a bounded part of a solid bottom through surface measurements of water waves. Using the classical water wave equations in an unbounded $d-$dimensional domain, we address this inverse problem focusing on the identifiability and the stability estimates issues. Our analysis builds on some well-established results about elliptic problems in Lipschitz domains to derive an identifiability result. Moreover, we derive a stability upper bound using the size estimation method. Importantly, we do not assume that the region delimited by two distinct bottom profiles fulfills the so-called fatness condition. In particular, we establish the uniqueness and stability of the bottom profile in any bounded open box I from knowledge of the free surface, its velocity potential, and the first derivative of the free surface at an instant t_0, within I, along with the bottom at ∂I.

A new result on the cost of fast controls for the 1D Schrödinger equation

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In this presentation, we will give a new upper bound on the fast control cost for the 1D Schrödinger equation controlled at the boundary, which significantly improves on the current literature on the subject. After contextualizing the problem and presenting the main result, I will digress at length on Beurling and Malliavin's multiplier theorem (BM1), as well as on a recent and very illuminating proof based on the Hilbert transform. I will explain how this proof allows us to obtain, in certain special cases, a completely quantitative version of BM1. In particular, we will provide evidence in the case of a very special weight, which is of interest for the problem under consideration. Finally, we will return to the control problem: the moments method will allow us to reduce the question to the following classical question of complex analysis. We have an entire function with imposed zeros, which grows very strongly on the real axis, and we want to multiply it by a weight that makes it L^2 on the real axis, and such that the Fourier transform of the product has as small a support as desired. We can then make the connection with BM1 and conclude.

Controllability of the 1D wave equation

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The wave equation with variable coefficients models the propagation of a physical quantity (such as wave, temperature, distortion, concentration, etc.) in a heterogeneous environment. The coefficients take into account conductivity, dissipation, density, reactions, velocity, and so on. In this talk, we discuss the controllability of such an equation. To do so, we used the backstepping method on the first-order system associated with this wave equation. Classically, we then obtain a finite-time stabilisable control and solution of our system. Moreover, this method gives us a direct method for the controllability (not based on the observability). As we have a direct method, we obtain an effective method to approximate the control and solution of the wave equation. Some illustrations will be presented to illustrate the numerical scheme for the stabilisation of the equation.

Approximation of the boundary controls for the 2D wave equation in a rectangle

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In this talk we extend previous results concerning the uniform boundary controllability properties of the finite-difference space semi-discretization of the 1-D wave equation to the case of a 2-D wave equation in a rectangular domain. It is already known that the constants on the discrete boundary observability inequality blow-up as the mesh-size tends to zero. Using controls acting on two adjacent edges of the rectangle we prove that, by filtering out the high frequencies of the initial data, the uniform controllability property can be restored. Our proof, based on Fourier expansion, moment problems and explicit construction of special biorthogonal sequences, allows us to estimate the control needed for each individual eigenfrequency.

Constructive proofs of controllability for semi-linear heat and wave equations

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I review some results obtained in the last years with mainly my colleague Jérome Lemoine, but also with Arthur Bottois, Kuntal Bhandari, Sue Claret, Sylvain Ervedoza, Irène Marin Gayte and Emmanuel Trélat, concerning the exact controllability of semilinear heat/wave equations. Uniform controllabilty (w.r.t. the initial condition) results are usually based on non constructive fixed point arguments assuming that the nonlinear function is smooth and does not grow too fast at infinity. Within analogous assumptions, i discuss briefly two methods yielding to strongly convergent approximations of the nonlinear controllability problem. The first one is based on a simple point fixed argument for which the nonlinear part is seen as a right hand side of the equation. The second one is based on a more elaborated least-squares type argument. Numerical illustrations will be given.

Null-controllability of the heat equation from very small sets

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Let Ω be a bounded regular domain in $\mathbb{R}^n$. If $\omega \subset \Omega$ is an open set or a set of positive Lebesgue measure, it is today well-known that the heat equation is null-controllable from ω in any time. On the first hand, we will see that this result still holds when ω is any measurable set with Hausdorff dimension strictly greater than $n-1$. On the other hand, even if this result is sharp with respect to the scale of Hausdorff dimension, we will see how to construct observable sets with codimension greater than 1 and how null-controllability is related to nodal sets of Laplace eigenfunctions. In particular, in the one-dimensional case, we give a characterization of observable subsets generalizing Dolecki's result.

Optimal stabilization rate for the wave equation with hyperbolic boundary condition

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We consider linear waves on a bounded domain where one part of the boundary is governed by a coupled lower-dimensional wave equation (i.e., dynamic Ventcel/Wentzell boundary condition) and is subject to viscous damping. The other (possibly empty) part is left at rest. When the dynamic boundary geometrically controls the domain, we show that the total energy of classical solutions decays like 1/t. The proof relies on an analysis of high-frequency quasimodes, suitable boundary estimates obtained in different microlocal regimes, and a special decoupling argument. Optimality is assessed via an appropriate quasimode construction.

Ongoing work with Nicolas Vanspranghe (Tampere University).

