Stochastic models for coagulation processes, large scale limits and phase transitions

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Particles with position (in some Polish space $\mathcal{S}$) and mass (in $\mathbb{N}$)

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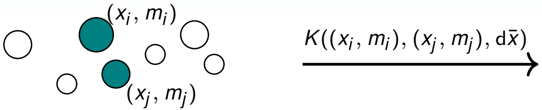

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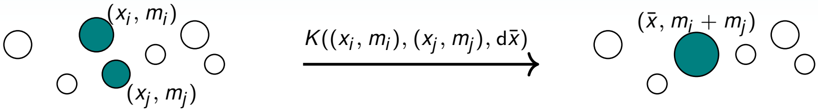

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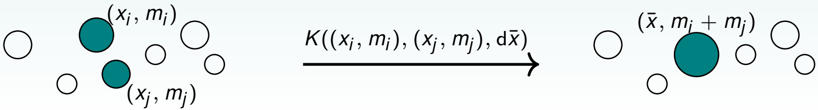

Examples: coagulation kernel

$\mathcal{S} \subseteq \mathbb{R}^d$

$$K((x, m), (y, n), \mathrm{d}\bar{x}) = (m\,n)^{\alpha}\ \rho(\|x - y\|)\ \delta_{\frac{mx+ny}{m+n}}(\mathrm{d}\bar{x})$$

Outline of the course

- Introduction to stochastic models for (spatially homogeneous) coagulation processes.
- Scaling limits for coagulation processes (generator approach).
- Connections with graph processes and combinatorics.
- Recent developments in inhomogeneous coagulation processes.

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$$\frac{\mathrm{d}}{\mathrm{d}t} n_k(t) = Q_k^+(n)(t) - Q_k^-(n)(t)$$

for all $k \geq 1$, where

$$Q_k^+(n)(t) = \frac{1}{2} \sum_{h=1}^{k-1} K(h, k-h) n_h(t) n_{k-h}(t) \qquad \text{gain term},$$

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Continuous version

$$\frac{\partial}{\partial t} n(x, t) = Q_x^+(n)(t) - Q_x^-(n)(t)$$

for all $x \in \mathbb{R}_+$, where

$$Q_x^+(n)(t) = \frac{1}{2} \int_0^x K(y, x-y) n(x-y, t) n(y, t) \mathrm{d}y \qquad \text{gain term},$$

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Weak formulation

For $\psi\colon \mathbb{N} \to \mathbb{R}$ test function

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{k \geq 1} \psi(k) n_k(t) = \frac{1}{2} \sum_{h,k \geq 1} K(h, k) n_h(t) n_k(t) \big(\psi(h+k) - \psi(h) - \psi(k)\big).$$

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Attention: if $\psi(k) = k$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{k \geq 1} k n_k(t) \leq 0.$$

Gelation (phase transition)

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Gelation criteria for spatially homogeneous coagulation processes

$K(m, n) \geq (mn)^\alpha, \quad \alpha > \frac{1}{2}$ [JEON (1998)]

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$$K(cm,cn) = c^{\gamma} K(m,n), \qquad \gamma > 1$$

[A., Iyer, Magnanini, (2024)]

[Fournier (2025)]

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$$M_1^{(N)}(t) \geq M_2^{(N)}(t) \geq M_3^{(N)}(t) \geq \cdots \geq M_{n(t)}^{(N)}(t) \geq 1, \quad \text{s.t.} \quad \sum_{i=1}^{n(t)} M_i^{(N)}(t) = N.$$

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Coagulation mechanism

To each pair of particle with masses (m, n) you associate exponential random clocks (independent) with parameter

$$K_N(m, n) := \frac{K(m, n)}{N}.$$

A common framework

Define

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$\{\mathbf{L}^{(N)}(t)\}_{t\in[0,\infty)}$ is a Markov process with infinitesimal generator

$$\mathcal{A}^{(N)} = \sum_{m,n=1}^{\infty} \mathcal{A}_{m,n}^{(N)}$$

where, for $F\colon \mathcal{E}_N \to \mathbb{R}$,

$$\mathcal{A}_{m,n}^{(N)} F(\ell) = \frac{1}{2N} K(m,n) \big(\ell_m \ell_n - \mathbf{1}(m=n)\ell_m\big) \big(F(\ell + \mathbf{e}_{n+m} - \mathbf{e}_n - \mathbf{e}_m) - F(\ell)\big).$$

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Recall: Dynkin's formula.

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. . . **blackboard!**

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Theorem (LLN, CLT and LDP)

Let (X_i) be i.i.d. $\mathbb{R}$-valued random variables satisfying

$$\varphi(t) = \mathbb{E}e^{tX_1} < \infty \qquad \forall t \in \mathbb{R}.$$

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- *(LDP)* *For all $a > \mu$,*

$$\lim_{n\to\infty} \frac{1}{n}\log \mathbb{P}(\frac{1}{n}S_n \geq a) = -\inf_{z\geq a} I(z),$$

where

$$I(z) = \sup_{t\in\mathbb{R}}\big[zt - \log\varphi(t)\big].$$

Back to our case: LLN

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Back to our case: **pathwise** LLN

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- weak topology: testing against bounded functions ψ:

$$\langle \psi, f \rangle;$$

- vague topology: testing against compactly supported functions ψ:

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Theorem (Norris, 99)

Assume that the coagulation kernel has at most linear growth:

$$\sup_{m,n\geq 1}\frac{K(m,n)}{m+n} < \infty.$$

The sequence of laws

$$\left\{\mathcal{P}_N = Law\left\{\frac{1}{N}\mathbf{L}^{(N)}(t)\right\}_{t\in[0,\infty)}\right\}_{N\in\mathbb{N}}$$

is tight. Moreover, every limit point $\mathcal{P}^*$ is concentrated on the unique solution of the **Smoluchowski equation** *with kernel K and monodisperse initial condition.*

And gelling kernels?

Flory: a modified Smoluchowski equation

$$\lim_{m\to\infty} \frac{K(m,n)}{m} = \beta(n), \qquad \forall n \in \mathbb{N}. \tag{1}$$

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Recall: Smoluchowski equation

$$\frac{\mathrm{d}}{\mathrm{d}t} n_k(t) = Q_k^+(n)(t) - Q_k^-(n)(t)$$

for all $k \geq 1$, where

$$Q_k^+(n)(t) = \frac{1}{2}\sum_{h=1}^{k-1} K(h, k-h) n_h(t) n_{k-h}(t) \qquad \text{gain term},$$

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with $g_\infty(t) = 1 - \sum_{k\geq 1} k n_k(t)$.

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Theorem (Fournier, Giet, 05)

Assume (1)*, then the sequence of laws* $\{\mathcal{P}_N\}_{N\in\mathbb{N}}$ *is tight. Moreover, every limit point* $\mathcal{P}^*$ *is concentrated on the set of solutions of the* **Flory equation** *with kernel* K *and monodisperse initial condition.*

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(CLT)

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where X is the solution of some **infinite dimensional SDE**.

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Some results:

- Non-gelling kernels
  [KOLOKOLTSOV, 2010], [AMORIM, ARELLANO, JARA, 2026]
- Multiplicative kernel (combinatorial methods)
  [BHAMIDI, BUDHIRAJA, SAKANAVEETI, 2024]

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- Theory well developed only for jump markov processes on **finite dimensional space**.
  $\mathcal{E}$ is infinite dimensional!