Branching Diffusions and generalized principal eigenvalues

MINT Summer School course, June 15-19, 2026

Pascal Maillard, Univ. Toulouse

0) INTRODUCTION

$E \subset \mathbb{R}^d$ domain. Consider elliptic operator

$$Lf = \sum_{i,j} \frac{1}{2} a_{ij}(x) \partial_{ij} f + \sum_i b_i(x) \partial_i f + c(x) f,$$

coeffs smooth, bounded, unif. elliptic, Dirichlet bdry cond.

Berestycki-Nirenberg-Varadhan 1994, Pinsky 1995:
Intrinsic notion of principal eigenvalue λ_1.
Useful mostly for bounded domains.
Unbounded domains: Berestycki-Rossi 2018: λ_1', λ_1''.

In this course: Approach to these e.v. through branching diffusions.

Diffusions: Consider $c \equiv 0$. L is the infinitesimal generator of a diffusion, i.e. the solution to the SDE

$$dX_t = \sqrt{A(X_t)}\, dW_t + B(X_t)\, dt, \quad (W_t)_{t \geq 0} \text{ d-dim Brownian Motion}$$

$A = (a_{ij})$, $B = (b_i)$, with killing at ∂E.

Then $u(t,x) := \mathbb{E}_x[f(X_t)]$ is solution of (f bdd, smooth)

$$(*)\begin{cases} \partial_t u = Lu, & E \\ u(t,\cdot) = 0, & \partial E \\ u(0,\cdot) = f \end{cases}$$

For general c, set $u(t,x) = \mathbb{E}_x\left[f(X_t)\exp\left(\int_0^t c(X_s)\,ds\right)\right]$ (Feynman-Kac formula). Alternatively: Branching Diffusion:

- particles diffuse independently according to $(X_t)_{t\geq 0}$,
- branch at rate $c(x)_+ = c(x) \vee 0$
- die at rate $c(x)_- = (-c(x)) \vee 0$

Then $u(t,x) = \mathbb{E}_x\left[\sum_{u \in N_t} f(X_t^u)\right]$ solves $(*)$

particles at time t — position of particle u at time t

Engländer-Kyprianou '05: Branching diffusion survives locally with positive prob. iff $\lambda_1 > 0$

(local survival = every $K \subset\subset E$ (compactly embedded) is visited infinitely often)

We will see:

<u>Theorem (M., Tough[125])</u>: ~~Branching~~
Branching diffusion survives <u>globally</u> if $\lambda_1' > 0$
— " — dies out " if $\lambda_1' < 0$

Moreover, we will discuss:

- criteria for $\lambda_1 = \lambda_1' = \lambda_1''$
- counterexamples
- implications for FKPP equation in unbounded domains.

I) DISCRETE-TIME RESULTS

1) Bienaymé-Galton-Watson processes (Harris 1963, Athreya-Ney 1972)

L: random variable taking values in $\mathbb{N}_0$.

$(L_{n,k})_{n\geq 1, k\geq 1}$: iid (independent, identically distributed) copies of L.

$$Z_0 = z_0 \in \mathbb{N}_0, \quad \forall n \geq 0: \; Z_{n+1} = \sum_{k=1}^{Z_n} L_{n+1,k}.$$

$(Z_n)_{n\geq 0}$ Bienaymé-Galton-Watson (BGW) process with offspring distribution law(L). ("L-BGW")

<u>Properties</u>:

- Markov chain on $\mathbb{N}_0$, 0 is absorbing state
- <u>Branching property</u>: If $(Z_n)_{n\geq 0}$, $(Z_n')_{n\geq 0}$ are independent L-BGW starting from $z_0, z_0' \in \mathbb{N}_0$, resp., then $(Z_n + Z_n')_{n\geq 0}$ is L-BGW starting from $z_0 + z_0'$.