Control and Hyperbolic Conservation Laws

Vincent Perrollaz, joint works with Rinaldo M. Colombo, Abraham Sylla and Thibault Liard

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We first recall the origins of hyperbolic conservation laws and provide some classical examples. We then introduce the framework of entropy solution which allows to deal with shock waves. In this functional setup the time evolution becomes irreversible.

In a second part, we are concerned with the problem of asymptotic stabilization by feedback control. We work in the scalar one dimensional case and show how to stabilize stationary shock waves in a first step. We then describe how to adapt those techniques to regulate traffic flow by controlling the flux at toll gates.

Finally, we showcase some results and techniques concerning the reachability problem and the inverse design problem for scalar unidimensional conservation laws. This is of course deeply related to the time irreversibility of the evolution.

Energy decay for Korteweg-de Vries-Burgers equations with delay feedback

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We consider a generalized KdV-Burgers equation with indefinite damping and time delay on the real line. Using semigroup arguments and suitable Lyapunov functionals, we establish the existence of a global solution when the exponent of the nonlinear term satisfies some growth conditions. Furthermore, we prove exponential stability estimates under suitable assumptions: first in the case of a positive damping coefficient, then within a more comprehensive framework, accommodating sign changes in the damping and delay feedback. In the two cases, we adopt refined conditions on the delay feedback coefficient, extending and enhancing existing results in the literature. In particular, our conditions are independent of the time delay size.

New Insights on the Stabilization of Generalized Serially Connected Piezoelectric and Elastic beams

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This study investigates the stability of a transmission problem involving alternating magnetizable piezoelectric and elastic beams subjected to various partial damping scenarios, including internal and electrical boundary damping. The system consists of $N+1$ piezoelectric components and N elastic components arranged in a serial configuration where N is a natural number and $N \geq 1$.

Decay properties of the Maxwell system with conductivity

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We discuss linear and quasilinear Maxwell systems with damping caused by a nonnegative conductivity σ. For the scalar wave equation it is well known that the location of the support of σ often determines the resulting decay behavior. The Maxwell case is far less studied and poses additional difficulties. For instance, the charges (or divergence conditions) play a crucial role as they have to counteract the large kernel of the curl operator.

We present several recent results. In the linear autonomous case one obtains strong stability for nontrivial damping in a rather general situation using the spectral criteria by Arendt–Batty–Lubich–Vũ. Morover, we show polynomial decay if σ is strictly positive on a strip of a cube, assuming that the permittivity ε and permeability μ are constant. This fact follows from a resolvent estimate which is established by means of the eigenfunctions of the undamped Maxwell problem.

We further look at the Maxwell system with matrix-valued ε and μ. Here we prove exponential stability if σ is strictly positive on its support that is a neighborhood of the connected boundary of a simply connected domain, assuming a nontrapping condition on ε and μ. Our proof is based on a Morawetz estimate (leading to an observability inequality) and a sophisticated Helmholtz-type decomposition. For small data the result can be extended to field-dependent, but isotropic ε and μ. To deal with such a quasilinear problem, we study the time-derived equations which contain several error terms forcing us to modify the arguments from the linear case significantly. For instance, the jump of σ now causes severe difficulties in form of interface terms which have to be controlled in subtle arguments.

The first part is joint work with Serge Nicaise (Valenciennes) and the second one with Richard Nutt (Karlsruhe).

Relaxation enhancement by controlling incompressible fluid flows

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We propose a PDE-controllability based approach to the enhancement of diffusive mixing for passive scalar fields. Unlike in the existing literature, our relaxation enhancing fields are not prescribed *ab initio* at every time and at every point of the spatial domain. Instead, we prove that time-dependent relaxation enhancing vector fields can be obtained as *state trajectories of control systems described by the incompressible Euler equations* either driven by finite-dimensional controls or by controls localized in space. The main ingredient of our proof is a new approximate controllability theorem for the incompressible Euler equations on the 2D flat torus, ensuring the approximate tracking of the full state all over the considered time interval. Combining this with a continuous dependence result yields enhanced relaxation for the passive scalar field. Another essential tool in our analysis is the exact controllability of the incompressible Euler system driven by spatially localized forces.

Determination of discontinuous diffusion coefficients for the heat equation on a tree-shaped network

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In this talk, we study the heat equation on a tree-shaped network with a piecewise regular diffusion coefficient. By developing new Carleman estimates, we establish stability results for the identification of the diffusion coefficient. These stability estimates are derived using either internal measurements or boundary observations, offering robust insights into the inverse problem for this class of equations.

High order Hardy type inequalities for strongly degenerate elliptic operators and applications

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In this talk , we adress the questions of unique continuation and approximate controllabity of a class of *strongly* degenerate parabolic equations with a *control acting at the degeneracy point*. The proofs are in particular based on the derivation of high order Hardy Type inequalities. Those inequalities are then used to obtain optimal imbeddings properties for the domain of the powers of the corresponding degenerate elliptic operator. This is a key point to show that the question of unique continuation makes senses here.