Cumulants, their evolution hierarchies, and how to use them to control chaos for kinetic theory

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based on joint works with

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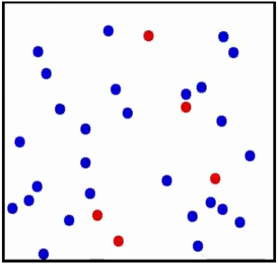
Introduction and motivation

Recently, important advances have been made in understanding of mathematical foundations of kinetic theory. For example,

1. For kinetic theory of waves (wave turbulence) in *nonlinear Schrödinger equation (NLS)*
   - Deng and Hani [arXiv:2104.11204, 2110.04565, ...]
   - Building on earlier works: Buckmaster, Germain, Shatah, ...
   - Other closely related results: JL & Spohn, Staffilani & Tran, Ampatzoglou & Collot & Germain, Dymov, ...
2. For kinetic theory of *rarefied gas* = original "billiard ball" model of Boltzmann
   - Deng, Hani, Ma [arXiv:2408.07818, 2503.01800, ...]
   - Earlier & related works: Lanford, Cercignani, ... Bodineau, Gallagher, Saint-Raymond, Pulvirenti, Simonella, ...

Original rarefied gas model



(Wikipedia / Kinetic theory of gases)

Deterministic evolution but random initial data:

- N particles, random initial position and velocity
- Between collisions, move in straight lines
- When two particles collide, update their velocities according to the elastic collision rules

Boltzmann equation from a scaling limit

Boltzmann–Grad scaling limit:

1. Particle radius $r_0 \to 0$
2. Assume $N(2r_0)^{d-1} \to c_0 > 0$ (*rarefied gas*) $\Rightarrow$ $N \to \infty$
3. Assume that the particle positions and velocities are initially ($t = 0$) sufficiently independent and "well-prepared"

Define the *collision operator*

$$\mathcal{C}_{\mathsf{b}}[h](\boldsymbol{v}_0) = 2c_0 \int_{(\mathbb{R}^d)^3} \mathrm{d}\boldsymbol{v}_1 \mathrm{d}\boldsymbol{v}_2 \mathrm{d}\boldsymbol{v}_3 \, \delta(\boldsymbol{v}_0 + \boldsymbol{v}_1 - \boldsymbol{v}_2 - \boldsymbol{v}_3) \\ \times \delta(|\boldsymbol{v}_0|^2 + |\boldsymbol{v}_1|^2 - |\boldsymbol{v}_2|^2 - |\boldsymbol{v}_3|^2) \left(h(\boldsymbol{v}_2)h(\boldsymbol{v}_3) - h(\boldsymbol{v}_0)h(\boldsymbol{v}_1)\right)$$

Theorem (Lanford, Deng–Hani–Ma)

In the limit $r_0 \to 0$, the "one-particle distribution function" f_t satisfies

$$\partial_t f_t(\boldsymbol{x}, \boldsymbol{v}) + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f_t(\boldsymbol{x}, \boldsymbol{v}) = \mathcal{C}_{\mathsf{b}}[f_t(\boldsymbol{x}, \cdot)](\boldsymbol{v})$$

(Scholarpedia / Boltzmann–Grad limit)

Key notion chaos, as statistical independence

This is also the key difficulty:

Need to control propagation of approximate independence

Even if initial state has complete independence of velocities,
it no longer holds after the first collision

- Earlier, control independence via "almost factorization" of moments and moment hierarchies (BBGKY)
- We show that cumulants can do this better, at least in a simplified model called stochastic Kac model

Propagation versus generation of chaos

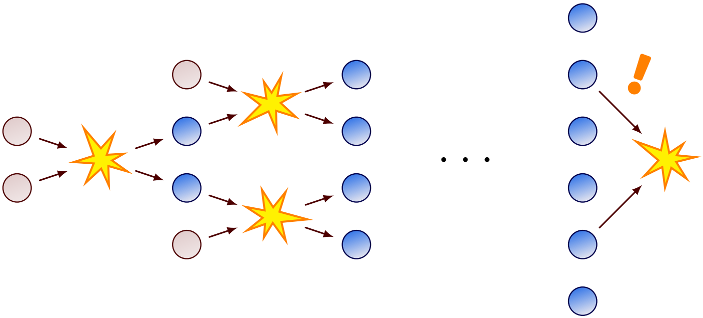
Chaos in kinetic theory

In most of the mathematical examples of derivation of a kinetic theory, starting from the rarefied gas Boltzmann equation, "chaos" refers to (approximate) **statistical independence** of some of the evolving quantities

Propagation of chaos means that if the above "evolving quantities" are (sufficiently) independent in the beginning, they remain so at least up to kinetic time scales

Generation of chaos means that even if the above "evolving quantities" are not independent in the beginning, they will become so later, at least in approximation and on some time-scale which need not be connected with kinetic theory

Stochastic Kac model



Stochastic evolution:

- N particles, consider only their velocities
- Poisson clock determines when particles collide
- Pick the colliding pair at random
- Outgoing velocities random, but preserve total energy

Accuracy of the Boltzmann–Kac evolution

1. Arbitrary initial distribution (total energy $= N$)
2. Wait until time t_0 such that the state is already chaotic, $t_0 = O(\ln N)$
3. At time t_0, compute the first marginal $f_0(v)$ of $F^N_{t_0}$ and solve the Boltzmann–Kac equation (yielding $\nu_t := f_t(v)\mathrm{d}v$)

$$\partial_t f_t(v) = 2\int_{-\pi}^{\pi}\int_{\mathbb{R}} \left(f_t(v')f_t(w') - f_t(v)f_t(w)\right) \mathrm{d}w \frac{\mathrm{d}\theta}{2\pi}$$

4. Construct product measures $\tilde{F}^N_T := \otimes_{i=1}^N \nu_T$ on $\mathbb{R}^N$

How close are the cumulants of particle energies, $e_i = v_i^2$, of these two measures for large N? (no scaling limits!)

Theorem (accuracy of the "Boltzmann–Kac" hierarchy)

$$|\kappa^{n,N}_{t_0+T}[e_r] - \tilde{\kappa}^{n,N}_T[e_r]| \le 2(N-1)^{-\alpha} C^{n^2} n! = O(N^{-\alpha})$$

Outline of the course

1. Basic notions from probability
   - Random variables and their moments and cumulants
   - Measures with δ-function constraints
2. Evolution hierarchies of cumulants
   - Wick polynomials
   - For deterministic evolution equations
   - For the stochastic Kac model
3. Exercise session (group work)
4. Analysis of the cumulant hierarchy of the stochastic Kac model

Lectures notes will be updated during the week, available online as a PDF