II) CONTINUOUS TIME

1) Construction of the Branching Markov Process (BMP)

- E : measurable space ("state space"), ∂ : cemetery symbol
- Markov process $X^0 = (X_t^0)_{t\geq 0}$ on E, possibly finite life time τ (at which it jumps to ∂).
  Remark: No topological regularity needed, only measurability
- $r \in p\mathcal{B}_b(E)$ ("branching rate")
- $(p_n)_{n\geq 1}$: "offspring distribution": $\forall n \geq 1$: $p_n \in pB_1(E)$, $\sum_{n\geq 1} p_n = 1$.
  Assume: $\lim_{n_0\to\infty} \sup_x \sum_{n=n_0}^{\infty} n p_n(x) \longrightarrow 0$ (uniform integrability)

Set: $m(x) := \sum_{n\geq 1} n p_n(x)$, $mf(x) = m(x) f(x)$, $f \in pB(E)$

$\Pi f(x) = \sum_{n\geq 1} p_n(x) f(x)^n$, $f \in pB_1(E)$

$P_t^0 f(x) := \mathbb{E}_x[f(X_t^0)]$, $f \in pB(E)$, where we extend f to $\bar{E}$ by $f(\partial) = 0$.

$(P_t^0)_{t\geq 0}$ is a semigroup on $pB(E)$: $P_t^0 P_s^0 f = P_{t+s}^0 f$.

Dynkin 1991 constructs a BMP $X = (X_t^u)_{t\geq 0, u\in\mathcal{U}}$, which satisfies, for a given starting point $x \in E$:

- $X^\emptyset$ follows the law of X^0 until a random time $d^\emptyset$ distributed as: $\mathbb{P}(d^\emptyset > t \mid X^\emptyset) = \exp\left(-\int_0^t r(X_s^\emptyset)\,ds\right)$
- $L^\emptyset$ is a r.v. s.t. $\mathbb{P}(L^\emptyset = n \mid X^\emptyset, d^\emptyset) = p_n(X^\emptyset_{d^\emptyset})$.
- Conditioned on $X^\emptyset, d^\emptyset, L^\emptyset$, we repeat this process starting with $L^\emptyset$ independent particles labeled $1, \dots, L_\emptyset$, at point $X^\emptyset_{d^\emptyset}$ at time $d^\emptyset$.

We set $\mathcal{N}_t \subset \mathcal{U}$ the set of particles alive at time t, $\underline{N_t := \#\mathcal{N}_t}$.

The BMP can be viewed as a Markov process

$$\Big(\sum_{u\in\mathcal{N}_t} \delta_{X_t^u}\Big)_{t\geq 0}$$ on the space of counting measures on E.

Branching property: $\forall t\geq 0$, conditioned on $\mathcal{F}_t = \sigma((X_s^u)_{s\leq t}, u\in\mathcal{U})$, the shifted processes $(X_s^{(u,t)})_{s\geq 0}$, $u\in\mathcal{N}_t$, defined by $X_s^{v(u,t)} = X_{t+s}^{uv}$, $v\in\mathcal{U}$ are independent and distributed as X under $\mathbb{P}_{X_t^u}$, respectively.

We define the semigroups:

$$P_t f(x) = \mathbb{E}_x\Big[\sum_{u\in\mathcal{N}_t} f(X_t^u)\Big], \quad f\in pB(E)$$

$$S_t f(x) = \mathbb{E}_x\Big[\prod_{u\in\mathcal{N}_t} f(X_t^u)\Big], \quad f\in pB_1(E)$$

We have:

$$P_t f(x) = P_t^0 f + \int_0^t P_{t-s}^0\, r(m-1) P_s f\, ds, \quad f\in pB(E)$$

$$S_t f(x) = P_t^0 f^{(1)} + \int_0^t P_{t-s}^0\, r(\pi-1) S_s f\, ds, \quad f\in pB_1(E),$$

where $f^{(1)}(x) = \begin{cases} f(x), & x\in E \\ 1, & x=\partial \end{cases}$

Exercise: P_t admits the Feynman-Kac representation

$$P_t f(x) = \mathbb{E}_x\Big[f(X_t^0)\exp\Big(\int_0^t r(X_s^0)(m(X_s^0)-1)\,ds\Big)\Big]$$

Def: $glSurv = \{N_t \not\to 0\}$, $glExt = \{N_t \to 0\}$.