- Notation: $\mathbb{P}_{Z_0}$, $\mathbb{E}_{Z_0}$
- $\mathbb{E}_{Z_0}[Z_n] = Z_0 \cdot m^n$, $m := \mathbb{E}[L] \in [0,\infty]$
- $f_n(s) := \mathbb{E}_1[s^{Z_n}]$, $s \in [0,1]$.

<u>Thm</u> (classical): a) $\mathbb{E}_{Z_0}[s^{Z_n}] = f_n(s)^{Z_0}$

b) $f_n(s) = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$, $f(s) = \mathbb{E}[s^L] = f_1(s)$.

<u>Proof</u>: a) By branching property, if $Z_n^{(1)}, Z_n^{(2)}, \ldots$ are iid copies of Z_n under $\mathbb{P}_1$,

$$\mathbb{E}_{Z_0}[s^{Z_n}] = \mathbb{E}[s^{Z_n^{(1)} + \cdots + Z_n^{(Z_0)}}]$$

$$= \mathbb{E}_1[s^{Z_n^{(1)}}] \times \cdots \times \mathbb{E}[s^{Z_n^{(Z_0)}}] \quad \text{(independence)}$$

$$= \mathbb{E}_1[s^{Z_n}]^{Z_0}$$

$$= f_n(s)^{Z_0}$$

b) $n=0$: $f_0(s) = s = \mathbb{E}_1[s^{Z_0}]$

$n \to n+1$: $f_{n+1}(s) = \mathbb{E}_1[s^{Z_{n+1}}]$

$$= \mathbb{E}_1[\mathbb{E}[s^{Z_{n+1}} \mid Z_1]]$$

$$= \mathbb{E}_1[\mathbb{E}_{Z_1}[s^{Z_n'}]] \quad \text{(Markov property)}$$

$$= \mathbb{E}_1[f_n(s)^{Z_1}] \quad \text{(part a))}$$

$$= \mathbb{E}[f_n(s)^L]$$

$$= (f \circ f_n)(s)$$

<u>Properties of f</u>: • $f(s) = \sum_{k=0}^{\infty} p_k s^k$, $p_k = \mathbb{P}(L=k)$

• $f(0) = p_0 \in [0,1]$, $f(1) = 1$

• f convex on $[0,1]$, non-decreasing, strictly convex if $p_0 + p_1 < 1$.

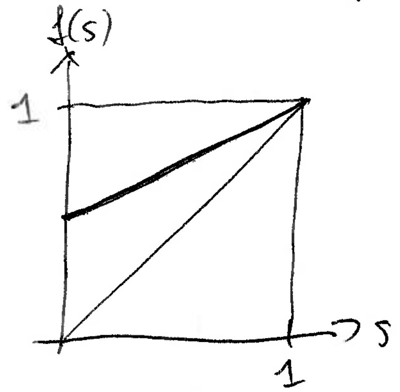
• $f'(1) = m$.

Three cases:

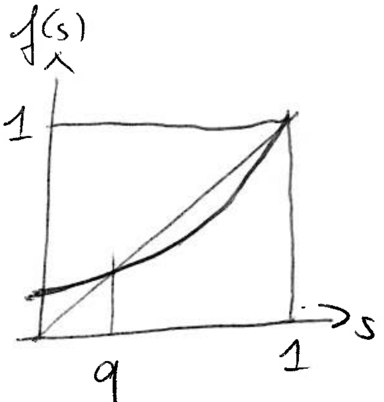
a) $m \leq 1$, $p_1 \neq 1$: 1 fixed point $(s=1)$

b) $m = 1$, $p_1 = 1$: $f = id$, every $s \in [0,1]$ is fixed point.

c) $m > 1$: 2 fixed points: $\{q, 1\}$, $q < 1$.



