Stochastic models for coagulation processes, large scale limits and phase transitions

Luisa Andreis
Università degli Studi di Torino

Summer School EUR MINT 2026
Particle systems and PDEs

Toulouse
June 15-19, 2026

And what about Smoluchowski?

Self-organised criticality

Multiplicative Smoluchowski equation

$$\partial_t n_k(t) = \frac{1}{2}\sum_{h=1}^{k-1} h(k-h)\, n_h(t) n_{k-h}(t) - k n_k(t) \sum_{h\geq 1} h n_h(t).$$

And what about Smoluchowski?

Self-organised criticality

Multiplicative Smoluchowski equation

$$\partial_t n_k(t) = \frac{1}{2}\sum_{h=1}^{k-1} h(k-h)\, n_h(t) n_{k-h}(t) - k n_k(t) \sum_{h\geq 1} h n_h(t).$$

Forest-fire models [RÁTH, TÓTH, 2009]

- graph grows by edge addition;
- large clusters burn after lightning events;
- the burning mechanism removes mass;
- the system self-organises near criticality.

And what about Smoluchowski?

Self-organised criticality

Multiplicative Smoluchowski equation

$$\partial_t n_k(t) = \frac{1}{2}\sum_{h=1}^{k-1} h(k-h)\, n_h(t) n_{k-h}(t) - k n_k(t) \sum_{h\geq 1} h n_h(t).$$

Forest-fire models [RÁTH, TÓTH, 2009]

- graph grows by edge addition;
- large clusters burn after lightning events;
- the burning mechanism removes mass;
- the system self-organises near criticality.

Limited aggregation / SOC models [MERLE, NORMAND, 2015]

- coagulation is allowed only below a size threshold;
- oversized clusters are removed or made inactive;
- the system self-organises near criticality.

And what about Smoluchowski?

Self-organised criticality

Multiplicative Smoluchowski equation

$$\partial_t n_k(t) = \frac{1}{2} \sum_{h=1}^{k-1} h(k-h)\, n_h(t) n_{k-h}(t) - k n_k(t) \sum_{h \geq 1} h n_h(t).$$

Forest-fire models [Ráth, Tóth, 2009]

- graph grows by edge addition;
- large clusters burn after lightning events;
- the burning mechanism removes mass;
- the system self-organises near criticality.

Limited aggregation / SOC models [Merle, Normand, 2015]

- coagulation is allowed only below a size threshold;
- oversized clusters are removed or made inactive;
- the system self-organises near criticality.

The explicit solution of the multiplicative Smoluchowski equation stays critical for all $t > T_{\text{GEL}}$:

$$\sum_k k n_k(t) = \frac{1}{t} \wedge 1$$

And what about Smoluchowski?

Self-organised criticality

Multiplicative Smoluchowski equation

$$\partial_t n_k(t) = \frac{1}{2} \sum_{h=1}^{k-1} h(k-h)\, n_h(t) n_{k-h}(t) - k n_k(t) \sum_{h \geq 1} h n_h(t).$$

Forest-fire models [Ráth, Tóth, 2009]

- graph grows by edge addition;
- large clusters burn after lightning events;
- the burning mechanism removes mass;
- the system self-organises near criticality.

Limited aggregation / SOC models [Merle, Normand, 2015]

- coagulation is allowed only below a size threshold;
- oversized clusters are removed or made inactive;
- the system self-organises near criticality.

The explicit solution of the multiplicative Smoluchowski equation stays critical for all $t > T_{\mathrm{GEL}}$:

$$\sum_k k n_k(t) = \frac{1}{t} \wedge 1$$

Difference with Flory:

$$\sum_k k n_k(t) = 1 - \rho(t) < \frac{1}{t}.$$

Adding inhomogeneities

Sparse inhomogeneous random graphs

Adding inhomogeneities

Sparse inhomogeneous random graphs

Inhomogeneous random graphs are a natural generalization of the Erdős-Rényi random graph ([SÖDERBERG 2002], [BOLLOBÁS, JANSON AND RIORDAN 2006]).

- $\mathcal{S}$ a metric space: the **type space**;
- $\mu \in \mathcal{M}(\mathcal{S})$ a probability on $\mathcal{S}$;
- $\mathbf{x}^N = (x_1, \dots, x_N) \in \mathcal{S}^N$ vector of vertices' type

$$\mu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \to \mu;$$

- a symmetric kernel $\kappa : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$.

Adding inhomogeneities

Sparse inhomogeneous random graphs

Inhomogeneous random graphs are a natural generalization of the Erdős-Rényi random graph ([SÖDERBERG 2002], [BOLLOBÁS, JANSON AND RIORDAN 2006]).

- $\mathcal{S}$ a metric space: the **type space**;
- $\mu \in \mathcal{M}(\mathcal{S})$ a probability on $\mathcal{S}$;
- $\mathbf{x}^N = (x_1, \dots, x_N) \in \mathcal{S}^N$ vector of vertices' type

$$\mu_N : = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \to \mu;$$

- a symmetric kernel $\kappa : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$.

The **sparse inhomogeneous random graph** $\mathcal{G}(N, \mathbf{x}^N, \kappa)$ is such that there is an edge between vertices i and j with probability that depends on their types:

$$i \sim j \text{ with probability } \frac{\kappa(x_i, x_j)}{N} \wedge 1.$$

Adding inhomogeneities

Sparse inhomogeneous random graphs

Inhomogeneous random graphs are a natural generalization of the Erdős-Rényi random graph ([SÖDERBERG 2002], [BOLLOBÁS, JANSON AND RIORDAN 2006]).

- $\mathcal{S}$ a metric space: the **type space**;
- $\mu \in \mathcal{M}(\mathcal{S})$ a probability on $\mathcal{S}$;
- $\mathbf{x}^N = (x_1, \dots, x_N) \in \mathcal{S}^N$ vector of vertices' type

$$\mu_N \colon = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \to \mu;$$

- a symmetric kernel $\kappa : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$.

