A class of quasi-unstable processes

Frédéric Proïa
Analyse des Séries Temporelles
Valenciennes

Nantes Université

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Summary

Autoregressive processes

- Consider an AR(p) process written in the VAR$_p(1)$ form, that is

$$\forall\, n \geqslant 1, \qquad \Phi_n = C_\theta\, \Phi_{n-1} + E_n$$

where $E_n = (\varepsilon_n, 0, \ldots, 0)^T$ is a p-vectorial noise and C_θ is the companion matrix

$$C_\theta = \begin{pmatrix} \theta_1 & \theta_2 & \ldots & \theta_p \\ \ddots & & & \vdots \\ & I_{p-1} & & 0 \\ & & \ddots & \vdots \end{pmatrix}.$$

- It is well-known that the asymptotic behavior of Φ_n is closely related to the spectrum of C_θ or, equivalently, to the complex roots of the polynomial

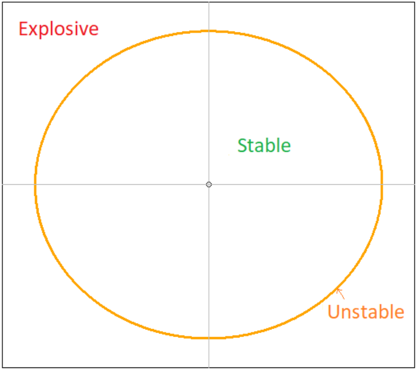
$$\Theta(z) = 1 - \theta_1 z - \ldots - \theta_p z^p$$

since

$$\forall\, \lambda \in \mathbb{C}^*, \qquad \det(C_\theta - \lambda\, I_p) = (-\lambda)^p\, \Theta(\lambda^{-1}).$$

Autoregressive processes

- Let $\rho(C_\theta) = |\lambda_1| \geqslant \ldots \geqslant |\lambda_p|$. According to the terminology of [Duflo, 1997], the process is stable when $|\lambda_1| < 1$, unstable when $|\lambda_1| = 1$ (purely unstable when in addition $|\lambda_p| = 1$), explosive when $|\lambda_1| > 1$, without mentioning all the resulting mixed cases...



- Stable = Stationary for $T = \mathbb{N}$ but not for $T = \mathbb{Z}$.

Autoregressive processes

OLS estimation

- The OLS estimator of θ is given by

$$\widehat{\theta}_n = \left(\sum_{k=1}^{n} \Phi_{k-1} \Phi_{k-1}^T \right)^{-1} \sum_{k=1}^{n} \Phi_{k-1} X_k.$$

Proposition (Consistency and Fluctuations)

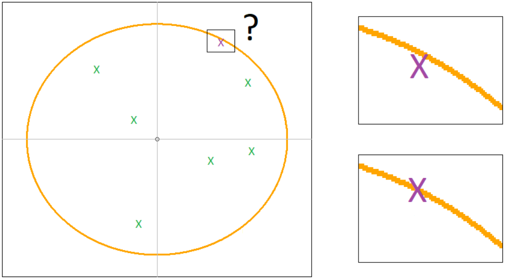
Assume that $(\varepsilon_n)_{n \geqslant 1}$ is a white noise of variance $\sigma^2 > 0$ and having a moment of order $2 + \nu$ $(\nu > 0)$. Then, $\widehat{\theta}_n$ is strongly consistent with

$$\begin{aligned}
\textit{(stable)} \quad & \sqrt{n}\,(\widehat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}_p(0, \sigma^2\, \Gamma_p^{-1}). \\
\textit{(explosive)} \quad & C_\theta^n\,(\widehat{\theta}_n - \theta) \xrightarrow{d} U_p. \\
\textit{(univariate unstable)} \quad & n\,(\widehat{\theta}_n - \theta) \xrightarrow{d} \mathrm{sgn}(\theta)\, \frac{\frac{1}{2}\,(W^2(1) - 1)}{\int_0^1 W^2(u)\,\mathrm{d}u}.
\end{aligned}$$

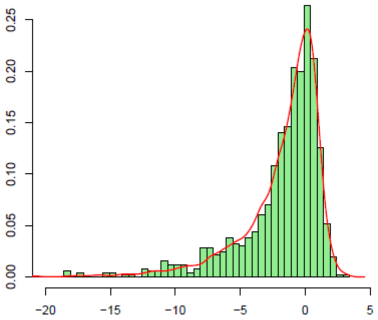
- See *e.g.* [Lai and Wei, 1983] for the strong consistency.
- See *e.g.* [Brockwell and Davis, 2006], [Stigum, 1974] or [Chan and Wei, 1988] for the weak convergences.

