

Cumulants, their evolution hierarchies, and how to use them to control chaos for kinetic theory

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based on joint works with

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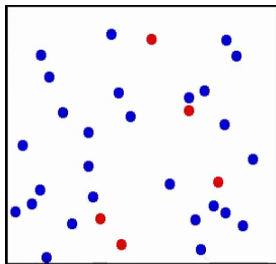
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IN RANDOMNESS
AND STRUCTURES
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Recently, important advances have been made in understanding of mathematical foundations of kinetic theory. For example,

- 1 For kinetic theory of waves (wave turbulence) in *nonlinear Schrödinger equation (NLS)*
 - Deng and Hani [arXiv:2104.11204, 2110.04565, ...]
 - Building on earlier works: Buckmaster, Germain, Shatah, ...
 - Other closely related results: JL & Spohn, Staffilani & Tran, Ampatzoglou & Collot & Germain, Dymov, ...

- 2 For kinetic theory of *rarefied gas*
 = original “billiard ball” model of Boltzmann
 - Deng, Hani, Ma [arXiv:2408.07818, 2503.01800, ...]
 - Earlier & related works: Lanford, Cercignani, ...
 Bodineau, Gallagher, Saint-Raymond, Pulvirenti, Simonella, ...

Original rarefied gas model



(Wikipedia / Kinetic theory of gases)

Deterministic evolution but random initial data:

- N particles, random initial position and velocity
- Between collisions, move in straight lines
- When two particles collide, update their velocities according to the elastic collision rules

Boltzmann equation from a scaling limit

Boltzmann–Grad scaling limit:

- 1 Particle radius $r_0 \rightarrow 0$
- 2 Assume $N(2r_0)^{d-1} \rightarrow c_0 > 0$ (*rarefied gas*) $\Rightarrow N \rightarrow \infty$
- 3 Assume that the particle positions and velocities are initially ($t = 0$) sufficiently independent and “well-prepared”

Define the *collision operator*

$$\begin{aligned} \mathcal{C}_b[h](\mathbf{v}_0) &= 2c_0 \int_{(\mathbb{R}^d)^3} d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{v}_3 \delta(\mathbf{v}_0 + \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3) \\ &\quad \times \delta(|\mathbf{v}_0|^2 + |\mathbf{v}_1|^2 - |\mathbf{v}_2|^2 - |\mathbf{v}_3|^2) (h(\mathbf{v}_2)h(\mathbf{v}_3) - h(\mathbf{v}_0)h(\mathbf{v}_1)) \end{aligned}$$

Theorem (Lanford, Deng–Hani–Ma)

In the limit $r_0 \rightarrow 0$, the “one-particle distribution function” f_t satisfies

$$\partial_t f_t(\mathbf{x}, \mathbf{v}) + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_t(\mathbf{x}, \mathbf{v}) = \mathcal{C}_b[f_t(\mathbf{x}, \cdot)](\mathbf{v})$$

Key notion **chaos, as statistical independence**

This is also the **key difficulty**:

Need to control propagation of **approximate** independence

Even if initial state has complete independence of velocities, it no longer holds after the first collision

- Earlier, control independence via “almost factorization” of moments and moment hierarchies (BBGKY)
- We show that cumulants can do this better, at least in a simplified model called stochastic Kac model

Propagation versus generation of chaos

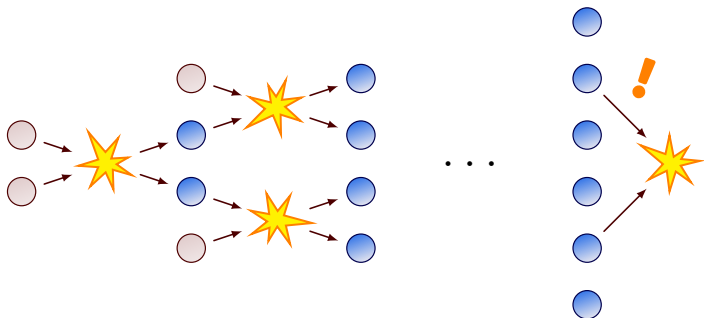
Chaos in kinetic theory

In most of the mathematical examples of derivation of a kinetic theory, starting from the rarefied gas Boltzmann equation, “chaos” refers to (approximate) **statistical independence** of some of the evolving quantities

Propagation of chaos means that if the above “evolving quantities” are (sufficiently) independent in the beginning, they remain so at least up to kinetic time scales

Generation of chaos means that even if the above “evolving quantities” are not independent in the beginning, they will become so later, at least in approximation and on some time-scale which need not be connected with kinetic theory