The **sparse inhomogeneous random graph** $\mathcal{G}(N, \mathbf{x}^N, \kappa)$ is such that there is an edge between vertices i and j with probability that depends on their types:

$$i \sim j \text{ with probability } \frac{\kappa(x_i, x_j)}{N} \wedge 1.$$

Sparse since the expected number of edges is proportional to N:

$$\mathbf{E}\#\{\text{ edges }\} \simeq N \int_{\mathcal{S}\times\mathcal{S}} \kappa(x, y)\mu(\mathrm{d}x)\mu(\mathrm{d}y).$$

Phase transition in inhomogeneous random graphs

Let $\mathcal{G}_N = \mathcal{G}(N, \mathbf{x}^N, \kappa)$, then the phase transition is in terms of the operator

$$T_{\kappa,\mu}\colon L^2(\mu) \to L^2(\mu), \qquad T_{\kappa,\mu} f(x) = \int_{\mathcal{S}} f(y)\, \kappa(x,y)\mu(\mathrm{d}y), \tag{0.1}$$

and its norm

$$\sigma(\kappa,\mu) = \|T_{\kappa,\mu}\|_{L^2(\mu)} = \sup_{f\in L^2(\mu)\colon \|f\|_{L^2(\mu)}=1} \|T_{\kappa,\mu} f\|_{L^2(\mu)}. \tag{0.2}$$

Phase transition in inhomogeneous random graphs

Let $\mathcal{G}_N = \mathcal{G}(N, \mathbf{x}^N, \kappa)$, then the phase transition is in terms of the operator

$$T_{\kappa,\mu} \colon L^2(\mu) \to L^2(\mu), \qquad T_{\kappa,\mu} f(x) = \int_{\mathcal{S}} f(y)\, \kappa(x,y)\mu(\mathrm{d}y), \tag{0.1}$$

and its norm

$$\sigma(\kappa,\mu) = \| T_{\kappa,\mu} \|_{L^2(\mu)} = \sup_{f \in L^2(\mu) \colon \|f\|_{L^2(\mu)} = 1} \| T_{\kappa,\mu} f \|_{L^2(\mu)}. \tag{0.2}$$

Theorem (BOLLOBAS, JANSON, RIORDAN (2006))

(i) *If* $\sigma(\kappa,\mu) \leq 1$,

size largest component of $\mathcal{G}_N = o(N)$

Phase transition in inhomogeneous random graphs

Let $\mathcal{G}_N = \mathcal{G}(N, \mathbf{x}^N, \kappa)$, then the phase transition is in terms of the operator

$$T_{\kappa,\mu} \colon L^2(\mu) \to L^2(\mu), \qquad T_{\kappa,\mu} f(x) = \int_{\mathcal{S}} f(y)\, \kappa(x,y)\mu(\mathrm{d}y), \tag{0.1}$$

and its norm

$$\sigma(\kappa,\mu) = \| T_{\kappa,\mu} \|_{L^2(\mu)} = \sup_{f \in L^2(\mu) \colon \ \|f\|_{L^2(\mu)}=1} \| T_{\kappa,\mu} f \|_{L^2(\mu)}. \tag{0.2}$$

Theorem (Bollobas, Janson, Riordan (2006))

(i) *If* $\sigma(\kappa,\mu) \le 1$,

$$\textit{size largest component of } \mathcal{G}_N = o(N)$$

(ii) *If* $\sigma(\kappa,\mu) > 1$, *then it has size* $\asymp N$. *More precisely,*

$$\textit{size largest component of } \mathcal{G}_N \lesssim N \int_{\mathcal{S}} \rho(x)\, \mu(\mathrm{d}x)$$

with $\rho \colon \mathcal{S} \to [0,\infty)$ *maximal solution of* $\rho = 1 - \mathrm{e}^{-T_{\kappa,\mu}\rho}$.

Phase transition in inhomogeneous random graphs

Let $\mathcal{G}_N = \mathcal{G}(N, \mathbf{x}^N, \kappa)$, then the phase transition is in terms of the operator

$$T_{\kappa,\mu}\colon L^2(\mu) \to L^2(\mu), \qquad T_{\kappa,\mu} f(x) = \int_{\mathcal{S}} f(y)\, \kappa(x,y)\mu(\mathrm{d}y), \tag{0.1}$$

and its norm

$$\sigma(\kappa,\mu) = \| T_{\kappa,\mu} \|_{L^2(\mu)} = \sup_{f \in L^2(\mu)\colon\ \|f\|_{L^2(\mu)}=1} \| T_{\kappa,\mu} f \|_{L^2(\mu)}. \tag{0.2}$$

Theorem (Bollobas, Janson, Riordan (2006))

(i) *If* $\sigma(\kappa,\mu) \leq 1$,

$$\textit{size largest component of } \mathcal{G}_N = o(N)$$

(ii) *If* $\sigma(\kappa,\mu) > 1$, *then it has size* $\asymp N$. *More precisely,*

$$\textit{size largest component of } \mathcal{G}_N \lesssim N \int_{\mathcal{S}} \rho(x)\, \mu(\mathrm{d}x)$$

with $\rho\colon \mathcal{S} \to [0,\infty)$ *maximal solution of* $\rho = 1 - \mathrm{e}^{-T_{\kappa,\mu}\rho}$.

Examples!

Counting microscopic and macroscopic components ($\mathcal{S}$ finite)

$\mathcal{C}_i$ connected component of $\mathcal{G}(N, \mathbf{x}^N, \kappa) \to \mathrm{types}(\mathcal{C}_i) \in \mathbb{N}^{\mathcal{S}}$.

Counting microscopic and macroscopic components ($\mathcal{S}$ finite)

$\mathcal{C}_i$ connected component of $\mathcal{G}(N, \mathbf{x}^N, \kappa) \to \mathrm{types}(\mathcal{C}_i) \in \mathbb{N}^{\mathcal{S}}$.

Microscopic empirical measure:

$$\mathrm{Mi}_N = \frac{\mathbf{1}}{\mathbf{N}} \sum_j \delta_{\mathrm{types}(\mathcal{C}_j)}$$

where $(\mathcal{C}_j)_j$ are the connected components of $\mathcal{G}(N, \mathbf{x}^N, \kappa)$.

Counting microscopic and macroscopic components ($\mathcal{S}$ finite)

$\mathcal{C}_i$ connected component of $\mathcal{G}(N, \mathbf{x}^N, \kappa) \to \mathrm{types}(\mathcal{C}_i) \in \mathbb{N}^{\mathcal{S}}$.