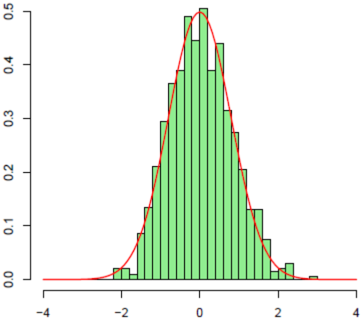
OLS estimation

- Depending on the spectrum of C_θ, we may obtain :
  - A stationary process whose estimation converges with rate $\sqrt{n}$.
  - A process growing (at least) like $\sqrt{n}$ whose estimation converges with rate n (or a power of n in case of multiple unit roots).
  - A process exploding at exponential rate whose estimation also converges with exponential rate.
- Unit root issue.
- In practice, we often encounter the case where $|\lambda_1| = 1 - \epsilon$ ($\epsilon \geqslant 0$) and $|\lambda_k| < 1$ for $2 \leqslant k \leqslant p$.



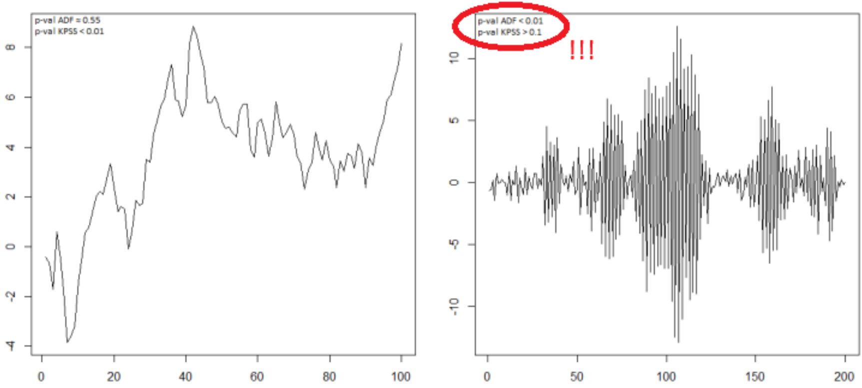
Unit root tests





- Unstable vs Stable ? Stable vs Unstable ?
- ADF : test of "$\theta_0 = 1$" where θ_0 is the (supposed) unit root.
- KPSS : test of "$\tau^2 = 0$" where τ^2 is the variance of a (supposed) hidden random walk in the noise. (Strictly speaking, this is *not* a unit root test).

Unit root tests



Summary

Time-varying coefficients

- Objective : make the boundary less abrupt.
- With time-varying coefficients :

$$\forall\, n \geqslant 1, \qquad X_n = \theta_{n,\,1}\, X_{n-1} + \ldots + \theta_{n,\,p}\, X_{n-p} + \varepsilon_n$$

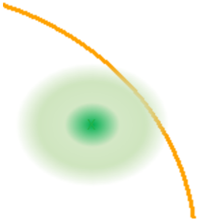
  where $(\theta_n)_{n \geqslant 1}$ is sequence of coefficients. Ex : random coefficients.
- With time-varying coefficients in a triangular form :

$$\forall\, n \geqslant 1,\ \forall\, 1 \leqslant k \leqslant n, \qquad X_{n,\,k} = \theta_{n,\,1}\, X_{n,\,k-1} + \ldots + \theta_{n,\,p}\, X_{n,\,k-p} + \varepsilon_k$$

  in which the AR(p) model is updated for each n. Ex : to focus on the boundary.

Random coefficients

- Principle : the vector of coefficients is written $\theta_n = \theta + \eta_n$ with $\mathbb{E}[\eta_n] = 0$ and $\mathbb{V}(\eta_n)$ usually small.
- Let
$$\forall\, n \geqslant 1, \qquad \Phi_n = (C_\theta + N_n)\,\Phi_{n-1} + E_n$$
where $(E_n)_{n\geqslant 1}$ is an i.i.d. p-vectorial sequence and $(N_n)_{n\geqslant 1}$ is a matrix of dimensions $p \times p$ with random i.i.d. fluctuations, independent of the noise of the model.
- Models introduced by [Nicholls and Quinn, 1981b, Nicholls and Quinn, 1981a].
- Estimations with good properties. Ex : [Aue and Horváth, 2011].
- Test of randomness in the coefficients. Ex : [Horváth and Trapani, 2019].



Nearly-unstable processes

- Principle : let the triangular scheme

$$\forall\, n \geqslant 1, \forall\, 1 \leqslant k \leqslant n, \qquad \Phi_{n,\,k} = A_n\, \Phi_{n,\,k-1} + E_k$$

where $(E_n)_{n \geqslant 1}$ is a p-vectorial i.i.d. sequence and

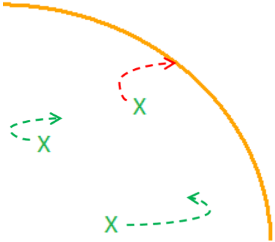
$$A_n = \begin{pmatrix} \theta_{n,\,1} & \theta_{n,\,2} & \ldots & \theta_{n,\,p} \\ \ddots & & & \vdots \\ & I_{p-1} & & 0 \\ & & \ddots & \vdots \end{pmatrix}$$

is the non-random but time-varying companion matrix.