Stochastic Kac model



Stochastic evolution:

- N particles, consider only their velocities
- Poisson clock determines when particles collide
- Pick the colliding pair at random
- Outgoing velocities random, but preserve total energy

Accuracy of the Boltzmann–Kac evolution

- 1 Arbitrary initial distribution (total energy = N)
- 2 Wait until time t_0 such that the state is already chaotic,
 $t_0 = O(\ln N)$
- 3 At time t_0 , compute the first marginal $f_0(v)$ of $F_{t_0}^N$ and solve the Boltzmann–Kac equation (yielding $\nu_t := f_t(v)dv$)

$$\partial_t f_t(v) = 2 \int_{-\pi}^{\pi} \int_{\mathbb{R}} (f_t(v')f_t(w') - f_t(v)f_t(w)) dw \frac{d\theta}{2\pi}$$

- 4 Construct product measures $\tilde{F}_T^N := \otimes_{i=1}^N \nu_T$ on \mathbb{R}^N

How close are the cumulants of particle energies, $e_i = v_i^2$, of these two measures for large N ? (no scaling limits!)

Theorem (accuracy of the “Boltzmann–Kac” hierarchy)

$$|\kappa_{t_0+T}^{n,N}[e_r] - \tilde{\kappa}_T^{n,N}[e_r]| \leq 2(N-1)^{-\alpha} C^{n^2} n! = O(N^{-\alpha})$$

Part I

Cumulant hierarchy and Wick polynomials

[JL and M. Marozzi, *J. Math. Phys.* **57** (2016) 083301]

Why study cumulants?

Observation: If y, z are **independent** (positive) random variables, we have

$$\mathbb{E}[y^n z^m] = \mathbb{E}[y^n] \mathbb{E}[z^m] \neq 0$$

whereas the corresponding cumulant is zero if both $n, m \neq 0$

Our key hypothesis

Smallness of joint cumulants is a good measure of “almost independence” for a collection of random variables

- NB: $\kappa[X_1, X_2, \dots, X_n] = 0$ if *for any* j the random variable X_j is independent from the rest
- For “almost independence”, we assume $\kappa[X_1, X_2, \dots, X_n] \approx 0$
- For corresponding moments ($j = 1$),
 $\mathbb{E}[X_1 X_2 \cdots X_n] \approx \mathbb{E}[X_1] \mathbb{E}[X_2 \cdots X_n]$ which can be harder to track

Cumulants and Wick polynomials

Moments and cumulants can be obtained from generating functions

$$g_c(\eta) := \ln g_{\text{mom}}(\eta), \quad g_{\text{mom}}(\eta) := \mathbb{E}[e^{\sum_{i=1}^n \eta_i X_i}]$$

$$\Rightarrow \kappa[X_J] = \partial_\eta^J g_c(0), \quad \mathbb{E}[X^J] = \partial_\eta^J g_{\text{mom}}(0)$$

$$X_J = (X_i)_{i \in J}, \quad \partial_\eta^J := \prod_{i \in J} \partial_{\eta_i}, \quad X^J = \prod_{i \in J} X_i$$

For a deterministic time-evolution, $y = y(t)$,

$$\begin{aligned} \partial_t \kappa[X(t)_J] &= \partial_\eta^J \partial_t g_c(\eta) \Big|_{\eta=0} = \partial_\eta^J \mathbb{E}[\eta \cdot (\partial_t X(t)) G_w(X(t), \eta)] \Big|_{\eta=0} \\ &= \sum_{\ell \in J} \mathbb{E}[\partial_t X(t)_\ell : X^{J \setminus \ell}] \Rightarrow \text{(evolution hierarchy of cumulants)} \end{aligned}$$

Involves Wick polynomials $:X^J:$ and their generating functions

$$G_w(z, \eta) := \frac{e^{\eta \cdot z}}{\mathbb{E}[e^{\eta \cdot X}]} = e^{\eta \cdot z - g_c(\eta)} \quad \Rightarrow \quad :X^J: = \partial_\eta^J G_w(X, 0)$$

- WP have been mainly used for Gaussian fields. They were introduced in quantum field theory where the unperturbed measure concerns Gaussian (free) fields
- **Gaussian case** has significant simplifications: If $C_{j'j} = \kappa[X_{j'}, X_j]$ denotes the *covariance matrix*,

$$G_w(y, \eta) = \exp[\eta \cdot (y - \langle y \rangle) - \eta \cdot C\eta/2].$$

- ⇒ Cumulants of order ≥ 3 are *zero*,
& Wick polynomials are *Hermite polynomials*