Microscopic empirical measure:

$$\mathrm{Mi}_N = \frac{\mathbf{1}}{\mathbf{N}} \sum_j \delta_{\mathrm{types}(\mathcal{C}_j)}$$

where $(\mathcal{C}_j)_j$ are the connected components of $\mathcal{G}(N, \mathbf{x}^N, \kappa)$.

$$\begin{aligned} \mathrm{Mi}_N \in \mathcal{N}(\mu_N) \quad &= \quad \left\{\lambda \in \mathcal{M}(\mathbb{N}^{\mathcal{S}}) \colon c(\lambda)(\cdot) = \mu_N(\cdot),\ \lambda(0) = 0\right\} \\ & \quad c(\lambda)(\cdot)\colon = \int_{\mathbb{N}^{\mathcal{S}}} k(\cdot)\, \lambda(\mathrm{d}k) \text{ integrated type configuration of } \lambda \end{aligned}$$

Counting microscopic and macroscopic components ($\mathcal{S}$ finite)

$\mathcal{C}_i$ connected component of $\mathcal{G}(N, \mathbf{x}^N, \kappa) \to \mathrm{types}(\mathcal{C}_i) \in \mathbb{N}^{\mathcal{S}}$.

Microscopic and **macroscopic** empirical measures:

$$\mathrm{Mi}_N = \frac{\mathbf{1}}{\mathbf{N}} \sum_j \delta_{\mathrm{types}(\mathcal{C}_j)} \qquad \text{and} \qquad \mathrm{Ma}_N = \sum_j \delta_{\frac{\mathbf{1}}{\mathbf{N}}\mathrm{types}(\mathcal{C}_j)},$$

where $(\mathcal{C}_j)_j$ are the connected components of $\mathcal{G}(N, \mathbf{x}^N, \kappa)$.

$$\begin{aligned}
\mathrm{Mi}_N \in \mathcal{N}(\mu_N) \quad &= \quad \left\{ \lambda \in \mathcal{M}(\mathbb{N}^{\mathcal{S}}) \colon c(\lambda)(\cdot) = \mu_N(\cdot),\ \lambda(0) = 0 \right\} \\
& \qquad c(\lambda)(\cdot) \colon = \int_{\mathbb{N}^{\mathcal{S}}} k(\cdot)\, \lambda(\mathrm{d}k) \text{ integrated type configuration of } \lambda \\
\mathrm{Ma}_N \in \mathcal{M}(\mu_N) \quad &= \quad \left\{ \alpha \in \mathcal{M}_{\mathbb{N}_0}((0,1]^{\mathcal{S}}) \colon c(\alpha)(\cdot) = \mu_N(\cdot) \right\} \\
& \qquad c(\alpha)(\cdot) \colon = \int_{(0,1]^{\mathcal{S}}} y(\cdot)\, \alpha(\mathrm{d}y) \text{ integrated type configuration of } \alpha
\end{aligned}$$

Counting microscopic and macroscopic components ($\mathcal{S}$ finite)

$\mathcal{C}_i$ connected component of $\mathcal{G}(N, \mathbf{x}^N, \kappa) \to \mathrm{types}(\mathcal{C}_i) \in \mathbb{N}^{\mathcal{S}}$.

Microscopic and **macroscopic** empirical measures:

$$\mathrm{Mi}_N = \frac{\mathbf{1}}{\mathbf{N}} \sum_j \delta_{\mathrm{types}(\mathcal{C}_j)} \qquad \text{and} \qquad \mathrm{Ma}_N = \sum_j \delta_{\frac{\mathbf{1}}{\mathbf{N}}\mathrm{types}(\mathcal{C}_j)},$$

where $(\mathcal{C}_j)_j$ are the connected components of $\mathcal{G}(N, \mathbf{x}^N, \kappa)$.

$$\mathrm{Mi}_N \in \mathcal{N} \quad = \quad \left\{\lambda \in \mathcal{M}(\mathbb{N}^{\mathcal{S}}) \colon c(\lambda)(\cdot) \leq \mu(\cdot),\ \lambda(0) = 0\right\} \qquad \text{vague topol.}$$

$$c(\lambda)(\cdot) \colon = \int_{\mathbb{N}^{\mathcal{S}}} k(\cdot)\, \lambda(\mathrm{d}k) \text{ integrated type configuration of } \lambda$$

$$\mathrm{Ma}_N \in \mathcal{M} \quad = \quad \left\{\alpha \in \mathcal{M}_{\mathbb{N}_0}((0,1]^{\mathcal{S}}) \colon c(\alpha)(\cdot) \leq \mu(\cdot)\right\} \qquad \text{vague topol.}$$

$$c(\alpha)(\cdot) \colon = \int_{(0,1]^{\mathcal{S}}} y(\cdot)\, \alpha(\mathrm{d}y) \text{ integrated type configuration of } \alpha$$

Large Deviation Principle

Theorem (A., König, Langhammer, Patterson (2023))

The pair of measures $(\mathrm{Mi}_N, \mathrm{Ma}_N)$ *satisfies a large deviations principle in* $\mathcal{N} \times \mathcal{M}$ *with rate N an* **explicitly given rate function**

$$I(\lambda, \alpha) = I_{\mathrm{Mi}}(\lambda) + I_{\mathrm{Ma}}(\alpha) + I_{\mathrm{Me}}(\mu - c(\lambda) - c(\alpha)) + C(\mu).$$

- If $c(\lambda) + c(\alpha) \not\leq \mu$

$$I(\lambda, \alpha) = +\infty$$

- Entropy form of the **microscopic part**:

$$I_{\mathrm{Mi}}(\lambda) = \sum_{k \in \mathbb{N}^{\mathcal{S}}} \lambda_k \log\left(\frac{\lambda_k}{\tau(k) \mathrm{e}^{1-|k|} \prod_r \frac{\mu_r^{k_r}}{k_r!}} \right).$$

Sketch of the proof

Space of configurations of the connected components of the graph:

$$\mathcal{E}_N := \Big\{ \ell = (\ell_k)_{k\in\mathbb{N}^{\mathcal{S}}} : \ell_k \in \mathbb{N} \text{ for all } k \text{ and } \sum_{k\in\mathbb{N}^{\mathcal{S}}} \ell_k k_r = N\mu_N(r) \text{ for all } r \in \mathcal{S} \Big\}.$$

Sketch of the proof

Space of configurations of the connected components of the graph:

$$\mathcal{E}_N := \left\{ \ell = (\ell_k)_{k\in\mathbb{N}^{\mathcal{S}}} : \ell_k \in \mathbb{N} \text{ for all } k \text{ and } \sum_{k\in\mathbb{N}^{\mathcal{S}}} \ell_k k_r = N\mu_N(r) \ \text{ for all } r \in \mathcal{S} \right\}.$$