$m \leq 1$



$m > 1$

Case a): Fixed point $s=1$ is stable $\Rightarrow f_n(0) = \mathbb{P}_1(Z_n = 0) \xrightarrow{n\to\infty} 1$

Case b): $\mathbb{P}_1(Z_n=0) = f_n(0) = 0 \quad \forall n$

Case c): Fixed point q is stable, 1 is unstable, since $f \nearrow$, $f_n(0) \xrightarrow{n\to\infty} q$.

In all cases: if we set $\mathrm{Surv} = \{Z_n \not\to 0\}$, $\mathrm{Ext} = \{Z_n \to 0\}$

$$\mathbb{P}_1(\mathrm{Surv}) > 0 \iff m > 1 \text{ or } p_1(1)$$

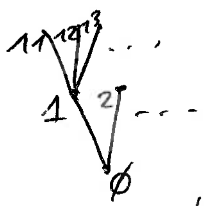
$$\mathbb{P}_1(\mathrm{Ext}) = \text{smallest fixed point of } f.$$

2) Branching Markov chains (following Jagers 1989)

E : measurable space ("state space")

$\partial \notin E$: isolated point, $\hat{E} = E \cup \{\partial\}$ (∂: "cemetery symbol")

Π : probability kernel from $\hat{E}$ to $\hat{E}^{\mathbb{N}}$ ("offspring distribution")
with $\Pi(\partial, \cdot) = \delta_{\partial^{\mathbb{N}}}$, $\partial^{\mathbb{N}} = (\partial, \partial, \dots)$



$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$ ("Ulam-Harris labels")

$u \in \mathcal{U}$, $u = u_1 \cdots u_n$, $|u| := n$. "length / generation of u"

<u>Def:</u> $(X_u)_{u\in\mathcal{U}}$ is a Branching Markov Chain (BMC) with offspring distribution Π starting from $x \in \hat{E}$, if $X_\emptyset = x$ and for all $n \in \mathbb{N}_0$, conditioned on $\sigma((X_u)_{|u|\le n})$, the random vectors $(X_{ui})_{i\in\mathbb{N}}$, $u \in \mathbb{N}^n$ are independent and follow the law $\Pi(X_u, \cdot)$.

<u>Existence</u>: Construct $(X_u)_{u\in\mathcal{U}, |u|\le n}$ by recurrence, extend to $(X_u)_{u\in\mathcal{U}}$ by Ionescu-Tulcea theorem

<u>Uniqueness</u>: Consequence of monotone class theorem.

<u>Notation</u>: • $\mathbb{P}_x$, $\mathbb{E}_x$.

• $N_n = \sum_{|u|=n} \mathbb{1}_{X_u \neq \partial}$: number of particles at time n.

• $glSurv = \{N_n \overset{n\to\infty}{\not\to} 0\}$, $glExt = \{N_n \xrightarrow{n\to\infty} 0\}$.

• In what follows, in $\prod_{i\in\mathbb{N}} f(X_i)$, $\prod_{|u|=n} f(X_u)$, $\sum_{|u|=n} f(X_u)$, etc...,
<u>we do not take product or sum over i, u s.t. $X_u = \partial$.</u>

- $pB(E) = \{ f : E \to [0,\infty] \text{ measurable} \}$

  $pB_b(E) = \{ f \in pB(E) : f \text{ bounded} \}$

  $pB_1(E) = \{ f \in pB(E) : f \leq 1 \}$

- $\Pi : pB_1(E) \to pB_1(E)$, $\Pi f(x) = \mathbb{E}_x \left[\prod_{i \in N} f(X_i) \right]$

  "probability generating functional"

- $m : pB(E) \to pB(E)$, $mf(x) = \mathbb{E}_x \left[\sum_{i \in N} f(X_i) \right]$

  "mean offspring kernel"

- Π^n, m^n: n-fold iterates,

  $\Pi^n f(x) = \mathbb{E}_x \left[\prod_{|u|=n} f(X_u) \right]$, $m^n f(x) = \mathbb{E}_x \left[\sum_{|u|=n} f(X_u) \right]$

Properties of Π^n, m^n, $n \geq 0$:

- positivity: If $f \leq g$, $\Pi^n f \leq \Pi^n g$ and $m^n f \leq m^n g$.