- The AR(p) model is updated for each n.
- The asymptotic behavior of $\rho(A_n) \triangleq \rho_n$ enables to focus on the interesting areas.

Nearly-unstable processes

- When $\rho_n < 1$ for all $n \geqslant 1$ but $\rho_n \to 1$, the process is nearly unstable.



- For a fixed n, the process is strictly stationary (provided that $\Phi_{n,\,0}$ is well chosen), causal and having variance

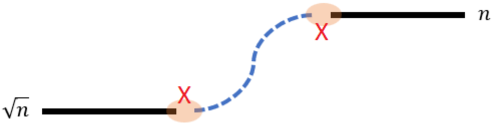
$$\Gamma_n = \sigma^2 \sum_{\ell=0}^{+\infty} A_n^{\ell}\, K_p\, (A_n^T)^{\ell} \qquad \text{with} \qquad K_p = \begin{pmatrix} 1 & 0 \\ 0 & 0_{p-1} \end{pmatrix}.$$

Example for $p = 1$

- [Phillips and Magdalinos, 2007] : when $\theta_n = 1 - \frac{c}{v_n}$ with $1 \ll v_n \ll n$ and $c > 0$,

$$\sqrt{n v_n}\,(\widehat{\theta}_n - \theta_n) \overset{d}{\longrightarrow} \mathcal{N}(0, 2c).$$

- Bridge (or almost) between stable and unstable.



- Take $v_n = n^{\alpha}$ for $0 \leqslant \alpha \leqslant 1$.
- $\alpha = 0$: $\sqrt{n}\,(\widehat{\theta}_n - \theta) \overset{d}{\longrightarrow} \mathcal{N}(0, 2c - c^2)$.
- $\alpha = 1$: $n\,(\widehat{\theta}_n - 1) \overset{d}{\longrightarrow} \mathcal{W}$ and $n\,(\widehat{\theta}_n - \theta_n) \overset{d}{\longrightarrow} \mathcal{W}^*$. Useless because 'too fast' ! Still in the unstable area.
- $0 < \alpha < 1$: $\sqrt{n^{1+\alpha}}\,(\widehat{\theta}_n - \theta_n) \overset{d}{\longrightarrow} \mathcal{N}(0, 2c)$.
    - Correct distribution but overestimated variance when $\alpha \to 0^+$ $(2c > 2c - c^2)$.
    - Wrong distribution when $\alpha \to 1^-$ ($\mathcal{N}$ vs $\mathcal{W}$).

Hypotheses

- Let us come back to

$$\forall\, n \geqslant 1, \forall\, 1 \leqslant k \leqslant n, \qquad \Phi_{n,\,k} = A_n\, \Phi_{n,\,k-1} + E_k$$

  with an arbitrary $\Phi_{n,\,0}$ having moments of order 2.

- Hypotheses :
  - (H_1) There exists a $p \times p$ matrix A such that

$$\lim_{n\,\to\,\infty} A_n = A$$

    with distinct eigenvalues $0 < |\lambda_p| \leqslant \ldots \leqslant |\lambda_2| \leqslant |\lambda_1| = \rho(A) = 1$, and the top-right element of A is non-zero ($\theta_p \neq 0$).
  - (H_2) There is exactly one unit root in A ($\lambda_1 = \pm 1$ but $|\lambda_2| < 1$ if $p \geqslant 2$), or two distinct unit roots ($\lambda_1 = 1$, $\lambda_2 = -1$ and $|\lambda_3| < 1$ if $p \geqslant 3$).
  - (H_3) The spectral radius of A_n is given by

$$\rho_n = 1 - \frac{c}{v_n}$$

    for some $c > 0$ and $1 \ll v_n \ll n$.

Memory

- For all $h \geqslant 0$, let

$$\Gamma_n(h) = \sigma^2 A_n^h \sum_{\ell \geqslant 0} A_n^\ell K_p (A_n^T)^\ell \qquad \text{and} \qquad \Gamma_n(-h) = \Gamma_n^T(h)$$

be the autocovariance function of the process (at fixed n).