Lemma (The distribution of $\mathrm{Mi}_{\mathbf{N}}$ and $\mathrm{Ma}_{\mathbf{N}}$)

For any $N \in \mathbb{N}$ and for any $\ell \in \mathcal{E}_N$ we have that

$$\mathbb{P}\left(\mathbf{N}\mathrm{Mi}_{\mathbf{N}}(\mathbf{k}) = \ell_k;\ \forall k \in \mathbb{N}_0^{\mathcal{S}}\right) = \mathbb{P}\left(\mathrm{Ma}_{\mathbf{N}}\,(\mathbf{Nk}) = \ell_k;\ \forall k \in \mathbb{N}_0^{\mathcal{S}}\right)$$

Sketch of the proof

Space of configurations of the connected components of the graph:

$$\mathcal{E}_N := \Big\{ \ell = (\ell_k)_{k\in\mathbb{N}^{\mathcal{S}}} : \ell_k \in \mathbb{N} \text{ for all } k \text{ and } \sum_{k\in\mathbb{N}^{\mathcal{S}}} \ell_k k_r = N\mu_N(r) \ \text{ for all } r \in \mathcal{S} \Big\}.$$

Lemma (The distribution of $\mathrm{Mi}_{\mathbf{N}}$ and $\mathrm{Ma}_{\mathbf{N}}$)

For any $N \in \mathbb{N}$ and for any $\ell \in \mathcal{E}_N$ we have that

$$\mathbb{P}\left(\mathbf{N}\mathrm{Mi}_{\mathbf{N}}(\mathbf{k}) = \ell_k;\ \forall k \in \mathbb{N}_0^{\mathcal{S}} \right) = \Big(\prod_{r\in\mathcal{S}} (N\mu_N(r))! \Big) \times \prod_{k\in\mathbb{N}^{\mathcal{S}}} \zeta^{(N)}(\ell, k),$$

where

$$\zeta^{(N)}(\ell, k) = \frac{p_N(k)^{\ell_k}}{\ell_k! \prod_{r\in\mathcal{S}} (k_r!)^{\ell_k}} \Big(\prod_{r,s\in\mathcal{S}} \Big(1 - \frac{\kappa(r,s)}{N} \wedge 1 \Big)^{\frac{1}{2} k_s [N\mu_N(r) - k_r]} \Big)^{\ell_k}.$$

Sketch of the proof

Space of configurations of the connected components of the graph:

$$\mathcal{E}_N := \Big\{ \ell = (\ell_k)_{k\in\mathbb{N}^{\mathcal{S}}} : \ell_k \in \mathbb{N} \text{ for all } k \text{ and } \sum_{k\in\mathbb{N}^{\mathcal{S}}} \ell_k k_r = N\mu_N(r) \ \text{ for all } r\in\mathcal{S} \Big\}.$$

Lemma (The distribution of $\mathrm{Mi}_{\mathbf{N}}$ and $\mathrm{Ma}_{\mathbf{N}}$)

For any $N \in \mathbb{N}$ and for any $\ell \in \mathcal{E}_N$ we have that

$$\mathbb{P}\left(\mathbf{N}\mathrm{Mi}_{\mathbf{N}}(\mathbf{k}) = \ell_k;\ \forall k \in \mathbb{N}_0^{\mathcal{S}}\right) = \Big(\prod_{r\in\mathcal{S}} (N\mu_N(r))!\Big) \times \prod_{k\in\mathbb{N}^{\mathcal{S}}} \zeta^{(N)}(\ell,k),$$

where

$$\zeta^{(N)}(\ell,k) = \frac{p_N(k)^{\ell_k}}{\ell_k! \prod_{r\in\mathcal{S}}(k_r!)^{\ell_k}} \Big(\prod_{r,s\in\mathcal{S}} \Big(1 - \frac{\kappa(r,s)}{N} \wedge 1\Big)^{\frac{1}{2}k_s[N\mu_N(r)-k_r]}\Big)^{\ell_k}.$$

Where, for any $k \in \mathbb{N}^{\mathcal{S}}$, given any vertex vector $x \in \mathcal{S}^{|k|}$ "compatible" with k:

$$p_N(k) = \mathbb{P}\big(\mathcal{G}(|k|, x, \tfrac{1}{N}\kappa) \text{ is connected}\big), \qquad N \in \mathbb{N}.$$

Sketch of the proof

We fix $R < \infty$ and $\epsilon > 0$.

For any pair $(\lambda, \alpha) \in \mathcal{N} \times \mathcal{M}$

$$\mathbb{P}\Big(\mathrm{Mi}_N \in \mathcal{B}_\delta(\lambda);\ \mathrm{Ma}_N \in \mathcal{B}_\rho(\alpha)\Big) =$$

$$\sum_{\ell \in \mathcal{A}_{\delta,\rho}(\lambda,\alpha) \subseteq \mathcal{L}_N} \Big(\prod_{r \in \mathcal{S}} (N\mu_N(r))!\Big) \times \prod_{|k| \leq R} \zeta^{(N)}(\ell, k) \prod_{|k| \geq \epsilon N} \zeta^{(N)}(\ell, k) \prod_{\text{otherwise}} \zeta^{(N)}(\ell, k)$$

Sketch of the proof

We fix $R < \infty$ and $\epsilon > 0$.

For any pair $(\lambda, \alpha) \in \mathcal{N} \times \mathcal{M}$

$$\mathbb{P}\Big(\mathrm{Mi}_N \in \mathcal{B}_\delta(\lambda);\ \mathrm{Ma}_N \in \mathcal{B}_\rho(\alpha)\Big) =$$

$$\sum_{\ell \in \mathcal{A}_{\delta,\rho}(\lambda,\alpha) \subseteq \mathcal{L}_N} \Big(\prod_{r \in \mathcal{S}} (N\mu_N(r))!\Big) \times \prod_{|k| \leq R} \zeta^{(N)}(\ell, k) \prod_{|k| \geq \epsilon N} \zeta^{(N)}(\ell, k) \prod_{\text{otherwise}} \zeta^{(N)}(\ell, k)$$

Key points:

- accurate estimates for $p_N(k) = \mathbb{P}\big(\mathcal{G}(|k|, x, \frac{1}{N}\kappa)$ is connected$\big)$ only in the cases: $|k| \leq R$ and $|k| \geq \epsilon N$;
- sufficient upper bound for $p_N(k) = \mathbb{P}\big(\mathcal{G}(|k|, x, \frac{1}{N}\kappa)$ is connected$\big)$ in the other regimes;
- combinatorial terms.