  ($\Rightarrow$ if $f = \lim_{k\to\infty} \uparrow f_k$, then $\Pi^n f \geq \lim_{k \to \infty} \uparrow \Pi^n f_k$)

- m^n is continuous (under increasing limits), by the monotone convergence theorem

---

Introduce assumption:

(C) Π^n is continuous under increasing limits

Then (C) holds for example if (exercise):

- $N_1 < \infty$ $\mathbb{P}_x$-a.s., for every $x \in E$
- E is finite

<u>Theorem</u>: Set $q(x) := \mathbb{P}_x(\mathcal{G}(Ext)$. Then $\Pi q \geq q$, with equality if (C) holds. Moreover, if $f \in pB_1(E)$ s.t. $\Pi f \leq f$, then $q \leq f$. In particular, if (C) holds, then q is smallest solution to $\Pi f = f$.

Proof: By definition, $q(x) = \lim_{n\to\infty} \uparrow P_x(N_n = 0) = \lim_{n\to\infty} \uparrow (\Pi^n 0)(x)$,
where 0 is the zero function $0(x) = 0$.

It follows that $\Pi q \geq \lim_{n\to\infty} \uparrow \Pi \Pi^n 0 = \lim_{n\to\infty} \uparrow \Pi^{n+1} 0 = q$.

Moreover, $\Pi q = q$ if Π is continuous under increasing limits.

Finally, if $f \in p\mathcal{B}_1(E)$ s.t. $\Pi f \leq f$, then $\Pi^n f \leq f$ $\forall n \geq 0$

and $q \leq \lim_{n\to\infty} \uparrow \Pi^n 0 \leq \lim_{n\to\infty} \uparrow \Pi^n f \leq f \Rightarrow q \leq f$. ∎

Q: Criterion in terms of m ??

3) Finite state space

Suppose E finite. Then m is a matrix $m(x,y)_{x,y\in E}$.

$m f(x) = \sum_{y\in E} m(x,y) f(y)$, m^n = n-th matrix power

m is irreducible if $\forall x,y \in E \; \exists n \geq 1 : m^n(x,y) > 0$.

Perron-Frobenius theorem: Suppose m irreducible.
then $\exists \rho_c = \rho_c(m) > 0$, $v, w \in (0,\infty)^E$, s.t.

- $m v = \rho_c v$, $w^T m = \rho_c w^T$
- $\rho_c = \limsup_{n\to\infty} (m^n(x,y))^{\frac{1}{n}}$, $\forall x,y \in E$.

Theorem (see Harris 1963):

If $\rho_c < 1$, then $q \equiv 1$
If $\rho_c > 1$, then $q(x) < 1$ $\forall x \in E$

(we ignore critical case $\rho_c = 1$ here)

Proof: If $\rho_c < 1$, then $\forall x \in E$:

$$\limsup_{n\to\infty} \mathbb{E}_x[N_n]^{\frac{1}{n}} = \limsup_{n\to\infty} \Big(\sum_{y\in E} m^n(x,y) \Big)^{\frac{1}{n}} \overset{E \text{ finite}}{=} \rho_c < 1$$

$$\Rightarrow \mathbb{E}_x[N_n] \xrightarrow{n\to\infty} 0 \Rightarrow \mathbb{P}_x(N_n = 0) \xrightarrow{n\to\infty} 1$$

If $\rho_c > 1$, two proofs:

a) Let $x \in E$. Since $\limsup_{n\to\infty} (m^n(x,x))^{\frac{1}{n}} > 1$, there exists $n_0 \geq 1$, such that $m^{n_0}(x,x) > 1$.

