Stochastic models for coagulation processes, large scale limits and phase transitions

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Summer School EUR MINT 2026
Particle systems and PDEs

Toulouse
June 15-19, 2026

Back to our case: **pathwise** LLN

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$$\left(\frac{1}{N}L_k^{(N)}(t)\right)_{k\in\mathbb{N}} \quad \in \mathcal{E} \quad \longleftrightarrow \quad \mu_t^{(N)} = \frac{1}{N}\sum_{i\in n(t)} \delta_{M_i^{(N)}(t)} \quad \in \mathcal{M}_{\leq 1}(\mathbb{N})$$

Back to our case: **pathwise** LLN

Weak formulation of Smoluchowski equation

For $\psi\colon \mathbb{N} \to \mathbb{R}$ test function

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \psi, n(t)\rangle = \frac{1}{2}\sum_{h,k\in\mathbb{N}_{\geq 1}} K(h,k)n_h(t)n_k(t)\big(\psi(h+k)-\psi(h)-\psi(k)\big)$$

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$$-\sum_{h\in\mathbb{N}}\beta(h)n_h(t)g_\infty(t)\psi_h$$

with $g_\infty(t) = 1 - \sum_{k\geq 1} kn_k(t)$.

Assume

$$\lim_{m\to\infty}\frac{K(m,n)}{m} = \beta(n), \qquad \forall n\in\mathbb{N}. \tag{1}$$

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Theorem (Fournier, Giet, 05)

Assume (1)*, then the sequence of laws* $\{\mathcal{P}_N\}_{N\in\mathbb{N}}$ *is tight. Moreover, every limit point* $\mathcal{P}^*$ *is concentrated on the set of solutions of the* **Flory equation** *with kernel* K *and monodisperse initial condition.*

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$$\sqrt{N}\left(\frac{1}{N}\mathbf{L}^{(N)}(t) - \mathbf{n}(t)\right)_{t\in[0,\infty)} \xrightarrow{n\to\infty} (X_t)_{t\in[0,\infty)},$$

where X is the solution of some **infinite dimensional SDE**.

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Some results:

- Non-gelling kernels
  [Kolokoltsov, 2010], [Amorim, Arellano, Jara, 2026]
- Multiplicative kernel (combinatorial methods)
  [Janson, 95], [Enriquez, Faraud,Lemaire, 24],[Bhamidi, Budhiraja, Sakanaveeti, 2024]

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(LDP) For some $(\mu_t)_{t\in[0,T]} \in \mathbb{C}([0,T],\mathcal{E})$, one expects

$$\mathbb{P}\left(\frac{1}{N}L^{(N)}(\cdot)\approx \mu_{\cdot}\right) = \mathrm{e}^{-N\mathcal{I}_T(\mu)+o(N)},$$

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- Partial results: [A., König, Langhammer, Patterson 2026], [A., Langhammer, 2026+]

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Multiplicative Smoluchowski equation

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- It is a function of a time-dependent version of the well-known ERDŐS-RÉNYI **random graph** model.

  The vector $(M_i^{(N)}(t))_{i=1}^{n(t)}$ is in distribution equal to the collection of sizes of all the connected components of the graph

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Additional edges

Edges can also appear inside an already existing component. These edges create surplus, but do not change the vector of component sizes.

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They matter only in large components!

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At exploration step r:

- explored E_r;
- active A_r;
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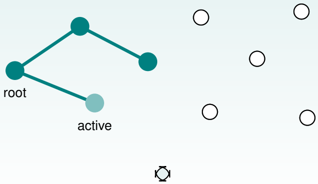

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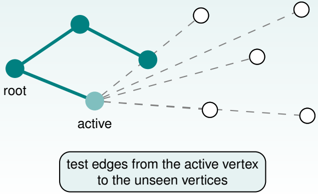

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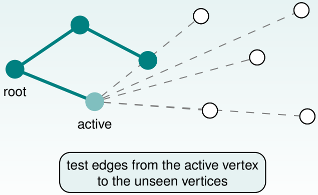

$$A_{r+1} = A_r - 1 + Y_r, \qquad Y_r \sim \mathrm{Bin}\left(U_r, \frac{t}{N}\right).$$

Given U_r, the number of new active vertices Y_r is independent from all the rest.

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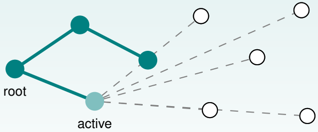

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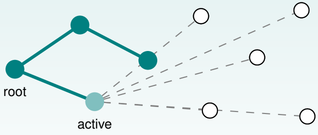

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When the exploration has only seen $o(N)$ vertices,

$$U_r = N - o(N).$$

Hence the number of new neighbours of the active vertex satisfies

$$Y_r \sim \mathrm{Bin}\left(U_r, \frac{t}{N}\right) \approx \mathrm{Poi}(t).$$

Local weak limit

Benjamini-Schramm convergence

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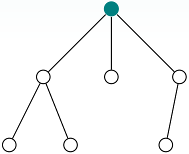

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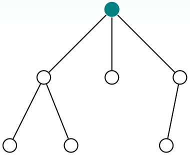

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The mean offspring is t.