The phase transition

$\sigma(\kappa,\mu)$ largest eigenvalue of the matrix $\{\kappa(r,s)\mu_s\}_{(r,s)\in\mathcal{S}^2}$

Theorem (Bollobas, Janson, Riordan (2006))

(i) *If* $\sigma(\kappa,\mu)\le 1$,

$$\textit{size largest component of } \mathcal{G}_N = o(N)$$

(ii) *If* $\sigma(\kappa,\mu) > 1$, *then it has size* $\asymp N$. *More precisely,*

$$\textit{size largest component of } \mathcal{G}_N \lesssim N\int_{\mathcal{S}}\rho(x)\,\mu(\mathrm{d}x)$$

with $\rho\colon \mathcal{S}\to[0,\infty)$ *maximal solution of* $\rho = 1-\mathrm{e}^{-T_{\kappa,\mu}\rho}$.

The phase transition

Minimization of $I_{\mathrm{Mi}}(\lambda) = \sum_{k \in \mathbb{N}^{\mathcal{S}}} \lambda_k \log\left(\frac{\lambda_k}{\tau(k) \mathrm{e}^{1-|k|} \prod_r \frac{\mu_r^{k_r}}{k_r!}}\right)$.

The phase transition

Minimization of $I_{\mathrm{Mi}}(\lambda) = \sum_{k\in\mathbb{N}^{\mathcal{S}}} \lambda_k \log\left(\frac{\lambda_k}{\tau(k)\mathrm{e}^{1-|k|}\prod_r \frac{\mu_r^{k_r}}{k_r!}}\right)$.

(i) If $\sigma(\kappa,\mu)\le 1$,
$\forall c\in\mathcal{M}(\mathcal{S})$ such that $c\preceq\mu$

$$\exists\lambda^*(c)\in\mathcal{N}(c) \qquad \text{such that } \lambda^*(c) = \arg\inf_{\substack{\lambda:\\ c(\lambda)=c}} I_{\mathrm{Mi}}(\lambda),$$

in particular

$$\lambda_k^*(c) = \tau(k)\prod_{r\in\mathcal{S}} \frac{c_r^{k_r}\,\mathrm{e}^{-k_r\kappa * c_r}}{k_r!}, \qquad k\in\mathbb{N}^{\mathcal{S}}.$$

Any microscopic mass is possible.

The phase transition

Minimization of $I_{\mathrm{Mi}}(\lambda) = \sum_{k\in\mathbb{N}^{\mathcal{S}}} \lambda_k \log\left(\frac{\lambda_k}{\tau(k)\mathrm{e}^{1-|k|}\prod_r \frac{\mu_r^{k_r}}{k_r!}}\right)$.

(i) If $\sigma(\kappa,\mu)\le 1$,
$\forall c\in\mathcal{M}(\mathcal{S})$ such that $c\preceq\mu$

$$\exists\lambda^*(c)\in\mathcal{N}(c) \qquad \text{such that } \lambda^*(c) = \arg\inf_{\substack{\lambda:\\ c(\lambda)=c}} I_{\mathrm{Mi}}(\lambda),$$

in particular

$$\lambda^*_k(c) = \tau(k)\prod_{r\in\mathcal{S}} \frac{c_r^{k_r}\mathrm{e}^{-k_r\kappa*c_r}}{k_r!}, \qquad k\in\mathbb{N}^{\mathcal{S}}.$$

Any microscopic mass is possible.

(ii) If $\sigma(\kappa,\mu)>1$,
$\forall c\in\mathcal{M}(\mathcal{S})$ such that $c\preceq\mu$ **but** $\sigma(\kappa,c)>1$

$I_{\mathrm{Mi}}(\lambda)$ does not admit minimizers in $\mathcal{N}(c)$.

The phase transition

Minimization of $I_{\mathrm{Mi}}(\lambda) = \sum_{k \in \mathbb{N}^{\mathcal{S}}} \lambda_k \log\left(\frac{\lambda_k}{\tau(k) e^{1-|k|} \prod_r \frac{\mu_r^{k_r}}{k_r!}} \right)$.

(i) If $\sigma(\kappa, \mu) \leq 1$,
$\forall c \in \mathcal{M}(\mathcal{S})$ such that $c \preceq \mu$

$$\exists \lambda^*(c) \in \mathcal{N}(c) \qquad \text{such that } \lambda^*(c) = \arg \inf_{\substack{\lambda: \\ c(\lambda)=c}} I_{\mathrm{Mi}}(\lambda),$$

in particular

$$\lambda_k^*(c) = \tau(k) \prod_{r \in \mathcal{S}} \frac{c_r^{k_r} e^{-k_r \kappa * c_r}}{k_r!}, \qquad k \in \mathbb{N}^{\mathcal{S}}.$$

Any microscopic mass is possible.

(ii) If $\sigma(\kappa, \mu) > 1$,
$\forall c \in \mathcal{M}(\mathcal{S})$ such that $c \preceq \mu$ **but** $\sigma(\kappa, c) > 1$
$\exists\, \lambda^{(n)}$ **minimizing sequence**

$$\lambda^{(n)} \searrow \lambda^*(b^*) \qquad \notin \mathcal{N}(c)$$

where $b^* = (1-f)c$ is such that $\sigma(\kappa, b^*) = 1$ and f is a positive non-trivial solution to

$$f = (1-f)\, T_{\kappa,c} f. \tag{0.3}$$

The phase transition

Minimization of $I_{\mathrm{Mi}}(\lambda) = \sum_{k\in\mathbb{N}^{\mathcal{S}}} \lambda_k \log\left(\frac{\lambda_k}{\tau(k)\mathrm{e}^{1-|k|}\prod_r \frac{\mu_r^{k_r}}{k_r!}}\right)$.