Wick polynomials and moments-to-cumulants formula 13

Truncated moments-to-cumulants formula

$$\mathbb{E}\left[X^{J'} : X^J\right] = \sum_{\pi \in \mathcal{P}(J' \cup J)} \prod_{A \in \pi} (\kappa[X_A] \mathbb{1}_{\{A \not\subset J\}}) \quad (1)$$

- $:X^J$: are μ -a.s. unique polynomials of order $|J|$ such that (1) holds for every J'

Multi-truncated moments-to-cumulants formula

Suppose $L \geq 1$ is given and consider a collection of $L + 1$ index sequences $J', J_\ell, \ell = 1, \dots, L$. Then with $I = J' \cup (\cup_{\ell=1}^L J_\ell)$

$$\mathbb{E}\left[X^{J'} \prod_{\ell=1}^L :X^{J_\ell}\right] = \sum_{\pi \in \mathcal{P}(I)} \prod_{A \in \pi} (\kappa[X_A] \mathbb{1}_{\{A \not\subset J_\ell, \forall \ell\}})$$

Part II

Stochastic Kac model and its cumulant hierarchy

[JL, A Vuoksenmaa, arXiv:2407.17068]

*It was pointed out to us by C. Mouhot that instead of a deterministic model, there is a better controlled test model for kinetic theory: **the stochastic Kac model with random velocity exchange***

- Toy model introduced by M. Kac in 1956 for deriving a Boltzmann equation
- N -particle system, where only velocities of the particles are tracked, and collisions between particles are stochastic
- Originally velocities taken to be 3-dimensional ($3N$ -dimensional phase space). Later analysis has also focused on the 1-dimensional case (N -dimensional phase space)

- System of N particles. Configurations are vectors $(v_i)_{i=1}^N$
- State is updated as follows: Randomly pick a pair of particles with indices (i, j) to collide. Pick a random collision angle $\theta \in 2\pi\mathbb{T}$ with some weight (say, uniform). Update velocities by *rotating*

$$\begin{aligned}v_i &\mapsto \cos(\theta)v_i + \sin(\theta)v_j \\v_j &\mapsto \cos(\theta)v_j - \sin(\theta)v_i\end{aligned}$$

After the update, the velocity vector is given by

$$R_{i,j}(\theta)v$$

where the diagonal and off-diagonal elements of the matrix $R_{i,j}(\theta)$ are

$$\begin{aligned}[R_{i,j}(\theta)]_{k,k} &= \mathbb{1}_{\{i,j \neq k\}} \mathbb{1} + \mathbb{1}_{\{i,j = k\}} \cos(\theta) \\[R_{i,j}(\theta)]_{k_1, k_2} &= \mathbb{1}_{\{i = k_1, j = k_2\}} \sin(\theta) + \mathbb{1}_{\{i = k_2, j = k_1\}} (-\sin(\theta))\end{aligned}$$

- Let $S^{N-1} := S^{N-1}(\sqrt{N}) = \{v \in \mathbb{R}^N : \|v\|_{\ell_N^2} = \sqrt{N}\}$
- If $v \in S^{N-1}$, then also $R_{i,j}(\theta)v \in S^{N-1}$ (**energy conservation**)
- *Markov transition operator* $Q = Q_N$ given by

$$Q\phi(v) = \frac{1}{N(N-1)} \sum_{i,j;i \neq j} \int_{-\pi}^{\pi} \phi(R_{i,j}(\theta)v) \frac{d\theta}{2\pi}$$

- Usually, start from a probability density f_0 on S^{N-1} with respect to the uniform probability measure μ_N on S^{N-1} . Evolve the density according to Kac's model, and denote the density at time t by f_t and the full measure by $F(t)$

- The collision times are chosen Poisson, and we scale time so that the collision rate is equal to N . Then the evolution of the density f_t is determined by the semigroup

$$f_t = S_t f_0 = e^{tN(Q_N - I)} f_0$$

- Writing the time evolution in terms of the generator, we have

$$\frac{d}{dt} f_t = N(Q_N - I) f_t$$

- Kac 1956: Showed that if the initial densities f_0^N are chaotic (comparing to a product state), also f_t^N are chaotic. Furthermore, the first marginal of f_t^N converges to f_t as $N \rightarrow \infty$, where f_t solves

$$\begin{cases} \partial_t f_t(v) = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{\mathbb{R}} (f_t(v') f_t(w') - f_t(v) f_t(w)) dw d\theta \\ f_0(v) = (\text{limit of marginal of } f_0^N) \end{cases}$$

Here

$$\begin{pmatrix} v' \\ w' \end{pmatrix} = R(\theta) \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