Proposition

Assume that (H_1), (H_2) and (H_3) hold, and that $\mathbb{E}[\varepsilon_1^2] = \sigma^2 < +\infty$. Then, for a sufficiently large n,

$$\Big\| \sum_{h \in \mathbb{Z}} \Gamma_n(h) \Big\| < +\infty.$$

Besides, as $n \to +\infty$,

$$\sum_{h \in \mathbb{Z}} \Gamma_n(h) \asymp \begin{cases} (1-\rho_n)^{-2} & \text{if } \lambda_1 = 1, \\ 1 & \text{if } \lambda_1 = -1. \end{cases}$$

- In other terms, $(\Phi_{n,\,k})$ has a short memory at fixed n whereas, as n tends to infinity, it turns to a long memory process in presence of a positive unit root but keeps a short memory in case of a negative unit root.

Summary

Consistency

- The 'OLS estimator' (at fixed n) is

$$\widehat{\theta}_n = S_{n,\,n-1}^{-1} \sum_{k=1}^{n} \Phi_{n,\,k-1}\, X_{n,\,k} \qquad \text{where} \qquad S_{n,\,n} = \sum_{k=0}^{n} \Phi_{n,\,k}\, \Phi_{n,\,k}^{T}.$$

- The empirical covariance satisfies

$$(1-\rho_n) \left\| \frac{S_{n,\,n}}{n} - \Gamma_n \right\| \overset{p}{\longrightarrow} 0 \qquad \text{where} \qquad \Gamma_n = \sigma^2 \sum_{\ell=0}^{+\infty} A_n^{\ell}\, K_p\, (A_n^{T})^{\ell}.$$

- Problem : $(1-\rho_n)\frac{S_{n,\,n}}{n}$ converges to a non-invertible matrix but

$$V_n^{1/2}\, P_n^{-1}\, \frac{S_{n,\,n-1}}{n}\, (P_n^{-1})^{T}\, V_n^{1/2} \overset{p}{\longrightarrow} H \text{ invertible, where } V_n = \mathrm{diag}(1-\rho_n, 1, \ldots, 1).$$

Theorem

Assume that (H_1), (H_2) and (H_3) hold, and that $\mathbb{E}[\varepsilon_1^2] = \sigma^2 < +\infty$. Then the OLS is asymptotically close to θ_n in the sense that

$$\left\| \widehat{\theta}_n - \theta_n \right\| \overset{p}{\longrightarrow} 0.$$

Asymptotic normality

- A closer look at H :

$$H = \sigma^2 \begin{pmatrix} \frac{\pi_{11}^2}{2} & 0 & \cdots & 0 \\ 0 & \frac{\pi_{21}^2}{1-\lambda_2^2} & \cdots & \frac{\pi_{21}\,\pi_{p1}}{1-\lambda_2\,\lambda_p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\pi_{p1}\,\pi_{21}}{1-\lambda_p\lambda_2} & \cdots & \frac{\pi_{p1}^2}{1-\lambda_p^2} \end{pmatrix} \triangleq \sigma^2\, H_0$$

  where π_{k1}, real and non-zero, is the k-th element of the first column of P^{-1}.
- Under (H_1), H is invertible and even positive definite if $\mathrm{sp}(A)$ is real.
- By direct calculation on the Cauchy-like matrix H_0,

$$\det(H_0) = \frac{\pi_{11}^2}{2} \left(\prod_{j=2}^{p} \frac{\pi_{j1}^2}{1-\lambda_j^2} \right) \left(\prod_{j=2}^{p} \prod_{i<j} \frac{(\lambda_i - \lambda_j)^2}{(1-\lambda_i\lambda_j)^2} \right).$$

Asymptotic normality

Theorem

Assume that (H_1), (H_2) and (H_3) hold, and that $\mathbb{E}[|\varepsilon_1|^{2+\nu}] = \eta_\nu < +\infty$ for some $\nu > 0$. Then, if the eigenvalues of A are real, we have the asymptotic normality

$$\sqrt{n}\, V_n^{-1/2} P_n^T \left(\widehat{\theta}_n - \theta_n\right) \xrightarrow{d} \mathcal{N}_p(0,\, H_0^{-1}).$$

If some eigenvalues of A are complex, we have the asymptotic normality

$$\sqrt{n\, v_n}\, \langle L_n,\, \widehat{\theta}_n - \theta_n \rangle \xrightarrow{d} \mathcal{N}(0,\, h_0^2)$$

where

$$L_n = \left(\frac{1}{(\lambda_1 \rho_n)^k} \right)_{0 \leqslant k \leqslant p-1} \qquad \text{and} \qquad h_0^2 = \frac{2c}{\pi_{11}^2} > 0.$$

- For $p = 1$: $\sqrt{n\, v_n} \left(\widehat{\theta}_n - \theta_n\right) \xrightarrow{d} \mathcal{N}(0,\, 2c)$. Result of [Phillips and Magdalinos, 2007] but under weaker hypotheses.
- In case of complex spectrum, we should have a $\mathcal{CN}_p$ distribution (still to investigate...)