(i) If $\sigma(\kappa,\mu) \le 1$,
$\forall c \in \mathcal{M}(\mathcal{S})$ such that $c \preceq \mu$

$$\exists \lambda^*(c) \in \mathcal{N}(c) \qquad \text{such that } \lambda^*(c) = \arg \inf_{\substack{\lambda:\\ c(\lambda)=c}} I_{\mathrm{Mi}}(\lambda),$$

in particular

$$\lambda^*_k(c) = \tau(k) \prod_{r\in\mathcal{S}} \frac{c_r^{k_r}\mathrm{e}^{-k_r\kappa * c_r}}{k_r!}, \qquad k \in \mathbb{N}^{\mathcal{S}}.$$

Any microscopic mass is possible.

(ii) If $\sigma(\kappa,\mu) > 1$,
$\forall c \in \mathcal{M}(\mathcal{S})$ such that $c \preceq \mu$ **but** $\sigma(\kappa, c) > 1$

Not every microscopic mass is admissible.

The phase transition

$$\text{Minimization of } I_{\mathrm{Mi}}(\lambda) = \sum_{k\in\mathbb{N}^{\mathcal{S}}} \lambda_k \log\left(\frac{\lambda_k}{\tau(k)\mathrm{e}^{1-|k|}\prod_r \frac{\mu_r^{k_r}}{k_r!}}\right).$$

(i) If $\sigma(\kappa,\mu) \le 1$,
$\forall c \in \mathcal{M}(\mathcal{S})$ such that $c \preceq \mu$

$$\exists \lambda^*(c) \in \mathcal{N}(c) \qquad \text{such that } \lambda^*(c) = \arg \inf_{\substack{\lambda:\\ c(\lambda)=c}} I_{\mathrm{Mi}}(\lambda),$$

in particular

$$\lambda^*_k(c) = \tau(k) \prod_{r\in\mathcal{S}} \frac{c_r^{k_r}\,\mathrm{e}^{-k_r \kappa * c_r}}{k_r!}, \qquad k \in \mathbb{N}^{\mathcal{S}}.$$

Any microscopic mass is possible.

(ii) If $\sigma(\kappa,\mu) > 1$,
$\forall c \in \mathcal{M}(\mathcal{S})$ such that $c \preceq \mu$ **but** $\sigma(\kappa, c) > 1$

Not every microscopic mass is admissible.

To get the typical behaviour you should minimize the **complete rate function**, but $I_{\mathrm{Mi}}(\lambda)$ already gives hint on the phase transition!

From random graphs back to coagulation processes

From random graphs back to coagulation processes

- LDP for $\{\mathrm{Mi}_N(t)\}_{t\in[0,T]}$

  Freidlin-Wentzell approach: $\mathcal{I}_{[0,T]}(n) = \int_0^T L(n_t, \dot{n}_t)\,\mathrm{d}\,t$.

  Problems: tilting argument in infinite dimensional space!

$$\mathbb{P}\Big(\mathrm{Mi}_N(t) \simeq n_t \Big| n_0\Big) \simeq e^{-NI_t(n_t|n_0)}.$$

From random graphs back to coagulation processes

- LDP for $\{\mathrm{Mi}_N(t)\}_{t\in[0,T]}$

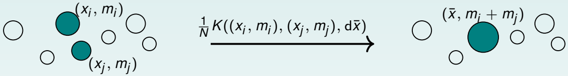
  Freidlin-Wentzell approach: $\mathcal{I}_{[0,T]}(n) = \int_0^T L(n_t, \dot{n}_t)\,\mathrm{d}\,t$.

  Problems: tilting argument in infinite dimensional space!

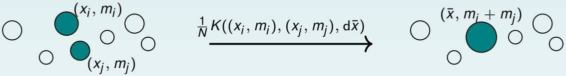
$$\mathbb{P}\Big(\mathrm{Mi}_N(t) \simeq n_t \Big| n_0\Big) \simeq e^{-NI_t(n_t|n_0)}.$$

$$\begin{aligned}\mathbb{P}\Big(\{\mathrm{Mi}_N(t)\}_{t\in[0,T]} \simeq \{n_t\}_{t\in[0,T]}\Big) \simeq &\prod_i \mathbb{P}\Big(\mathrm{Mi}_N(t_{i+1}) \simeq n_{t_{i+1}} \Big| n_{t_i}\Big)\\ \simeq &e^{-N\sum_i I_t(n_{t_{i+1}}|n_{t_i})}\end{aligned}$$

What about more general kernels?



What about more general kernels?

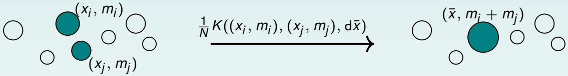


Poisson monodisperse initial condition: for $\mu \in \mathcal{M}_1(\mathcal{S})$

$$\mu_0^{(N)} := \frac{1}{N} \sum_i \delta_{(X_i,1)},$$

where $(X_i) \sim$ Poisson Point Process on $\mathcal{S}$ with intensity measure $N\mu$ ($\mathrm{Poi}_{N\mu}$).

What about more general kernels?



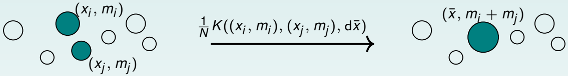
Poisson monodisperse initial condition: for $\mu \in \mathcal{M}_1(\mathcal{S})$

$$\mu_0^{(N)} := \frac{1}{N} \sum_i \delta_{(X_i, 1)},$$

where $(X_i) \sim$ Poisson Point Process on $\mathcal{S}$ with intensity measure $N\mu$ $(\mathrm{Poi}_{N\mu})$.

If there is **no coagulation**

What about more general kernels?



Poisson monodisperse initial condition: for $\mu \in \mathcal{M}_1(\mathcal{S})$

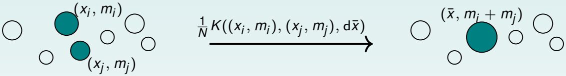
$$\mu_0^{(N)} :\, = \frac{1}{N}\sum_i \delta_{(X_i,1)},$$

where $(X_i) \sim$ Poisson Point Process on $\mathcal{S}$ with intensity measure $N\mu$ $(\mathrm{Poi}_{N\mu})$.

If there is **no coagulation**

$$\mu_T^{(N)} = \frac{1}{N}\sum_i \delta_{\left(X_i^{(N)}(T),\,1\right)} \in \mathcal{M}(\mathcal{S} \times \mathbb{N}).$$

What about more general kernels?