- Evolution equation similar to the spatially homogeneous Boltzmann equation

- For each N , $f_t^N \mu_N \rightarrow \mu_N$, so the density equilibrates. Can this be quantified?
- Janvresse [*Ann. Probab.* 2001] proved that the operator Q_N has a *spectral gap*. This shows that

$$\|f_t^N - 1\|_{L^2(S^{N-1}, \mu_N)} \leq e^{-ct} \|f_0^N - 1\|_{L^2} \rightarrow 0$$

with the rate c uniform in N . Unfortunately, the prefactor depends on the L^2 size of the initial data which is often $\propto C_0^N$

\Rightarrow **Observable relaxation time estimated to be $\propto N$**

- Carlen, Carvalho, Loss [*Acta Math.* 2003] (cf. also Maslen [*Math. Z.* 2003]) provided the spectral gaps of Q_N explicitly,

$$\Delta_N = \frac{1}{2} \frac{N+2}{N-1}$$

$$\Rightarrow \|f_t^N - 1\|_{L^2} \leq e^{-\frac{t}{2}} \|f_0^N - 1\|_{L^2}$$

- Carlen et. al [Kinetic & related models 2009] studied the *entropy production* of the Kac model. Showed that there is no universal lower bound for the entropy production (bad news for fast equilibration).
- Einav [Kinetic & related models 2011] showed that entropy production behaves almost as badly as $\frac{1}{N}$.
- Mischler, Mouhot [*Invent. math.* 2013] prove several results concerning quantitative uniform in time propagation of chaos, propagation of entropic chaos, and quantitative estimates (independent of the number N of particles) on relaxation times

All of the previous results assume a permutation invariant (initial) state and so do we.

Evolution of observables

Instead the full probability density f_t^N , we may instead consider the correlation structure of the time-evolved random variables:

Let $\phi : S^{N-1} \rightarrow \mathbb{R}$ be an observable. Since Q_N preserves μ_N , the evolution is self-adjoint

$$\begin{aligned} \mathbb{E}_{F(t)}[\phi] &= \int_{S^{N-1}} \phi(v) f(t, v) \mu_N(\mathrm{d}v) \\ &= \int_{S^{N-1}} (e^{tN(Q-I)} \phi(v)) f_0(v) \mu_N(\mathrm{d}v) \end{aligned}$$

- We can then consider evolution of, for instance, moments $\phi(v) := v^I$ for any index sequence I
- The moments will satisfy a “closed” hierarchy of evolution equations (the rate of change of moment of order n will only depend on moments of order $\leq n$)

Because of the (almost sure) conservation law $\sum_{i=1}^N v_i^2 = N$,
it is easier to analyse the local energy variables

Energies as random variables

For each particle i , its energy $e_i := v_i^2$ is a random variable whose cumulant hierarchy is closed, similarly to v_i before. At equilibrium and as $N \rightarrow \infty$, these should converge to i.i.d. χ^2 -random variables

How to measure “chaoticity” of a cumulant hierarchy? 24

Assumption (chaos bounds)

Let $c \geq 0$, $0 < \alpha < 1$, $N \geq 2$, and $n^* \geq 1$ be given. We say that $B > 0$ is a constant for the *chaos bound* of type α, c up to order n^* if for every $n \in [n^*]$ the joint cumulants are bounded as

$$|\kappa_0^{n,N}[e_r]| \leq B^{n-1} (n-1)! N^{c(n-1) - \alpha(\text{len}(r)-1)}$$

- Here, r is a sequence of n labels ($|r| = n$)
- $\text{len}(r) \leq n$ denotes the number of *different* energy labels in r
- Such a constant B can be found but B might need be very large for large N . The idea here is to find c, α such that B is “order one”
- We say $c = 0$ corresponds to a *chaotic state*
- In fact, for any symmetric state and α we can always satisfy the bounds with $c = 1$ and $B = 4$: namely,

$$|\kappa_0^{n,N}[e_r]| \leq 4^{n-1} (n-1)! N^{n-\text{len}(r)}, \quad n \leq N/2 + 1$$

These bounds are saturated by symmetrizing deterministic data where one particle has all the energy

Let $c \geq 0$ and $\alpha \in (0, 1)$. Let $n^* \in \mathbb{N}$ be a maximal order of cumulants. Then there exists a $N_0 = N_0(n^*, \alpha, c) \geq 2$ such that

Theorem (Generation of chaotic bounds)

Consider some fixed $N \geq N_0$ and some permutation invariant initial data F_0^N on $S^{N-1}(\sqrt{N})$. Denote the corresponding joint cumulants at order $n \in [n^*]$ and at time $t \geq 0$ by $\kappa_t^n[e_r]$, $|r| = n$. Assume that the initial data satisfies *chaos bound* of type α, c up to order n^* with a constant $B \geq 1$.