Remarks

- In the stable case (stationary and ergodic) and in the unstable case, respectively,

$$\frac{S_n}{n} \xrightarrow{as} \Gamma \qquad \text{and} \qquad \frac{S_n}{n^2} \xrightarrow{d} \sigma^2 \int_0^1 W^2(t)\mathrm{d}t \qquad (p = 1).$$

  To be compared with

$$n^{-\alpha} \left\| \frac{S_{n,\,n}}{n} - \Gamma_n \right\| \xrightarrow{p} 0.$$

- The triangle inequality directly implies that, under the same hypotheses,

$$\widehat{\theta}_n \xrightarrow{p} \theta.$$

  But the corresponding asymptotic normality cannot hold.
- As a consequence of the consistency of $\widehat{\theta}_n$, we also have

$$\|\widehat{A}_n - A_n\| \xrightarrow{p} 0 \qquad \text{and even} \qquad \widehat{A}_n \xrightarrow{p} A$$

  so that H_0 may be consistently estimated.

Moderate deviations

Definition ([Dembo and Zeitouni, 1998])

A sequence of random variables (V_n) *on a topological space* $(\mathcal{X}, \mathcal{B})$ *satisfies a large deviation principle (LDP) with speed* (a_n) *if there exists a lower semi-continuous application* $I : \mathcal{X} \to \bar{\mathbb{R}}^+$ *such that :*

- *For any closed set* $F \in \mathcal{B}$,

$$\limsup_{n \to +\infty} \frac{1}{a_n} \ln \mathbb{P}(V_n \in F) \leqslant - \inf_{x \in F} I(x).$$

- *For any open set* $G \in \mathcal{B}$,

$$- \inf_{x \in G} I(x) \leqslant \liminf_{n \to +\infty} \frac{1}{a_n} \ln \mathbb{P}(V_n \in G).$$

In particular, if the infimum of I *coincides on the interior* $\overset{o}{H}$ *and the closure* $\bar{H}$ *of a set* $H \in \mathcal{B}$*, then*

$$\lim_{n \to \infty} \frac{1}{a_n} \ln \mathbb{P}(V_n \in H) = - \inf_{x \in H} I(x).$$

Example of LDP

$$\lim_{n \to +\infty} \frac{1}{n} \ln \mathbb{P}\big(|\bar{X}_n| \geqslant r\big) = -\frac{r^2}{2} \qquad \text{with } X_i \overset{iid}{\sim} \mathcal{N}(0,1)$$

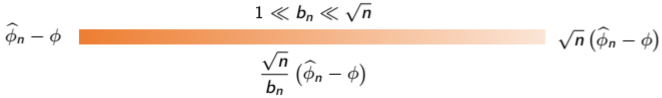
so that

$$\mathbb{P}\big(|\bar{X}_n| \geqslant r\big) = \mathrm{e}^{-\frac{r^2}{2} n + o(n)} \qquad \text{when } n \to +\infty.$$

Moderate deviations

Definition ([Dembo and Zeitouni, 1998])

A sequence of random variables (V_n) on a topological space $(\mathcal{X}, \mathcal{B})$ satisfies a moderate deviation principle (MDP) with speed (b_n^2) and rate I if there exists a speed (a_n) with $b_n \ll a_n$ such that $(\frac{a_n}{b_n} V_n)$ satisfies a large deviation principle with speed (b_n^2) and rate I.



Moderate deviations

Proposition (MDP, simplified version)

Assume that... Then, the sequence

$$\left(\frac{\sqrt{n}}{b_n \sqrt{1-\rho_n}} \, (\widehat{\theta}_n - \theta_n) \right)_{n \geqslant 1}$$

satisfies an LDP on $\mathbb{R}^p$ *with speed* (b_n^2) *and rate* $I_\theta : x \mapsto \frac{1}{2} \langle x, \Sigma x \rangle$, *with*

$$\Sigma \propto \lim_{n \to \infty} (1-\rho_n) \, \Gamma_n.$$

- In concrete terms, for all $r > 0$,

$$\mathbb{P}\big(\|\widehat{\theta}_n - \theta_n\| \,>\, r_n\big) = \mathrm{e}^{-\frac{m_r}{2} b_n^2 + o(b_n^2)} \qquad \text{for } n \to +\infty$$

