Summer School EUR MINT Exercises (Lukkarinen)

June 2026 16.6.2026

In the exercise session, we will work, one by one, on the problems in small groups, and then summarize the solutions together. It is possible to start thinking about the problems in advance but this is not necessary to attend the session. In particular, there will be no need to submit written solutions at any stage. In the Exercises, some necessary regularity assumptions are not stated explicitly: you can always assume that the unspecified probability distributions are sufficient regular for the computations to work, for example, that the joint distributions of the random variables have exponential moments.

Exercise 1

Consider a collection of random variables X_j, $j = 0, 1, 2, \ldots, N$. Suppose that X_0 is *deterministic*, i.e., there is a constant $c \in \mathbb{R}$ such that $X_0 = c$ almost surely. Compute explicitly the cumulants

$$\kappa[X_0, X_1, \ldots, X_m]\,, \qquad m = 0, 1, \ldots, N\,.$$

Exercise 2

Cumulants of "nearly independent" exponentially distributed random variables

Consider two random variables X, Y whose joint distribution is given by the probability density function

$$\Phi_\varepsilon(x, y) := \frac{1}{4}\mathrm{e}^{-|x|-|y+\varepsilon x|}\,, \qquad x, y \in \mathbb{R}\,,$$

where $\varepsilon > 0$ is a parameter which controls the dependence between the two variables.

(a) Convince yourself that after the limit $\varepsilon \to 0$ the random variables X and Y are independent.

(b) Compute the cumulant generating function

$$g_c(u, v) := \ln \mathbb{E}[\mathrm{e}^{uX+vY}] = \ln\left(\frac{1}{4}\int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}y\, \mathrm{e}^{ux+vy-|x|-|y+\varepsilon x|}\right)$$

explicitly (assuming that $|u|$ and $|v|$ are small enough so that the integrals converge).

(c) Use the above formula (e.g. either by induction or by Taylor expansion for small values of u, v) to compute

$$\kappa[\underbrace{X, \ldots, X}_{n \text{ times}}, \underbrace{Y, \ldots, Y}_{n \text{ times}}]\,,$$

explicitly for all $n \in \mathbb{N}$.

(d) Use Stirling's approximation of factorials to estimate the magnitude of the above cumulant. Conclude that if $\varepsilon \ll 1$, the cumulants are small up to $n \lesssim \frac{1}{\sqrt{\varepsilon}}$. What happens if $n \gg \frac{1}{\sqrt{\varepsilon}}$?

(Continues on the next page…)

Exercise 3

Recall from the lecture notes the definition of the measure $\nu_{N,R}(\mathrm{d}v)$, $v \in \mathbb{R}^N$, the uniform probability distribution on the sphere $S^{N-1}(R)$. Consider the marginal distribution $\mu_{N,R}$ of the corresponding "energy variables", $e_i := v_i^2$, $i = 1, 2, \ldots, N$, defined by requiring that

$$\int_{\mathbb{R}^N} \mu_{N,R}(\mathrm{d}e)\, f(e) = \int_{\mathbb{R}^N} \nu_{N,R}(\mathrm{d}v)\, f(v_1^2, \ldots, v_N^2)\,, \qquad f \in C_c(\mathbb{R}^N)\,. \tag{1}$$

Derive a weight function $\Phi : \mathbb{R}^N \to \mathbb{R}_+$ such that

$$\mu_{N,R}(\mathrm{d}e) = \Phi(e)\, \delta\left(\sum_{i=1}^{N} e_i - R^2 \right) \mathrm{d}^N e\,,$$

with the same definition of δ-functions as in the Lecture notes.

(*Hint:* It might be easier to start from μ for an arbitrary Φ and then see if it can fixed so that the integrals in (1) match. You can require $\Phi(e) = 0$ if $e_i \leq 0$ for any i.)

Exercise 4

Consider the following standard Ornstein–Uhlenbeck process, given by the stochastic differential equation

$$\mathrm{d}X_t = -\frac{1}{2} X_t \mathrm{d}t + \mathrm{d}W_t$$

where W_t denotes the standard unit variance Wiener process (Brownian motion). The generator L of this process is given by

$$(Lf)(x) = -\frac{1}{2} x f'(x) + \frac{1}{2} f''(x)\,,$$

in order words, for all sufficiently nice functions $f : \mathbb{R} \to \mathbb{R}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}[f(X_t)] = \mathbb{E}[(Lf)(X_t)]\,.$$

Derive the cumulant hierarchy of this process, i.e., the evolution equations for the collection of n:th order cumulants $\kappa_n[X_t]$. Solve the hierarchy and compute $\lim_{t\to\infty} \kappa_n[X_t]$ for $n \in \mathbb{N}$. Is there a distribution whose cumulants coincide with the above limiting values?