Consider $Z_k = \sum_{|u| = k \cdot n_0} \mathbb{1}(X_{u|_{jn_0}} = x \quad \forall j \leq k) \leq N_{kn_0}$,

where $u|_m = u_1 \cdots u_m$, if $u = u_1 \cdots u_n$, $m \leq n$.

Then $(Z_k)_{k\geq 0}$ is a BGW starting from 1, with mean offspring $m^{n_0}(x,x) > 1$. Hence, $\mathbb{P}_x(N_n \xrightarrow{n\to\infty} 0) \leq \mathbb{P}_x(Z_k \xrightarrow{k\to\infty} 0) < 1$.

---

b) Let $v > 0$ s.t. $mv = \rho_c v$. Set: $f_t(x) := e^{-tv(x)}$,

$W_n = \sum_{|u|=n} v(X_u)$, $M_n(t) := e^{-tW_n}$. Then,

$\mathbb{E}_x[W_n] = m^n v(x)$, $\mathbb{E}_x[M_n(t)] = \pi^n f_t(x)$.

We have $\forall x \in E$: $-\frac{d}{dt} \pi f_t(x)\Big|_{t=0} = -\frac{d}{dt} \mathbb{E}_x[e^{-tW_1}]\Big|_{t=0} = \mathbb{E}_x[W_1] = \rho_c v(x)$.

$\Rightarrow \forall x \in E$: $\pi f_t(x) = 1 - t\rho_c v(x) + o(1)$, $t \to 0$

$\overset{\rho_c > 1}{\Rightarrow} \exists t > 0$: $\forall x \in E$: $\pi f_t(x) \leq e^{-tv(x)} = f_t(x) \Rightarrow \forall n \geq 0$: $\pi^n f_t \leq f_t$.

$\overset{\forall x \in E}{\Rightarrow} \forall n \in \mathbb{N}$: $\mathbb{P}_x(N_n = 0) \leq \mathbb{E}_x[M_n(t)] = \pi^n f_t(x) \leq f_t(x) = e^{-tv(x)} < 1$.

$\Rightarrow \forall x \in E$: $q(x) = \lim_{n\to\infty} \uparrow \mathbb{P}_x(N_n = 0) < 1$. ∎

4) General state space

(Back to E general)

Theorem: a) Suppose (C) holds.
Let $x_0 \in E$, suppose $\mathbb{P}_{x_0}(\text{gl Surv}) > 0$. Then there exists
$f \in pB_1(E)$: $mf \geq f$, $f(x_0) > 0$.

b) Suppose: $\exists K$ constant, s.t. $N_1 \leq K$ $\mathbb{P}_x$-a.s. $\forall x \in E$. (K)
Let $x_0 \in E$, suppose $\exists f \in pB_1(E)$; $\exists \rho > 1$: $mf \geq \rho f$, $f(x_0) > 0$.
Then $\mathbb{P}_x(\text{gl Surv}) > 0$.

Proof: a) Since $\prod(1-x_i) \geq 1 - \sum x_i$ $(x_i \geq 0)$, we have
$\prod(1-f) \geq 1 - mf \Rightarrow mf \geq 1 - \prod(1-f)$.
Set $f(x) := \mathbb{P}_x(\text{gl Surv}) = 1 - q(x)$. Then $\prod(1-f) = 1-f$
by theorem from Section 2. $\Rightarrow mf \geq f$.

b) Similar to proof of finite state space theorem:
Set $M_n(t) = \exp(-t \sum_{|u|=n} f(X_u))$. We have

$$M_1(t) = 1 - t \sum_{i \in \mathbb{N}} f(X_i) + O\left(t^2 \left(\sum_{i \in \mathbb{N}} f(X_i)\right)^2\right)$$
$$= 1 - (t + O(Kt^2)) \sum f(X_i)$$