Then there exists a constant C , **depending only on** B , such that for all $n \leq n^*$, the time-evolved cumulants satisfy

$$|\kappa_t^{n,N}[e_r]| \leq \frac{1}{(N-1)^{\alpha(\text{len}(r)-1)}} C^{n^2} n! \left(N^c e^{-\frac{1}{4}t} + 1 \right)$$

Convergence to equilibrium

Let c, α, n^*, N_0 be given as above. Similarly, assume $N \geq N_0$ and take some symmetric initial data F_0^N , whose cumulants are $\kappa_t^n[e_r]$.

Theorem (Equilibration)

Denote the stationary cumulants (corresponding to the uniform probability distribution on $S^{N-1}(\sqrt{N})$) by $\bar{\kappa}^n[e_r]$. Assume that the initial data satisfies *chaos bound* of type α, c up to order n^* with a constant B .

Then there exists a constant C , depending only on B , such that the time-evolved energy cumulants satisfy the following bound for all $|r| = n \leq n^*$ and $t \geq 0$

$$\left| \kappa_t^{n,N}[e_r] - \bar{\kappa}^{n,N}[e_r] \right| \leq \frac{1}{(N-1)^{\alpha(\text{len}(r)-1)}} n! C^{n^2} N^{c(n-1)} e^{-\frac{t}{4}} \quad (2)$$

- If the initial state is chaotic ($c = 0$), equilibration in time $O(1)$
- If the initial state is not chaotic ($c > 0$), need a time $\propto \ln N$ to equilibrate

Accuracy of kinetic theory

- 1 Take symmetrized initial data as above.
- 2 Wait until time t_0 such that the state is already chaotic: Choose $t_0 = 0$ if $c = 0$, and $t_0 = 4c(n^* - 1) \ln N$, otherwise
- 3 At time t_0 , compute the first marginal $f_0(v)$ of $F_{t_0}^N$ and solve the Boltzmann–Kac equation (yielding $\nu_t := f_t(v)dv$)

$$\partial_t f_t(v) = 2 \int_{-\pi}^{\pi} \int_{\mathbb{R}} (f_t(v')f_t(w') - f_t(v)f_t(w)) dw \frac{d\theta}{2\pi}$$

- 4 Construct product measures $\tilde{F}_T^N := \otimes_{i=1}^N \nu_T$ on \mathbb{R}^N

How close are the cumulants of \tilde{F}_T^N to the “real” ones, of $F_{t_0+T}^N$?

Theorem (accuracy of the “Boltzmann–Kac” hierarchy)

$$|\kappa_{t_0+T}^{n,N}[e_r] - \tilde{\kappa}_T^{n,N}[e_r]| \leq 2(N-1)^{-\alpha} C^{n^2} n! = O(N^{-\alpha})$$

Part III

Outline of the tools and proofs

[JL, A Vuoksenmaa, arXiv:2407.17068]

Let $F_0^N \in \mathcal{P}_{\text{sym}}(S^{N-1}(\sqrt{N}))$. Consider $F_t^N := S_t^N(F_0^N)$, satisfying

$$\frac{d}{dt} \langle \phi, F_t^N \rangle = \langle N(Q_N - I)\phi, F_t^N \rangle, \quad \forall \phi \in C_b(S^{N-1})$$

$$(Q_N \phi)(v) = \frac{1}{N(N-1)} \sum_{i,j=1}^N \mathbb{1}_{\{i \neq j\}} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \phi(R_{i,j}(\theta)v)$$

Here we focus on the case where

- $N \geq 2$ is fixed but sufficiently large and
- $t \geq 0$ is arbitrary (varied independently of N)

We study the chaos and equilibration of the *kinetic energies*. Random variables on $(S^{N-1}(\sqrt{N}), F_t^N)$, defined by $e_i(v) = v_i^2$.

These satisfy the conservation law
$$\sum_{i=1}^N e_i = N$$

Study their time-dependent joint cumulants:

$$\kappa_t^N(e_I) \text{ of } \{e_i\}_{i=1}^N. I = (i_j)_{j=1}^n, i_j \in [N],$$

Goal is to show that energies will become chaotic when $t \rightarrow \infty$
 \Leftrightarrow mixed cumulants become small ($\ll_N 1$).

Partition classifiers

The measure F_t^N is symmetric.