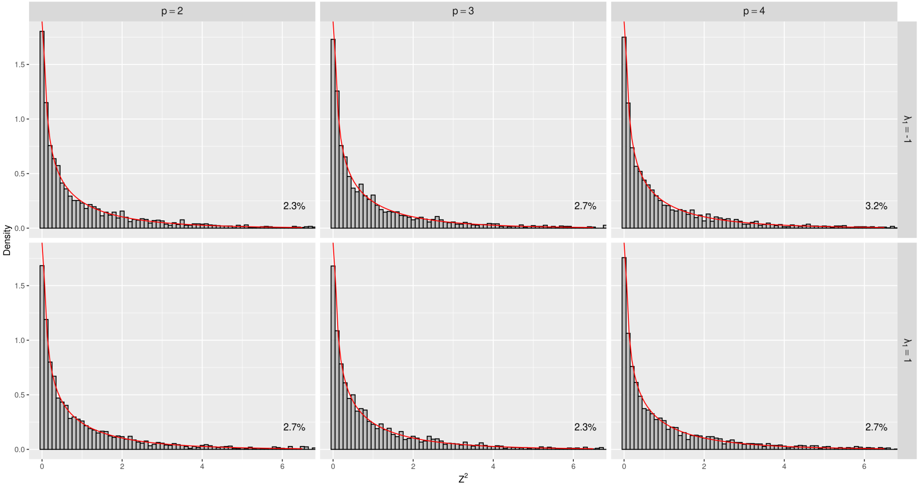
  where

$$r_n = \frac{b_n \sqrt{1-\rho_n}}{\sqrt{n}} \, r \to 0 \qquad \text{and} \qquad m_r = \inf_{\|x\|=r} \langle x, \Sigma x \rangle.$$

  (Consider for it the open set $H = \mathbb{R}^p \backslash \mathcal{B}_r(\mathbb{R}^p)$ and its closure on which $\inf I_\theta = \frac{m_r}{2}$.)
- Equivalent to [Miao et al., 2015] for $p = 1$.

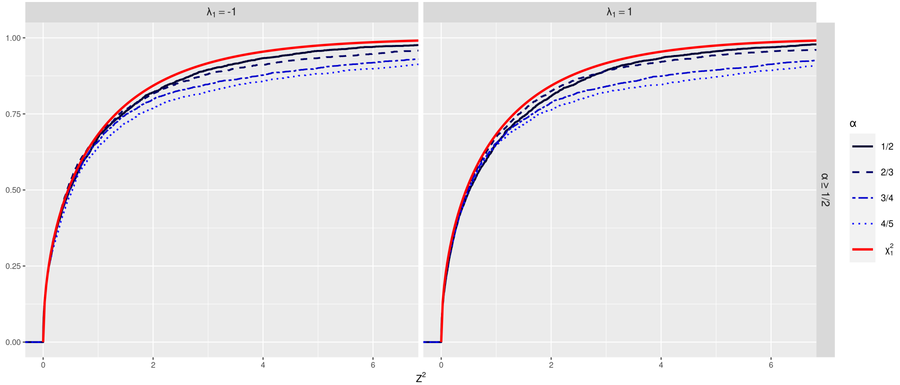
Simulations

Based on $Z_n^2 = \left(\frac{\sqrt{n\, v_n}}{h_0} \left\langle L_n, \widehat{\theta}_n - \theta_n \right\rangle \right)^2 \xrightarrow{d} \chi_1^2$ with $\rho_n = 1 - 1/\sqrt{n}$ and $n = 5000$.



Simulations

Based on $Z_n^2 = \left(\frac{\sqrt{n\, v_n}}{h_0} \left\langle L_n, \, \widehat{\theta}_n - \theta_n \right\rangle \right)^2 \xrightarrow{d} \chi_1^2$ with $\rho_n = 1 - 1/n^{\alpha}$ and $n = 5000$.



Summary

Other procedure of estimation

- If there's only one quasi-unit root (say, $\lambda_1 = 1$) then we have

$$\forall\, n \geqslant 1, \forall\, 1 \leqslant k \leqslant n, \qquad \left\{ \begin{array}{lcl} X_{n,\,k} & = & \rho_n\, X_{n,\,k-1} + Y_{n,\,k} \\ Z_{n,\,k} & = & \bar{A}_n\, Z_{n,\,k-1} + \bar{E}_k \end{array} \right.$$

  where $Z_{n,\,k} = (Y_{n,\,k}, \ldots, Y_{n,\,k-p+2})^T$ and $\bar{A}_n$ is a $(p-1) \times (p-1)$ companion matrix such that $\rho(\bar{A}_n) \to \rho(\bar{A}) < 1$.
- Hence, $(Y_{n,k})$ is a stable AR$(p-1)$ with time-varying coefficients $\beta_{n,\,1}, \ldots, \beta_{n,\,p-1}$, and $(X_{n,k})$ is a quasi-random walk generated by $(Y_{n,k})$.
- Consider $\rho_n^{(\alpha)} = 1 - \frac{c}{n^\alpha}$ with $0 < \alpha < 1$.
- Let $\Psi_{n,\,k}(\alpha) = (X_{n,\,k}, X_{n,\,k} - \rho_n^{(\alpha)} X_{n,\,k-1}, \ldots, X_{n,\,k-p+2} - \rho_n^{(\alpha)} X_{n,\,k-p+1})^{\,T}$ so that