Let $t > 0$ small enough, s.t. $t + O(Kt^2) \geq \frac{t}{\rho}$. Then,

$$\mathbb{E}_x[M_1(t)] \leq 1 - \frac{t}{\rho}\mathbb{E}_x\left[\sum f(X_i)\right] = 1 - \frac{t}{\rho} mf(x) \leq 1 - tf(x) \leq e^{-tf(x)}$$

As before (finite state space):

$$\mathbb{E}_x[M_n(t)] \leq e^{-tf(x)} \Rightarrow q(x_0) \leq e^{-tf(x_0)} < 1. \quad \blacksquare$$

Remarks:

- b) holds also under weaker assumption:
  $\exists f \in pB_1(E), f(x_0) > 0, \exists \varepsilon > 0$:
  $\forall x \in E: \mathbb{E}_x\left[\sum_{i \in \mathbb{N}} f(X_i) e^{-\varepsilon f(X_i)}\right] \geq f(x)$
- Exercise: Under b), prove that $\mathbb{P}_x(\liminf N_n^{\frac{1}{n}} \geq \rho) > 0$.

5) Generalized principal eigenvalues

Fix $x \in E$. We set

$\rho_{c,x} = \rho_{c,x}(m) = \inf\{\rho \geq 0 : \exists u \in pB(E) : mu \leq \rho u,\ u > 0,\ u(x) < \infty\}$

$\rho_{c,x}' = \rho_{c,x}'(m) = \sup\{\rho \geq 0 : \exists u \in pB_1(E) : mu \geq \rho u,\ u(x) > 0\}$

$\rho_{c,x}'' = \rho_{c,x}''(m) = \inf\{\rho \geq 0 : \exists u \in pB(E) : mu \leq \rho u,\ u \geq 1,\ u(x) < \infty\}$

Remark: Obviously, $\rho_{c,x} \leq \rho_{c,x}''$, but relation with $\rho_{c,x}'$ a priori not clear!

Corollary (of previous theorem): Let $x \in E$. Suppose (C) holds.

a) Suppose $\rho_{c,x}' < 1$. Then $\mathbb{P}_x(\text{glSurv}) = 0$.

b) Suppose $\rho_{c,x}' > 1$, and (K) holds. Then $\mathbb{P}_x(\text{glSurv}) > 0$.

Theorem:

$$\forall x \in E: \quad \rho_{c,x}' \leq \liminf_{n \to \infty} (\mathbb{E}_x[N_n])^{\frac{1}{n}} \leq \limsup_{n \to \infty} (\mathbb{E}_x[N_n])^{\frac{1}{n}} = \rho_{c,x}''$$

Proof: Exercise.

Definition:

$$\rho_c(m) := \sup_{x\in E} \rho_{c,x}(m) = \inf\{\rho\geq 0 : \exists u\in pB(E) : mu \leq \rho u,\ 0<u<\infty\}$$

$$\rho_c'(m) := \inf_{x\in E} \rho_{c,x}'(m) = \sup\{\rho\geq 0 : \exists u\in pB_1(E) : mu \geq \rho u,\ u>0\}$$

$$\rho_c''(m) := \sup_{x\in E} \rho_{c,x}''(m) = \inf\{\rho\geq 0 : \exists u\in pB(E) : mu \leq \rho u,\ 1\leq u<\infty\}$$

Corollary: $\rho_c' \leq \inf_{x\in E} \liminf_{n\to\infty} \mathbb{E}_x[N_n]^{\frac{1}{n}} \leq \sup_{x\in E} \limsup_{n\to\infty} \mathbb{E}_x[N_n]^{\frac{1}{n}} \leq \rho_c''$.

6) Countable state space, local survival.