\Rightarrow Instead of sequences I of labels, the joint cumulants *of order n* can be indexed using *partition classifiers* in \mathcal{C}_n :

Multi-index $r \in \mathcal{C}_n \subset \mathbb{N}_0^n$ if and only if

- 1 $\sum_{\ell=1}^n r_\ell = n$, and
- 2 $r_\ell \geq r_{\ell+1}$

$$\kappa_t^N(e_1, e_1, e_1, e_1, e_1)$$

$$\kappa_t^N(e_1, e_1, e_1, e_1, e_2)$$

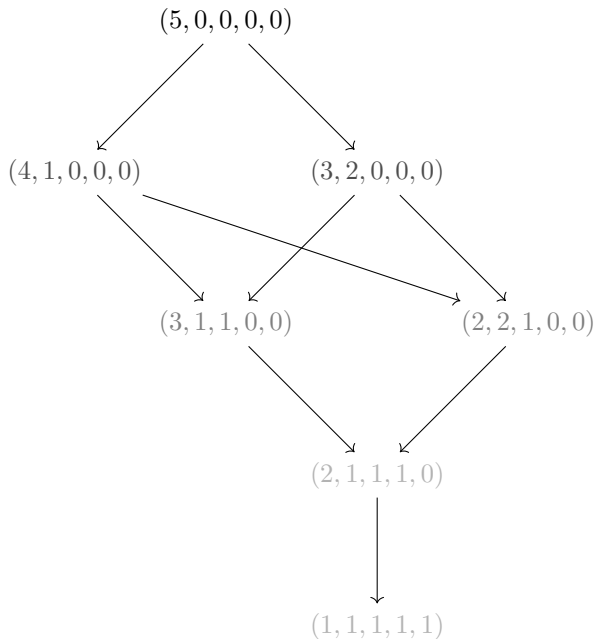
$$\kappa_t^N(e_1, e_1, e_1, e_2, e_2)$$

$$\kappa_t^N(e_1, e_1, e_1, e_2, e_3)$$

$$\kappa_t^N(e_1, e_1, e_2, e_2, e_3)$$

$$\kappa_t^N(e_1, e_1, e_2, e_3, e_4)$$

$$\kappa_t^N(e_1, e_2, e_3, e_4, e_5)$$



Time-evolution of the joint cumulants

The time-evolution of the joint cumulants is given by ($n \geq 2$)

$$\frac{d}{dt}(\kappa_t^{n,N})(r) = (L_{n,N}\kappa_t^{n,N})(r) + (\mathcal{N}_{<}^{n,N}[\kappa_t^{[\cdot],N}])(r)$$

The linear part splits into $L_{n,N} = L_n + R_{n,N}$.

The operator L_n couples to *more mixed cumulants* and generates exponentially decaying semigroup in the correct norm, $R_{n,N}$ is a perturbation.

The nonlinear term includes only cumulants of strictly lower order.

Let $\alpha \in (0, 1)$.

- Consider the vector of all joint energy cumulants of order n :

$$\kappa^{n,N} = (\kappa^{n,N}(e_r))_{r \in \mathcal{C}_n}$$

- And the space $X_\alpha = (\mathbb{R}^{\mathcal{C}_n}, \|\cdot\|_{\alpha,n,N})$, with the norm

$$\|\kappa^{n,N}\|_\alpha = \|\kappa^{n,N}\|_{\alpha,n,N} = \sup_{r \in \mathcal{C}_n} (N-1)^{\alpha(\text{len}(r)-1)} |\kappa^{n,N}(e_r)|$$

- Now the task is to analyse $\|\kappa_t^{n,N}\|_\alpha$

Theorem (generation of chaos)

Assume that we have a sequence of initial data $(F_0^N)_{N \geq 2}$. Assume the initial data has cumulants that satisfy

$$\|\kappa_0^{n,N}\|_\alpha \leq B^{n^2} N^{n-1}.$$

Then there exists a constant C , depending only on B , such that for every $n^* \in \mathbb{N}_1$, we have a threshold $N_0 = N_0(n^*, \alpha)$, such that for all $N \geq N_0$ and $n \leq n^*$, the time-evolved cumulants satisfy

$$\|\kappa_t^{n,N}\|_\alpha \leq C^{n^2} \left(N^{n-1} e^{-\frac{1}{4}t} + 1 \right). \quad (3)$$

$$\frac{d}{dt} \kappa_t^{n,N}(e_r) = \sum_{s \in \mathcal{C}_n} C_N(r, s) \kappa_t^{n,N}(e_s) + \mathcal{N}_{<n,N}[\kappa_t^{[\cdot],N}](e_r)$$

- Prove inductively on the order of cumulants n that the *totally nonrepeated energy cumulants* $\kappa_t^{n,N}(e_1, \dots, e_n)$ are controlled nicely.
- For the rest of the cumulants prove inductively on the order n , that
 - The linear part minus the totally nonrepeated cumulants leads to exponential decay in the X_α -norm. \Leftarrow Identify an **explicitly dissipative term** and show that the rest is its perturbation
 - With Duhamel's formula, the bounds propagate in the norm when the source term (nonlinear expression of lower order cumulants + nonrepeated cumulants) are taken into account.