2) Generalized principal eigenvalues

Fix $x \in E$.

$$\lambda_{c,x} = \lambda_{c,x}((P_t)_{t\geq 0}) = \inf\{\lambda \in \mathbb{R} : \exists u \in pB(E) : P_t u \leq e^{\lambda t} u \ \forall t \geq 0, u > 0, u(x) < \infty\}$$

$$\lambda'_{c,x} = \lambda'_{c,x}((P_t)_{t\geq 0}) = \sup\{\lambda \in \mathbb{R} : \exists u \in pB_1(E) : P_t u \geq e^{\lambda t} u \ \forall t \geq 0, u(x) > 0\}$$

$$\lambda''_{c,x} = \lambda''_{c,x}((P_t)_{t\geq 0}) = \inf\{\lambda \in \mathbb{R} : \exists u \in pB(E) : P_t u \leq e^{\lambda t} u \ \forall t \geq 0, u \geq 1, u(x) < \infty\}$$

<u>Theorem:</u>

a) $u(x) := \mathbb{P}_x(gl\,Ext)$ is the smallest solution to $S_t f = f \ \forall t \geq 0$, $f \in pB_1(E)$

b) Let $x \in E$. If $\lambda'_{c,x} < 0$, then $\mathbb{P}_x(gl\,Surv) = 0$

If $\lambda'_{c,x} > 0$, then $\mathbb{P}_x(gl\,Surv) > 0$

$\forall x \in E$;

c) $\lambda'_{c,x} \leq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x[N_t] \leq \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x[N_t] = \lambda''_{c,x}$

d) $\forall t > 0$: $\lambda'_{c,x} = \frac{1}{t} \log \rho'_{c,x}(P_t)$, $\lambda''_{c,x} = \frac{1}{t} \log \rho''_{c,x}(P_t)$.

<u>Remark:</u> No assumption of bounded offspring distribution here. Instead, the crucial assumption is the <u>local branching</u>.

<u>Proof</u> (only b)): The heart of the proof is to reduce to a discrete-time BMC with <u>bounded</u> number of offspring, then apply discrete-time results.

<u>Claim:</u> For every $\varepsilon > 0$, there exists a (discrete-time) BMC $Y = (Y_u)_{u\in\mathcal{U}}$ coupled to $(X^u_t)_{t\geq 0}$, such that:

1) For every $x \in E$, $\mathbb{P}_x\left(\sum_{|u|=n} \delta_{Y_u} \leq \sum_{u \in N_t} \delta_{X^u_n} \ \forall n \geq 0\right) = 1$

2) $m_Y \geq (1-\varepsilon) P_1$.

Proof of claim:

Step 1: Reduce to bounded offspring distribution:

By uniform integrability, there exists n_0, s.t.

$$\sup_x \sum_{n \geq n_0} n\, p_n(x) \leq \frac{\varepsilon}{2\|r\|_\infty}.$$

Set $\tilde{p}_n = \begin{cases} p_n, & n < n_0 \\ \sum_{n=n_0}^{\infty} p_n, & n = n_0 \end{cases}$, so that $\tilde{m}(x) \geq m(x) - \frac{\varepsilon}{2\|r\|_\infty}$

Now construct $\tilde{X}$ from X by removing children of label $n > n_0$. We have $\forall f \in pB(E)$:

$$\tilde{P}_1 f(x) = \mathbb{E}_x\left[\exp\left(\int_0^1 r(X_s^\circ)(\tilde{m}(X_s^\circ)-1)\,ds\right) f(X_1^\circ)\right]$$

$$\geq e^{-\frac{\varepsilon}{2}}\, \mathbb{E}_x\left[\exp\left(\int_0^1 r(X_s^\circ)(m(X_s^\circ)-1)\,ds\right) f(X_1^\circ)\right]$$

$$= e^{-\frac{\varepsilon}{2}} P_1 f(x) \geq \left(1-\tfrac{\varepsilon}{2}\right) P_1 f(x).$$

Step 2: Assume now offspring distribution is bounded by n_0.

We first construct X as follows:

- Consider Z a BMP with spatial motion X°, branching at rate $\|r\|_\infty$ into n_0 offspring
- Construct X from Z as follows: At every branchpoint:
  - with prob. $\frac{r}{\|r\|_\infty}$, sample L according to offspring distribution $(p_n(x))_{n \leq n_0}$, keep only L first branches
  - otherwise: keep only 1^{st} branch.

The resulting process has the same law as X.

Now let n_1, s.t. $\mathbb{E}\left[N_1^Z \mathbb{1}_{(N_1^Z > n_1)}\right] \le \frac{\varepsilon}{2}$.

Defin Y a BMC, s.t. the first generation satisfies:

$$\sum_{|u|=1} \delta_{Y_u} = \begin{cases} \sum_{u \in N_1} \delta_{X_1^u}, & \text{if } N_1^Z \le n_1 \\ 0, & \text{if } N_1^Z > n_1 \end{cases}$$

This BMC obviously satisfies (C) and bounds X from below.