Assume E countable or finite. Then m is an infinite matrix $m(x,y)$ and $mf(x) = \sum_{y\in E} m(x,y) f(y)$. Assume:

- m is irreducible: $\forall x,y\in E\ \exists n\geq 1 : m^n(x,y) > 0$
- $m^n(x,y) < \infty \quad \forall n\geq 0,\ x,y\in E$

Exercise: $\forall x\in E : \rho_c'' = \rho_{c,x}''$. If (K) holds, $\forall x\in E : \rho_c' = \rho_{c,x}'$

Theorem (see Seneta 1973):

1. $\forall x,y\in E$: $\rho_c = \limsup_{n\to\infty} (m^n(x,y))^{\frac{1}{n}}$
2. For $F\subset E$, define $m_F(x,y) = m(x,y)\mathbb{1}_{x,y\in F}$. Then,

$$\rho_c(m) = \sup_{F\subset E,\ |F|<\infty,\ m_F \text{ irreducible}} \rho_c(m_F).$$

Remark: ρ_c is called "spectral radius"

Def: For $x\in E$, $\mathrm{locSurv}_x := \left\{ \sum_{|u|=n} \mathbb{1}_{X_u = x} > 0 \text{ infinitely often} \right\}$

$$\mathrm{locSurv} := \bigcap_{x\in E} \mathrm{locSurv}_x.$$

<u>Exercise</u>: $\forall x,y \in E$: $\mathbb{P}_x(\text{locSurv}) > 0 \iff \mathbb{P}_y(\text{locSurv}) > 0$.

<u>Theorem</u>: (Müller '08, Zucca '11)

$$\mathbb{P}(\text{locSurv}) > 0 \iff \rho_c > 1.$$

<u>Proof</u>: Two proofs: 1. Use $\rho_c = \limsup_{n\to\infty} (m^n(x,x))^{\frac{1}{n}}$ to find embedded BGW process with mean > 1 (see proof for finite E).

2. Let $F \subset E$ finite, m_F irreducible, s.t. $\rho_c(m_F) > 1$. Now use survival result for finite F. ∎

<u>Corollary</u>: $\rho_c \leq \rho_c'$

<u>Proof</u>: By linearity, it is enough to show that $\rho_c > 1$ implies $\rho_c' \geq 1$. First assume (C) holds. Then,

$\rho_c > 1 \Rightarrow \mathbb{P}(\text{locSurv}) > 0 \Rightarrow \mathbb{P}(\text{glSurv}) > 0 \overset{\text{Thm}}{\Rightarrow} \rho_c' \geq 1$.

In general, restrict to large enough finite $F \subset E$, s.t. m_F is irreducible and $\rho_c(m_F) > 1$, so that (C) holds.

<u>Corollary</u>: $\rho_c \leq \rho_c' \leq \rho_c''$.

<u>Theorem</u>: Suppose m is symmetric ($m(x,y) = m(y,x)\ \forall x,y \in E$), and that there exists $x \in E$, s.t. $\limsup_{n\to\infty} |A_n|^{\frac{1}{n}} \leq 1$, where $A_n = \{y \in E : m^n(x,y) > 0\}$.

Then, $\rho_c = \rho_c' = \rho_c''$

Proof: We only have to show $\rho_c \geq \rho_c''$. For $n \geq 0$,

$$m^{2n}(x,x) = \sum_{y \in A_n} m^n(x,y)\, m^n(y,x)$$

$$= \sum_{y \in A_n} \left(m^n(x,y)\right)^2 \qquad \text{(symmetry)}$$

$$\geq \frac{1}{|A_n|} \left(\sum_{y \in A_n} m^n(x,y) \right)^2 \qquad \text{(Cauchy-Schwarz)}$$

$$= \frac{1}{|A_n|} \mathbb{E}_x[N_n]^2.$$

$$\Rightarrow \rho_c \geq \limsup_{n\to\infty} m^{2n}(x,x)^{\frac{1}{2n}} \geq \limsup_{n\to\infty} \frac{1}{|A_n|^{\frac{1}{2n}}} \mathbb{E}_x[N_n]^{\frac{1}{n}} \geq \rho_c'',$$

by assumption on $|A_n|$ and expression for ρ_c''. ∎