$$\widehat{\vartheta}_n^{(\alpha)} = \left(\sum_{k=1}^{n} \Psi_{n,\,k-1}(\alpha)\, \Psi_{n,\,k-1}^{\,T}(\alpha) \right)^{-1} \sum_{k=1}^{n} \Psi_{n,\,k-1}(\alpha)\, X_{n,\,k}$$

  is the 'OLS' of $\vartheta_n^{(\alpha)} = (\rho_n^{(\alpha)}, \beta_{n,\,1}, \ldots, \beta_{n,\,p-1})^{\,T}$.
- Suppose (for the moment) that c is known.

Other procedure of estimation

- Let

$$\widehat{J}_n = \begin{pmatrix} 1 & & \\ -\widehat{\vartheta}_{n,\,1} & \ddots & \\ & \ddots & 1 \\ & & -\widehat{\vartheta}_{n,\,1} \end{pmatrix}_{p \times (p-1)} \qquad (= 0 \text{ if } p = 1)$$

and use it to build $\widetilde{\theta}_n = \widehat{J}_n \, \widehat{\vartheta}_{n,\,2:p} + \widehat{\vartheta}_{n,\,1} \, e_p$.

Proposition

Assume that (H_1), (H_2) and (H_3) hold, and that $\mathbb{E}[|\varepsilon_1|^{2+\nu}] = \eta_\nu < +\infty$ for some $\nu > 0$. Then,

$$\big\|\widetilde{\theta}_n - \theta_n\big\| \xrightarrow{p} 0 \qquad \text{and} \qquad \sqrt{n}\,\big(\widetilde{\theta}_n - \theta_n\big) \xrightarrow{d} \mathcal{N}\big(0,\, J\,\Sigma_{p-1}^{-1} J^{\,T}\big)$$

where

$$\Sigma_{p-1} = \sum_{k \geqslant 0} \bar{A}^{\,k}\, K_{p-1}\, (\bar{A}^{T})^{k}.$$

- Expected problem : $J\,\Sigma_{p-1}^{-1} J^{\,T}$ is singular.
- $\widetilde{\theta}_n$ is consistent for θ but it is asymptotically normal only if $\alpha > 1/2$.

(Quasi-)unit root test (extent of instability)

Theorem

Assume that (H_1), (H_2) and (H_3) hold, and that $\mathbb{E}[|\varepsilon_1|^{2+\nu}] = \eta_\nu < +\infty$ for some $\nu > 0$. Then, uniformly in $\alpha \in]0, 1[$,

$$C_n^{1/2}\,(\widehat{\vartheta}_n^{\,(\alpha)} - \vartheta_n^{(\alpha)}) \ \xrightarrow{d}\ \mathcal{N}_p(0,\, \Upsilon_p)$$

where $C_n = \mathrm{diag}(n^{1+\alpha}, n, \ldots, n)$ and Υ_p is positive definite. In addition,

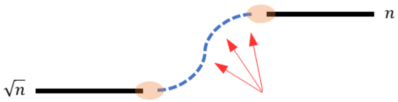
$$(\ln n)\sqrt{n^{1-\alpha}}\,(\widehat{\alpha}_n(\alpha) - \alpha) \ \xrightarrow{d}\ \mathcal{N}(0,\, \tau^2)$$

with $\tau^2 > 0$ and

$$\widehat{\alpha}_n(\alpha) = \frac{\ln c - \ln\left(1 - \widehat{\vartheta}_{n,1}^{\,(\alpha)}\right)}{\ln n}.$$

- Two huge improvements compared to the asymptotic normality of $\widehat{\theta}_n$:
  - The mixing matrix P_n has vanished.
  - The first component is the spectral radius.

(Quasi-)unit root test (extent of instability)



- Under $\mathcal{H}_0$: "$\alpha = \alpha_0$", the test statistic satisfies

$$(\ln n)\,\frac{\sqrt{n^{1-\alpha_0}}}{\widehat{\tau}_n}\,\left(\widehat{\alpha}_n(\alpha_0) - \alpha_0\right) \xrightarrow{d} \mathcal{N}(0,1).$$