Furthermore, we have

$$m_Y f(x) = P_1 f(x) - E_x\left[\mathbb{1}_{(N_1^Z > n_1)} \sum_{u \in N_t} f(X_1^u)\right], \text{ and}$$

$$\mathbb{E}_x\left[\mathbb{1}_{(N_1^Z > n_1)} \sum_{u \in N_t} f(X_1^u)\right] \le \mathbb{E}_x\left[\mathbb{1}_{(N_1^Z > n_1)} \sum_{u \in N_t} f(Z_1^u)\right]$$

$= \mathbb{E}_x\left[\mathbb{1}_{(N_1^Z > n_1)} \cdot N_1^Z\right] \cdot P_1^0 f(x)$ (since spatial motion and branching are independent for Z.)

$\le \frac{\varepsilon}{2} P_1 f(x)$ (since $P_1^0 \le P_1$)

$\Rightarrow m_Y \ge (1 - \frac{\varepsilon}{2}) P_1$ ∎

Proof of Thm (part b)) from claim:

The first implication follows from part a), which is proven as in the discrete-time case.

For the second implication, by the claim, for every $\varepsilon > 0$, we can define a discrete-time BMC Y that bounds X from below at integer times and such that $m_Y \ge (1-\varepsilon) P_1$. Choose ε small enough, so that $(1-\varepsilon) e^{\lambda'_{c,x}} > 1$. It follows that $\rho'_{c,x}(m_Y) > 1$, and we can apply the discrete-time theorem, so that $\mathbb{P}_x(\text{gl Surv}) > 0$. ∎

3) Irreducibility

Assumption (I):

1. E is a topological space
2. X^0 is strong Feller: $\forall t>0$: $P_t^0(B_b(E)) \subset C_b(E)$
3. Topological irreducibility: $\forall x \in E$, $U \subset E$ open, $\exists t \geq 0$: $P_t^0 \mathbb{1}_U(x) > 0$

Theorem: Assume (I). Then $\forall x, y \in E$:

$$\mathbb{P}_x(\mathrm{glSurv}) > 0 \iff \mathbb{P}_y(\mathrm{glSurv}) > 0$$

Proof: Exercise

III) BRANCHING DIFFUSIONS AND FKPP EQUATION

$E \subset \mathbb{R}^d$, $\neq \emptyset$, open, connected, smooth boundary

Recall: $Lu = \frac{1}{2} \sum_{ij} a_{ij}(x) \partial_{ij} u + \sum_i b_i(x) \partial_i u + c(x) u$,

a, b, c bounded, locally Hölder, a symmetric, unif. pos. def.

$$L^0 u = Lu - c_+ u = \frac{1}{2} \sum_{ij} a_{ij}(x) \partial_{ij} u + \sum_i b_i(x) \partial_i u - c_-(x) u,$$

where $c_+ = \max(c, 0)$, $c_- = \max(-c, 0)$.

L^0 is the generator (in the Stroock-Varadhan sense) of a continuous, strong (sub-) Markov process X^0: for every smooth $\varphi : E \to \mathbb{R}$, for every $x \in E$: with $X_0^0 = x$,

$$\varphi(X_t^0) - \int_0^t L^0 \varphi(X_s^0)\, ds \quad \text{is a local martingale.}$$

We furthermore kill X^0 upon exiting E.

We set $\quad P_t^0 f(x) = \mathbb{E}_x [f(X_t^0)]$, $f \in p\mathcal{B}(E)$.

It is known (see e.g. Krylov '21), that X^0 satisfies Assumption (I).

We consider the BMP from Section , with X^0 like here, $r \equiv c_+$, $p_n \equiv \delta_{n=2}$ (binary branching). Note that

$$P_t f(x) = \mathbb{E}_x \left[f(X_t^0) \exp\left(\int_0^t c_+(X_s^0)\, ds \right) \right]$$

Berestycki - Rossi '18 define generalized principal e.v.'s as follows (modulo a minus sign):

$$\lambda_1 = \lambda_1(L) = \inf\{\lambda\in\mathbb{R} : \exists u \in W^{2,d}_{loc}(E) : (L-\lambda)u \le 0,\ u>0\}$$

$$\lambda_1' = \lambda_1'(L) = \sup\{\lambda\in\mathbb{R} : \exists u\in W^{2,d}_{loc}(E)\cap L^\infty(E) : (L-\lambda)u \ge 0,\ u>0,\ u(x)\to 0,\ x\to\partial E\}$$

$$\lambda_1'' = \lambda_1''(L) = \inf\{\lambda\in\mathbb{R} : \exists u\in W^{2,d}_{loc}(E) : (L-\lambda)u\le 0,\ u\ge 1\}$$

Theorem: $\lambda_1'(L) = \lambda_c'((P_t)_{t\ge 0})$, $\lambda_1''(L) = \lambda_c''((P_t)_{t\ge0})$