Poisson monodisperse initial condition: for $\mu \in \mathcal{M}_1(\mathcal{S})$

$$\mu_0^{(N)} := \frac{1}{N}\sum_i \delta_{(X_i,1)},$$

where $(X_i) \sim$ Poisson Point Process on $\mathcal{S}$ with intensity measure $N\mu$ $(\mathrm{Poi}_{N\mu})$.

If there is **no coagulation**

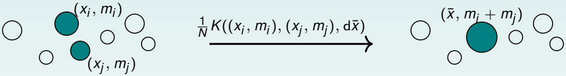
$$\mu_T^{(N)} = \frac{1}{N}\sum_i \delta_{\left(X_i^{(N)}(T),\, 1\right)} \in \mathcal{M}(\mathcal{S} \times \mathbb{N}).$$

Sanov's theorem

$(\mu_T^{(N)})_{N\in\mathbb{N}}$ satisfies an LDP on $\mathcal{M}(\mathcal{S} \times \mathbb{N})$ with speed N and rate function

$$\nu \mapsto H(\nu \mid \mu \times \delta_1).$$

What about more general kernels?



Poisson monodisperse initial condition: for $\mu \in \mathcal{M}_1(\mathcal{S})$

$$\mu_0^{(N)} : = \frac{1}{N} \sum_i \delta_{(X_i, 1)},$$

where $(X_i) \sim$ Poisson Point Process on $\mathcal{S}$ with intensity measure $N\mu$ $(\mathrm{Poi}_{N\mu})$.

If there is **no coagulation**

$$\mu_T^{(N)} = \frac{1}{N} \sum_i \delta_{\left(X_i^{(N)}(T), 1\right)} \in \mathcal{M}(\mathcal{S} \times \mathbb{N}).$$

Sanov's theorem

$(\mu_T^{(N)})_{N \in \mathbb{N}}$ satisfies an LDP on $\mathcal{M}(\mathcal{S} \times \mathbb{N})$ with speed N and rate function

$$\nu \mapsto H(\nu \mid \mu \times \delta_1).$$

What if we add the **coagulation interaction**?

LDP for cluster coagulation process

[A., König, Langhammer, Patterson (2026)]

LDP for cluster coagulation process

[A., KÖNIG, LANGHAMMER, PATTERSON (2026)]

We are interested in describing, under $\mathbb{P}^{(N)}_{\mathrm{Poi}_{N\mu}}(\cdot)$,

$$\mu_T^{(N)} = \frac{1}{N}\sum_i \delta_{\left(X_i^{(N)}(T), M_i^{(N)}(T)\right)} \in \mathcal{M}(\mathcal{S}\times\mathbb{N}).$$

LDP for cluster coagulation process

[A., König, Langhammer, Patterson (2026)]

We are interested in describing, under $\mathbb{P}^{(N)}_{\mathrm{Poi}_{N\mu}}(\cdot)$,

$$\mu_T^{(N)} = \frac{1}{N}\sum_i \delta_{\left(X_i^{(N)}(T), M_i^{(N)}(T)\right)} \in \mathcal{M}(\mathcal{S} \times \mathbb{N}).$$

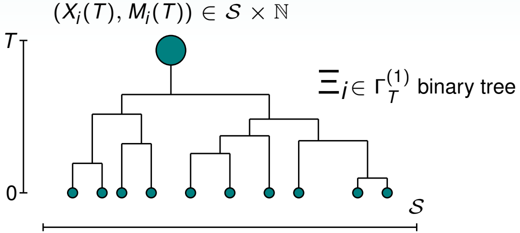
$$(X_i(T), M_i(T)) \in \mathcal{S} \times \mathbb{N}$$

LDP for cluster coagulation process

[A., König, Langhammer, Patterson (2026)]

We are interested in describing, under $\mathbb{P}^{(N)}_{\mathrm{Poi}_{N\mu}}(\cdot)$,

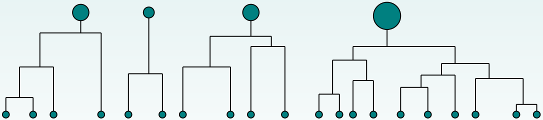
$$\mu_T^{(N)} = \frac{1}{N} \sum_i \delta_{\left(X_i^{(N)}(T), M_i^{(N)}(T)\right)} \in \mathcal{M}(\mathcal{S} \times \mathbb{N}).$$



$(X_i(T), M_i(T)) \in \mathcal{S} \times \mathbb{N}$

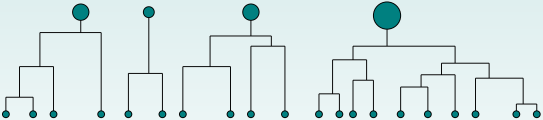
$\Xi_i \in \Gamma_T^{(1)}$ binary tree

Description via binary trees

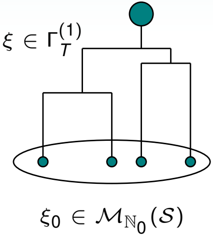


$\Gamma_T^{(1)}$ set of binary trees embedded in $[0, T]$: a subset of $\mathbb{D}([0, T], \mathcal{M}(\mathcal{S} \times \mathbb{N}))$.

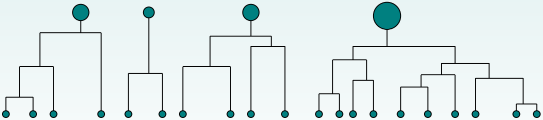
Description via binary trees



$\Gamma_T^{(1)}$ set of binary trees embedded in $[0, T]$.



Description via binary trees

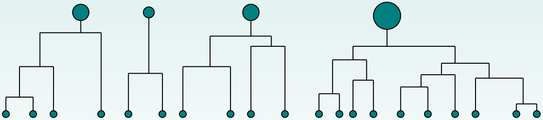


$\Gamma_T^{(1)}$ set of binary trees embedded in $[0, T]$.

The empirical measure on trees is

$$\mathcal{V}_N^{(T)} :\ = \frac{1}{N} \sum_i \delta_{\Xi_i} \quad \in \mathcal{M}(\Gamma_T^{(1)}).$$

Description via binary trees



$\Gamma_T^{(1)}$ set of binary trees embedded in $[0, T]$.