Subcritical for $t < 1$
Critical for $t = 1$
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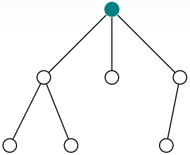

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The gelation time is the branching-process critical point.

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Subcritical regime: $t < 1$

- Let $\mathcal{T}_k^N$ be the number of tree-like components of fixed size $k \leq N$, then

$$\frac{\mathcal{T}_k^N}{N} \to \frac{1}{k} q_k(t),$$

where $q_k(t) = \mathbb{P}(\mathbf{T}_t = k)$ and $\mathbf{T}_t$ is the total progeny of the $\mathrm{PGW}(t)$, which has **Borel distribution** with parameter t. [BOLLOBÁS 2001].

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- The largest component is logarithmic:

$$|\mathcal{C}_{\max}| = O(\log N) \qquad \text{with high probability.}$$

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Supercritical regime: $t > 1$

- There is a unique giant component:

$$|\mathcal{C}_{\max}| \sim \rho(t)N,$$

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$$|\mathcal{C}_{(2)}| = O(\log N) \qquad \text{with high probability.}$$

- The finite size components are still trees and

$$\frac{\mathcal{T}_k^N}{N} \to n_k(t) = \frac{k^{k-2}}{k!} t^{k-1} e^{-tk},$$

where $\mathbf{n}(t)$ is the solution of the Flory equation.

True also for $t > 1$, due to the fact that $te^{-t} = t^* e^{-t^*}$ and $t^* < 1$

$$n_k(t) = \frac{t^*}{t}\frac{1}{k}\mathbb{P}(\mathbf{T}_{t^*} = k).$$

In particular: $g_\infty(t) = 1 - \sum_k kn_k(t) = \rho(t)$.

Fluctuations around the typical limit (dynamical)

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Finite tree components [JANSON, 1995]

Then, for every fixed subset of indexes $\mathbf{k} = (k_1, k_2, \dots, k_n)$,

$$\frac{\mathcal{T}_{\mathbf{k}}^N(t) - N n_{\mathbf{k}}(t)}{\sqrt{N}} \implies V_{\mathbf{k}}(t) \qquad \text{in } \mathbb{D}([0,\infty),\mathbb{R}),$$

where $(V_{\mathbf{k}}(t))_{k\geq 1}$ is a centered continuous Gaussian process with explicit covariance

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Largest component above criticality [Enriquez, Faraud, Lemaire, 2025]

Set

$$u(t) = \frac{1}{1-\rho(t)} - t, \qquad v(t) = \frac{\rho(t)}{1-\rho(t)}.$$

Then, in $\mathbb{D}((1,\infty),\mathbb{R})$,

$$\left(\frac{|\mathcal{C}_{\max}| - \rho(t)N}{\sqrt{N}} : t > 1\right) \Longrightarrow \left(\frac{1}{u(t)} B(v(t)) : t > 1\right),$$

where B is a standard Brownian motion.

Critical window: zooming around $T_{\mathrm{GEL}} = 1$

The critical window is

$$t_N = 1 + \frac{s}{N^{1/3}}, \qquad s \in \mathbb{R}.$$

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Theorem (Critical component sizes [ALDOUS, 97])

In the critical window,

$$|\mathcal{C}_{(1)}|, |\mathcal{C}_{(2)}|, \dots \quad \text{are of order } N^{2/3}.$$

More precisely,

$$\left(N^{-2/3}|\mathcal{C}_{(i)}|\right)_{i\geq 1}$$

converges in distribution in $\ell^2_{\geq}$ to a sequence of excursion lengths of a 1-dimensional Brownian motion with drift $(t - s)$ at time t, reflected in zero.

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This shows that in this time frame the largest components scale like $N^{2/3}$ but they do not have a deterministic limit!