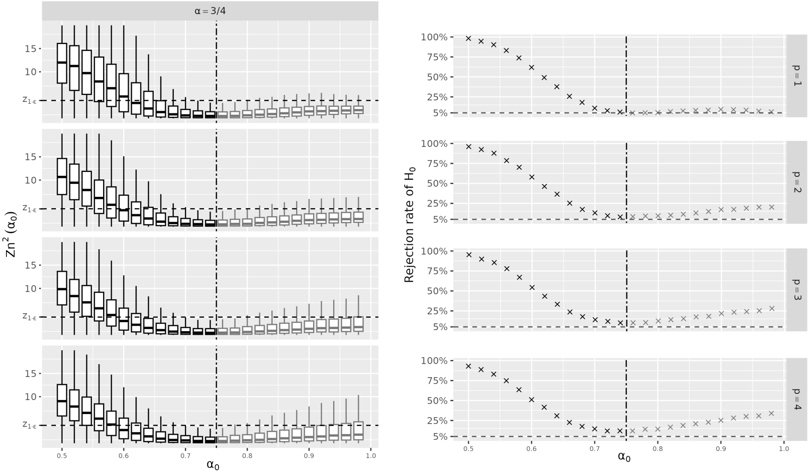
- Under $\mathcal{H}_1$: "$\alpha > \alpha_0 > \alpha/2$", the test statistic satisfies

$$\left[(\ln n)\,\frac{\sqrt{n^{1-\alpha_0}}}{\widehat{\tau}_n}\,\left(\widehat{\alpha}_n(\alpha_0) - \alpha_0\right)\right]^2 \xrightarrow{p} +\infty.$$

- Selection of $\alpha \in \mathcal{A} \subset [1/2, 1[$:

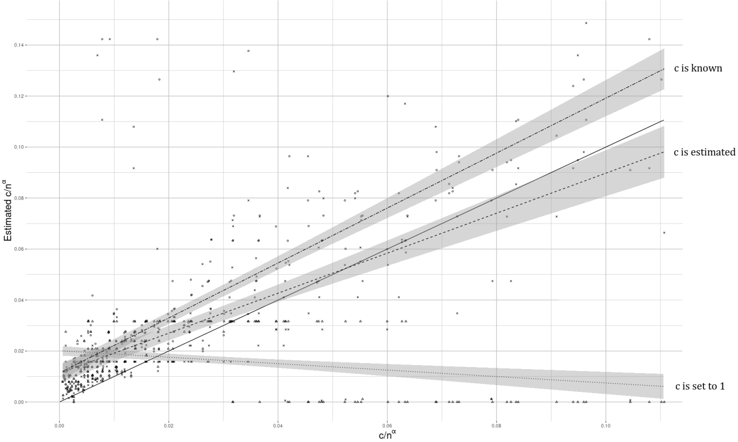
$$\widehat{\alpha}_c = \inf\{\alpha_0 \in \mathcal{A} : \mathcal{H}_0 \text{ is not rejected}\} \qquad (\inf \varnothing = +\infty).$$

Simulations (examples)



Selection of c

- For each $c_0 \in \mathcal{C}$, compute $\widehat{\alpha}_{c_0}$ based on the first $(n - n_{val})$ values of the series.
- Choose $(\widehat{c}_0, \widehat{\alpha}_{\widehat{c}_0})$ as the pair that optimizes some criterion stemming from the one-step predictions of the n_{val} remaining values.



Real data

- NelPlo dataset [Nelson and Plosser, 1982].

Series	n	p	$\widehat{c}$	$\widehat{\alpha}_{\widehat{c}}$	$\widehat{\rho}_n = 1 - \widehat{c}/n^{\widehat{\alpha}_{\widehat{c}}}$
Velocity	120	1	0.80	$\in [0.50, 0.66]$	$\in [0.93, 0.97]$
Industrial production	129	6	0.80	$\in [0.50, 0.72]$	$\in [0.93, 0.98]$
Nominal GNP	80	2	0.80	$\in [0.50, 0.57]$	$\in [0.91, 0.93]$
Consumer prices	129	4	1.08	$\in [0.73, 0.98]$	$\in [0.97, 0.99]$
Employment	99	5	0.80	$\in [0.50, 0.59]$	$\in [0.92, 0.95]$
Interest rate	89	4	0	$+\infty$	1
Wages	89	2	0.80	$\in [0.50, 0.56]$	$\in [0.92, 0.94]$
GNP deflator	100	6	1.20	$\in [0.50, 0.53]$	$\in [0.88, 0.90]$
Money stock	100	3	0.80	$\in [0.50, 0.53]$	$\in [0.92, 0.93]$
Real GNP	80	4	1.10	$\in [0.50, 0.74]$	$\in [0.88, 0.96]$
Common stock prices	118	6	0.82	$\in [0.50, 0.98]$	$\in [0.92, 0.99]$
Real per capita GNP	80	2	0.80	$\in [0.50, 0.62]$	$\in [0.91, 0.95]$
Real wages	89	2	1.04	$\in [0.50, 0.79]$	$\in [0.89, 0.97]$
Unemployment rate	99	3	1.08	$\in [0.50, 0.65]$	$\in [0.89, 0.95]$

Thank you for your attention !

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