Proof: Omitted. We remark that is uses Itô's lemma from Krylov's book (1980), which works (precisely) for $W^{2,d}_{loc}$ functions. ∎

Local survival: We set

$$\mathrm{locSurv} = \bigcap_{U \text{ open}, \neq\emptyset, \text{connected}} \mathrm{locSurv}_U, \quad \mathrm{locSurv}_U := \{\sup\{t : \sum_{u\in N_t} \mathbb{1}(X_t^u\in U) > 0\} = \infty\}$$

FKPP equation: We consider the PDE

$$(*)\quad Lu - c_+u^2 = 0, \qquad u\equiv 0 \text{ on } \partial E$$

This is the equation satisfied by stationary solutions of an FKPP-type reaction-diffusion equation.

Theorem: Set $u(x) = \mathbb{P}_x(\mathrm{glSurv})$, $v(x) = \mathbb{P}_x(\mathrm{locSurv})$.

Then: • u is the largest (classical) solution to $(*)$
• if $v>0$, then v is the smallest non-trivial (classical) sol. to $(*)$

Corollary: Suppose $\mathbb{P}_x(\mathrm{glSurv}\setminus\mathrm{locSurv}) \equiv 0$. Then there exists at most one non-trivial solution to $(*)$.

Proof of Theorem (sketch):

The fact that u and v are (classical) solutions to (∗) follows from the integral equation for $S_t f$, but one needs to prove the required regularity, see aside.

The fact that u is the largest solution follows as in the corresponding discrete-time result.

To prove the statement for v, let $\bar{u}$ be another non-trivial solution, and define $M_t := \prod_{u \in N_t} (1-\bar{u})(X_t^u)$.

Then $(M_t)_{t \geq 0}$ is a $[0,1]$-valued martingale

$\Rightarrow M_t \xrightarrow{t \to \infty} M_\infty$, a.s. and in L^1.

We show that $M_\infty = 0$ on locSurv: Fix a ball $B(x_0, \delta)$, compactly contained in E, s.t. $\inf_{B(x_0,\delta)} \bar{u} > 0$. On locSurv, $B(x_0,\delta)$ is visited ∞ many times, at each one we have a positive probability of having many descendants in $B(x_0,\delta)$ shortly after. $\Rightarrow M_\infty = 0$.

We get $v(x) \leq 1 - E_x[M_\infty] = 1-(1-\bar{u}(x)) = \bar{u}(x)$ ▯

Two applications of the theorem (pour $L = \frac{1}{2}\Delta + c$):

Theorem: Assume $E = \mathbb{R}^d$, $c \not\equiv 0$, suppose $L = \frac{1}{2}\Delta + c$, c compactly supported. Then, there exists at most one (non-trivial) solution to (∗) if $d = 1,2$, but more than one if $d \geq 3$.

Proof (sketch): If $d = 1,2$, by recurrence of BM, on glSurv, we must also have locSurv. $\Rightarrow P_x(\text{glSurv} \setminus \text{locSurv}) = 0$.

If $d \geq 3$, with pos. prob., there exists a unique particle going to ∞ and surviving forever $\Rightarrow P(\text{glSurv} \setminus \text{locSurv}) > 0$. ▯

Theorem: Suppose $\limsup_{r\to\infty} \lambda_1'\left(\frac{1}{2}\Delta+c, E\setminus\overline{B(0,r)}\right)<0$.
Then there exists at most one (non-trivial) solution to $(*)$
$\left(L=\frac{1}{2}\Delta+c\right)$

Proof (sketch) Technical lemma: Define r-locSurv :=
$$\text{glSurv} \cap \left\{\liminf_{t\to\infty} \inf_{u\in N_t} (X_t^u) < \infty\right\}.$$
Then $\mathbb{P}(\text{r-locSurv} \setminus \text{locSurv}) = 0$.

With this lemma, it is enough to show that
$$\mathbb{P}_x(\text{glSurv} \setminus \text{r-locSurv}) = 0.$$
But for all R, for all $x\in E\setminus\overline{B(0,R)}$, we have
$$\mathbb{P}_x(\text{glSurv} \setminus \text{r-locSurv}) \leq \mathbb{P}_x\left(\text{glSurv}_{E\setminus\overline{B(0,R)}}\right),$$
which is $=0$ for R large enough, so that $\lambda_1'\left(\frac{1}{2}\Delta+c, E\setminus\overline{B(0,R)}\right)<0$. ∎

CONCLUSION:

1) Topics we haven't touched; see article:
   - bounded irregular domains
   - maximum principle
2) Research venues:
   - processes with jumps ($\leadsto$ integral operators)
   - survival in subsets, Martin boundary, ...
   - Homogenization, random environment, ...