

Exercises for the course
*Stochastic models for coagulation processes, large scale limits
and phase transitions*

EUR MINT School 2026

18 June 2026

Exercises

Exercise 1 (Large deviations for coin tossing). Let X_1, X_2, \dots be iid Bernoulli(p), with $p \in (0, 1)$, and set

$$\bar{X}_N = \frac{1}{N} \sum_{i=1}^N X_i.$$

(a) Use Stirling's formula to show that

$$\mathbb{P}(\bar{X}_N \approx q) \asymp e^{-NI(q)}, \quad q \in [0, 1],$$

where

$$I(q) = q \log \frac{q}{p} + (1 - q) \log \frac{1 - q}{1 - p}.$$

(b) Show that $I(q) \geq 0$, with equality if and only if $q = p$.

(c) Show that for $a > p$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}(\bar{X}_N \geq a) = -I(a),$$

and state the analogous result for $a < p$.

(d) Show that

$$I(q) = \sup_{t \in \mathbb{R}} \{tq - \log \phi(t)\}, \quad \phi(t) = \mathbb{E}[e^{tX_1}].$$

Exercise 2 (Sanov-type large deviations on a finite space). Let $\Gamma = \{1, \dots, r\}$. Let $(X_i)_{i \geq 1}$ be iid with law $\rho \in \mathcal{M}_1(\Gamma)$, where $\rho_s > 0$ for all s . Define the empirical measure

$$L_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}.$$

For $\nu \in \mathcal{M}_1(\Gamma)$, define

$$I_\rho(\nu) = \sum_{s=1}^r \nu_s \log \left(\frac{\nu_s}{\rho_s} \right),$$

with the convention $0 \log 0 = 0$. Let

$$B_a(\rho) = \{\nu \in \mathcal{M}_1(\Gamma) : d(\nu, \rho) \leq a\}, \quad B_a^c(\rho) = \mathcal{M}_1(\Gamma) \setminus B_a(\rho).$$

Show that, for every $a > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(L_n \in B_a^c(\rho)) = - \inf_{\nu \in B_a^c(\rho)} I_\rho(\nu),$$

with the convention $\inf_{\emptyset} I_\rho = \infty$.

Exercise 3 (Properties of relative entropy). In the setting of Exercise 2, prove that:

- (i) I_ρ is finite, continuous and strictly convex on $\mathcal{M}_1(\Gamma)$.
- (ii) $I_\rho(\nu) \geq 0$, with equality if and only if $\nu = \rho$.

Exercise 4 (Edges in sparse Erdős–Renyi graphs). Let E_N be the number of edges in the Erdős–Renyi graph $\mathcal{G}(N, t/N)$, where $t > 0$ is fixed. Recall the Gärtner–Ellis theorem and assume the connection probability asymptotic

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}(\mathcal{G}(N, t/N) \text{ is connected}) = \log(1 - e^{-t}).$$

- (a) Prove that E_N/N satisfies an LDP and identify its rate function.
- (b) Prove that E_N/N , conditioned on $\mathcal{G}(N, t/N)$ being connected, satisfies an LDP and identify its rate function.
- (c) Show that the typical number of edges per vertex under this conditioning is strictly larger than in the unconditioned graph.

Exercise 5 (Borel distribution and Poisson Galton–Watson trees). Let $(Z_n)_{n \geq 0}$ be a Galton–Watson process with $Z_0 = 1$ and offspring distribution $\text{Poisson}(\lambda)$. Let

$$T = \sum_{n \geq 0} Z_n$$

be the total progeny.

- (a) Show that the offspring generating function is

$$f(s) = e^{\lambda(s-1)}.$$

- (b) Show that $G(s) = \mathbb{E}[s^T]$ satisfies

$$G(s) = sf(G(s)) = se^{\lambda(G(s)-1)}.$$

- (c) Recall the following result

- **Law of total progeny** (Theorem 3.13 in [Van der Hofstad, Random graphs and complex networks, 2016])

For a branching process with i.i.d. offspring distribution $Z_1 = X$,

$$\mathbb{P}(T = n) = \frac{1}{n} \mathbb{P}(X_1 + \cdots + X_n = n - 1), \tag{3.5.1}$$

where $(X_i)_{i=1}^n$ are i.i.d. copies of X .

Using the above, prove that

$$\mathbb{P}(T = n) = \frac{(\lambda n)^{n-1}}{n!} e^{-\lambda n}, \quad n \geq 1.$$

(d) Determine when $T < \infty$ almost surely and compute $\mathbb{E}[T]$ in the subcritical case.

Exercise 6 (Asymptotics and maxima for the Borel distribution). Let T have the Borel distribution with parameter λ , namely

$$\mathbb{P}(T = n) = \frac{(\lambda n)^{n-1}}{n!} e^{-\lambda n}, \quad n \geq 1.$$

(a) Use Stirling's formula to show that

$$\mathbb{P}(T = n) \sim \frac{1}{\lambda \sqrt{2\pi}} n^{-3/2} \left(\lambda e^{1-\lambda} \right)^n.$$

(b) Discuss the subcritical case $\lambda < 1$ and the critical case $\lambda = 1$.

(c) Let X_1, X_2, \dots be iid random variables with law $n_k(t) = \frac{1}{k} \mathbb{P}(T = k)$, for any $k \in \mathbb{N}$. Set $M_N = \max_{1 \leq i \leq N} T_i$. Discuss the order of M_N in the subcritical and critical cases.