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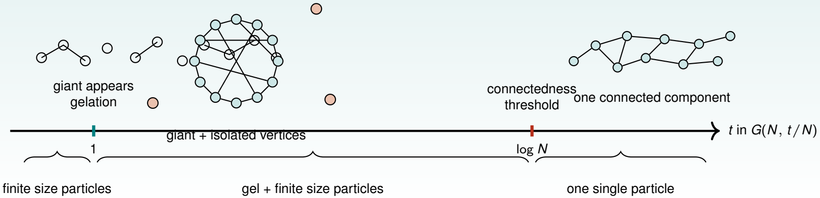

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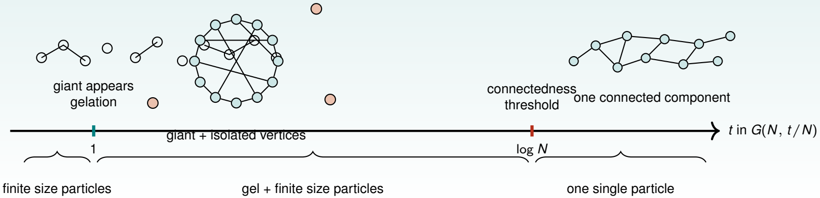

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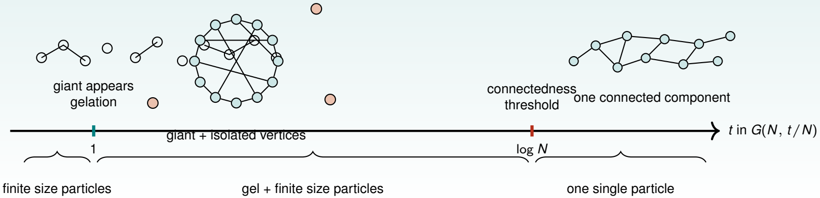

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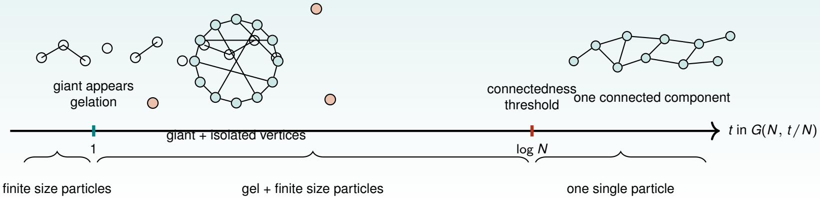

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- $t \simeq s \log N$ with $s > 1$: a unique particle is present with probability tending to 1.

Multiplicative Smoluchowski equation

$$\partial_t n_k(t) = \frac{1}{2}\sum_{h=1}^{k-1} h(k-h)\, n_h(t) n_{k-h}(t) - k n_k(t) \sum_{h\geq 1} h n_h(t).$$

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- graph grows by edge addition;
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Difference with Flory:

$$\sum_k k n_k(t) = 1 - \rho(t) < \frac{1}{t}.$$

Adding inhomogeneities

Sparse inhomogeneous random graphs

Inhomogeneous random graphs are a natural generalization of the Erdős-Rényi random graph ([Söderberg 2002], [Bollobás, Janson and Riordan 2006]).

- $\mathcal{S}$ a metric space: the **type space**;
- $\mu \in \mathcal{M}(\mathcal{S})$ a probability on $\mathcal{S}$;
- $\mathbf{x}^N = (x_1, \ldots, x_N) \in \mathcal{S}^N$ vector of vertices' type

$$\mu_N := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i} \to \mu;$$

- a symmetric kernel $\kappa : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$.

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The **sparse inhomogeneous random graph** $\mathcal{G}(N, \mathbf{x}^N, \kappa)$ is such that there is an edge between vertices i and j with probability that depends on their types:

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Sparse since the expected number of edges is proportional to N:

$$\mathbf{E}\#\{\text{ edges }\} \simeq N \int_{\mathcal{S}\times\mathcal{S}} \kappa(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y).$$

Phase transition in inhomogeneous random graphs

Let $\mathcal{G}_N = \mathcal{G}(N, \mathbf{x}^N, \kappa)$, then the phase transition is in terms of the operator

$$T_{\kappa,\mu}\colon L^2(\mu) \to L^2(\mu), \qquad T_{\kappa,\mu} f(x) = \int_{\mathcal{S}} f(y)\, \kappa(x,y)\mu(\mathrm{d}y), \tag{0.1}$$

and its norm

$$\sigma(\kappa,\mu) = \| T_{\kappa,\mu} \|_{L^2(\mu)} = \sup_{f \in L^2(\mu)\colon \|f\|_{L^2(\mu)}=1} \| T_{\kappa,\mu} f\|_{L^2(\mu)}. \tag{0.2}$$

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(i) *If $\sigma(\kappa, \mu) \leq 1$,*

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Theorem (Bollobas, Janson, Riordan (2006))

(i) *If* $\sigma(\kappa,\mu) \le 1$,

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(ii) *If* $\sigma(\kappa,\mu) > 1$, *then it has size* $\asymp N$. *More precisely,*

$$\textit{size largest component of } \mathcal{G}_N \lesssim N \int_{\mathcal{S}} \rho(x)\, \mu(\mathrm{d}x)$$

with $\rho \colon \mathcal{S} \to [0,\infty)$ *maximal solution of* $\rho = 1 - \mathrm{e}^{-T_{\kappa,\mu}\rho}$.