The empirical measure on trees is

$$\mathcal{V}_N^{(T)} := \frac{1}{N} \sum_i \delta_{\Xi_i} \quad \in \mathcal{M}(\Gamma_T^{(1)}).$$

$$N \int \xi_0 \mathcal{V}_N^{(T)}(\mathrm{d}\xi) \quad \in \mathcal{M}_{\mathbb{N}_0}(\mathcal{S})$$

$\Rightarrow$ initial distribution of points in $\mathcal{S}$

Does the law of $\mathcal{V}_N^{(T)}$ satisfy a **large deviations principle**?

$$\mathbb{P}^{(N)}_{\mathrm{Poi}_{N\mu}}(\mathcal{V}_N^{(T)} \in \mathrm{d}\nu) \approx e^{-NI(\nu)+o(N)}$$

Many-body system approach

Theorem [A., König, Langhammer, Patterson, 2026]

For $\nu \in \mathcal{M}(\Gamma_T^{(1)})$

$$\mathbb{P}^{(N)}_{\mathrm{Poi}_{N\mu}}(\mathcal{V}_N^{(T)} \in \mathrm{d}\nu) = \exp\Big\{ -\frac{1}{2N}\sum_{i\neq j} R^{(T)}(\xi_i,\xi_j)\Big\}\, \mathtt{P}^{(N)}_{NM^{(T)}_{\mu,N}}\left(\frac{Y}{N} \in \mathrm{d}\nu\right)$$

where Y is a Poisson Point process on the space $\Gamma_T^{(1)}$ with intensity measure $NM^{(T)}_{\mu,N}$.

Many-body system approach

Theorem [A., König, Langhammer, Patterson, 2026]

For $\nu \in \mathcal{M}(\Gamma_T^{(1)})$

$$\mathbb{P}^{(N)}_{\mathrm{Poi}_{N\mu}}(\mathcal{V}_N^{(T)} \in \mathrm{d}\nu) = \exp\Big\{ -\frac{1}{2N} \sum_{i \neq j} R^{(T)}(\xi_i, \xi_j) \Big\} \, \mathtt{P}^{(N)}_{NM^{(T)}_{\mu,N}} \Big(\frac{Y}{N} \in \mathrm{d}\nu \Big)$$

where Y is a Poisson Point process on the space $\Gamma_T^{(1)}$ with intensity measure $NM^{(T)}_{\mu,N}$.

For any $\xi, \xi' \in \Gamma_T^{(1)}$,

$$R^{(T)}(\xi, \xi') = \int_0^T \mathrm{d}t \sum_{m, m' \in \mathbb{N}} \int_{\mathcal{S}} \xi_t(\mathrm{d}x, m) \int_{\mathcal{S}} \xi'_t(\mathrm{d}x', m') \, K\big((x, m), (x', m')\big).$$

Many-body system approach

Theorem [A., König, Langhammer, Patterson, 2026]

For $\nu \in \mathcal{M}(\Gamma_T^{(1)})$

$$\mathbb{P}^{(N)}_{\mathrm{Poi}_{N\mu}}(\mathcal{V}_N^{(T)} \in \mathrm{d}\nu) = \exp\Big\{ -\frac{1}{2N} \sum_{i \neq j} R^{(T)}(\xi_i, \xi_j) \Big\} \, \mathbb{P}^{(N)}_{NM^{(T)}_{\mu,N}} \left(\frac{Y}{N} \in \mathrm{d}\nu \right)$$

where Y is a Poisson Point process on the space $\Gamma_T^{(1)}$ with intensity measure $NM^{(T)}_{\mu,N}$.

For any $\xi, \xi' \in \Gamma_T^{(1)}$,

$$R^{(T)}(\xi, \xi') = \int_0^T \mathrm{d}t \sum_{m,m' \in \mathbb{N}} \int_{\mathcal{S}} \xi_t(\mathrm{d}x, m) \int_{\mathcal{S}} \xi'_t(\mathrm{d}x', m') \, K\big((x,m),(x',m')\big).$$

Interpretation:
a many body system with interaction R.

LDP for $\mathcal{V}_N^{(T)}$

For all $k \in \mathbb{N}$

$$M_{\mu,N}^{(T)}(\mathrm{d}\xi) \xrightarrow{N\to\infty} M_{\mu}^{(T)}(\mathrm{d}\xi) \qquad \xi \in \Gamma_T^{(1)},\ |\xi_0| = k.$$

Conditional LDP for $\mathcal{V}_N^{(T)}$

For all $k \in \mathbb{N}$

$$M_{\mu,N}^{(T)}(\mathrm{d}\xi) \xrightarrow{N\to\infty} M_{\mu}^{(T)}(\mathrm{d}\xi) \qquad \xi \in \Gamma_T^{(1)},\ |\xi_0| = k.$$

Take $\mathcal{A} \subseteq \mathcal{M}(\Gamma_T^{(1)})$ such that $\nu \mapsto \int \nu(\mathrm{d}\xi)|\xi_0|$ is continuous in $\mathcal{A}$, then

Conditional LDP for $\mathcal{V}_N^{(T)}$

For all $k \in \mathbb{N}$

$$M_{\mu,N}^{(T)}(\mathrm{d}\xi) \xrightarrow{N\to\infty} M_{\mu}^{(T)}(\mathrm{d}\xi) \qquad \xi \in \Gamma_T^{(1)},\ |\xi_0| = k.$$

Take $\mathcal{A} \subseteq \mathcal{M}(\Gamma_T^{(1)})$ such that $\nu \mapsto \int \nu(\mathrm{d}\xi)|\xi_0|$ is continuous in $\mathcal{A}$, then

Conditional LDP holds

$$\mathbb{P}^{(N)}_{\mathrm{Poi}_{N\mu}}(\mathcal{V}_N^{(T)} \in \mathrm{d}\nu | \mathcal{A}) \approx \mathrm{e}^{-N\,(I_\mu(\nu) - \chi_A)}$$

where

$$I_\mu(\nu) = \left\langle \nu, \log \frac{\mathrm{d}\nu}{\mathrm{d}M_\mu^{(T)}} \right\rangle + 1 - |\nu| + \frac{1}{2}\langle \nu, \mathfrak{R}^{(T)}(\nu)\rangle,$$

$$\mathfrak{R}^{(T)}(\nu)(\xi) = \int_{\Gamma_T^{(1)}} R^{(T)}(\xi,\xi')\,\nu(\mathrm{d}\xi'), \qquad \xi \in \Gamma_T^{(1)}, \nu \in \mathcal{M}(\Gamma_T^{(1)}),$$

and χ_A is a constant.

Merci!