Evolution of the energy cumulants

The energy cumulants evolve according to

$$\begin{aligned} \frac{d}{dt} \kappa_t^N(e_I) &= \frac{d}{dt} (\partial_\eta^I \log \mathbb{E}_t[e^{\eta \cdot e}]) |_{\eta=0} \\ &= \left(\partial_\eta^I \left(\mathbb{E}_t[e^{\eta \cdot e}]^{-1} \frac{d}{dt} \mathbb{E}_t[e^{\eta \cdot e}] \right) |_{\eta=0} \right) \end{aligned}$$

The time-evolution of the mgf $\mathbb{E}_t[e^{\eta \cdot e}]$ is given by

$$\frac{d}{dt} \mathbb{E}_t[e^{\eta \cdot e}] = \frac{1}{N-1} \sum_{i \neq j} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int e^{\eta \cdot e} \left(e^{\eta \cdot P_{i,j}^\theta(v)} - 1 \right) F_t^N(dv)$$

Here $P_{i,j}^\theta(v) \in \mathbb{R}^N$, with $(P_{i,j}^\theta(v))_k = 0$, for $k \neq i, j$, and

$$\begin{aligned} (P_{i,j}^\theta(v))_i &= -\sin(\theta)^2 e_i + 2 \cos(\theta) \sin(\theta) v_i v_j + \sin(\theta)^2 e_j \\ (P_{i,j}^\theta(v))_j &= -\sin(\theta)^2 e_j - 2 \cos(\theta) \sin(\theta) v_i v_j + \sin(\theta)^2 e_i \end{aligned}$$

Evolution of the energy cumulants

$$\begin{aligned} & \mathbb{E}_t[\mathbf{e}^{\eta \cdot e}]^{-1} \frac{d}{dt} \mathbb{E}_t[\mathbf{e}^{\eta \cdot e}] \\ &= \mathbb{E}_t[\mathbf{e}^{\eta \cdot e}]^{-1} \frac{1}{N-1} \sum_{i \neq j} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int \mathbf{e}^{\eta \cdot e} \left(\mathbf{e}^{\eta \cdot P_{i,j}^\theta(v)} - 1 \right) F_t^N(dv) \end{aligned}$$

Recognizing that $\mathbb{E}_t[\mathbf{e}^{\eta \cdot e}]^{-1} \mathbf{e}^{\eta \cdot e} = G_w(\eta; e)$ is the *Wick polynomial generating function*, we obtain

$$\frac{d}{dt} \kappa_t(e_I) = \frac{1}{N-1} \sum_{i \neq j} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \mathbb{E}_t[\partial_\eta^I \left(G_w(\eta; e) (\mathbf{e}^{\eta \cdot P_{i,j}^\theta(v)} - 1) \right) |_{\eta=0}]$$

Evolution of the energy cumulants

$$\frac{d}{dt} \kappa_t(e_I) = \frac{1}{N-1} \sum_{i \neq j} \sum_{\emptyset \neq J \subseteq I} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \mathbb{E}_t[:e_{I \setminus J} : \partial_\eta^J (e^{\eta \cdot P_{i,j}^\theta(v)}) |_{\eta=0}]$$

Those resulting $(P_{i,j}^\theta(v))^J$ terms which are odd in v are also odd in θ
 \Rightarrow *polynomial in e*

Since the expectations of products of Wick polynomials and monomials of random variables can be written in terms of cumulants as follows:

$$\mathbb{E}_t[:e_{I'} : e_{J'}] = \sum_{\pi \in \mathcal{P}(I'+J')} \left(\prod_{A \in \pi} \mathbb{1}_{\{A \cap J' \neq \emptyset\}} \kappa_t(e_A) \right)$$

we arrive at

$$\frac{d}{dt} \kappa_t(e_I) = \sum_{|J|=|I|} C_N(I, J) \kappa_t(e_J) + \mathcal{N}_{<|I|}[\kappa_t](e_I)$$

Proof outline (linear part)

- In the vector form, we have

$$\kappa_t^{n,N} = e^{tM_{n,N}} \kappa_0^{n,N} + e^{tM_{n,N}} \int_0^t ds e^{-sM_{n,N}} \mathcal{N}_{<,n}[\kappa_s^{[:,N]}] \quad (4)$$

- The linear part splits into two: $M_{n,N} = M_n + R_{n,N}$, where M_n looks at *more mixed cumulants* and $R_{n,N}$ is an $O(N^{\alpha-1})$ perturbation in $\|\cdot\|_\alpha$.
- It follows that for sufficiently large $N(n^*, \alpha)$

$$\|e^{tM_{n,N}}\|_\alpha \leq 10e^{-\frac{1}{4}t}, \quad (5)$$

Proof outline (equilibration)

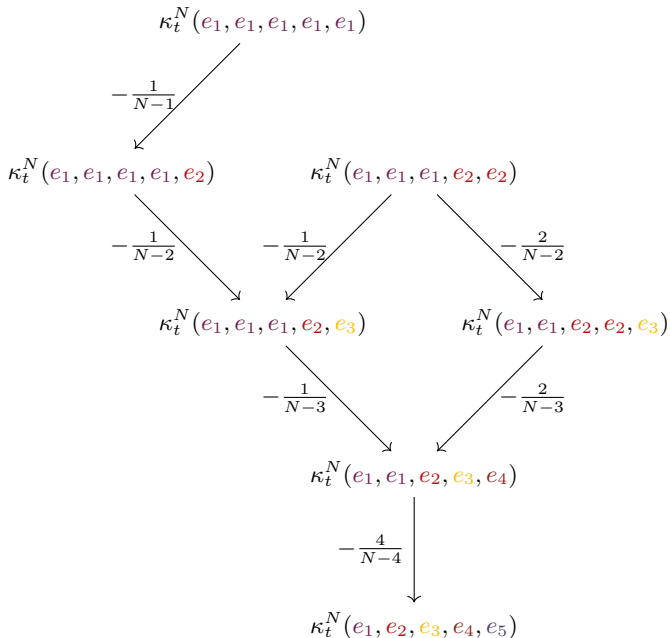
Let $h_t^{n,N}(e_r) := \kappa_t^{n,N}(e_r) - \bar{\kappa}^{n,N}(e_r)$. Note that we have for all $n \leq n^*$ and $N \geq N_0(n^*, \alpha)$

$$\|\bar{\kappa}^{n,N}\|_\alpha \leq Bn^2 \quad (6)$$

for some constant B .

$$\frac{d}{dt} h_t^{n,N}(e_r) = \sum_{s \in \mathcal{C}_n} \tilde{L}_N(s, r) h_t^{n,N}(e_s) + \tilde{\mathcal{N}}_{<n,N}[h_t^{[\cdot],N}](e_s)$$

- Similar to the generation of chaos but now the source terms contain $(\bar{\kappa}^{m,N})_{m < n}$.



Part IV

Generalized Kac model

- More “physical” with faster generation of chaos

[JL, J. Miñarro, A Vuoksenmaa, *work in progress*]

One might find it surprising that even a model as “mixing” as the Kac model would need $O(\ln N)$ times to reach sufficiently chaotic states for kinetic theory.

The following *generalization of the Kac model* was pointed out to us by Eric Carlen:

- In more realistic kinetic models, faster particles have a faster collision rate
- The simplest way to implement this is to let the Kac rate increase proportionally to the sum of the energies of the colliding particles:

Generalized Kac model

Instead of constant collision rate, let the collision of particles i and j , with velocities v_i and v_j , occur at a rate

$$1 + v_i^2 + v_j^2 = 1 + e_i + e_j$$

- “Entropic chaos” emerges faster in this model, on timescales of order one [Carlen, Carvalho, Einav: Kin Rel Models, 2018]
- How about cumulants? Technical difficulty:
The resulting cumulant hierarchy is no longer closed
- Hence, direct recursive estimates used earlier do not work (the linear operator feeds back from higher order cumulants)
- Our preliminary results (relying on control of moments), indicate that cumulants should also equilibrate at order one, not $\ln N$, times:

Our current best bound, generalized Kac

Let α be a partition classifier of order $n \geq 2$. Then for the same class of initial data as before,

$$|\kappa_t(e_\alpha)| \leq 15^n 2^{n^2} (n-1)!, \quad t \geq 1$$

uniformly in N . (\Rightarrow worst case now better by a factor of $1/N^{n-1}$)

Summary: Cumulants were surprisingly good in measuring independence of energies in this model (a factor of $1/N$ for each new roughly independent variable)

Next: Control also the cumulant evolution in the generalized Kac model

Further: Extend the analysis to other models, perhaps even weakly perturbed wave evolution?

Thank you for your attention!