Wandering Domains for Skew-Products

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Joint work with Jasmin Raissy

Overview

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Theorem

The following properties hold:

- *$\mathcal{F}(f)$ is open and $\mathcal{J}(f)$ is closed.*
- *Both $\mathcal{F}(f)$ and $\mathcal{J}(f)$ are completely invariant.*

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Classification of Fatou Components

Definition

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Let U be a Fatou component of f and define U_n as the Fatou component such that $f^n(U) \subset U_n$, $n \in \mathbb{N}_0$. Then U is called

- **periodic** if $U_0 = U_n$ for some $n \neq 0$, and **invariant** if for $n = 1$;
- **preperiodic** if $U_n = U_m$ for some $n \neq m$;
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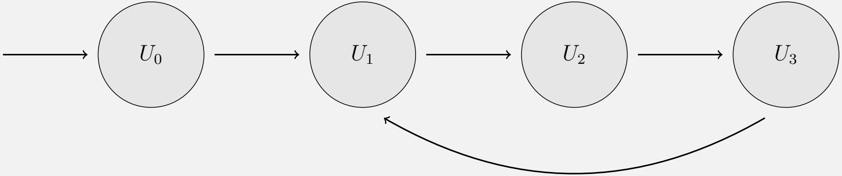

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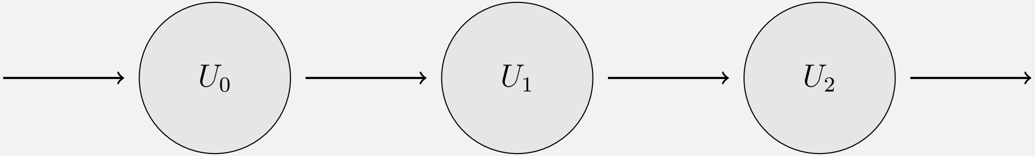

Periodic Fatou Components

Theorem (Fatou Classification Theorem; Fatou, Cremer, Baker-Kotus-Lü, ...)

Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and U an invariant Fatou component of f. Then one of the following holds:

(i) *U is the* ***immediate attractive basin*** *of some $z_0 \in U$.*

(ii) *U is the* ***parabolic domain*** *of some $z_0 \in \partial U$.*

(iii) *U is a* ***rotation domain****.*

(iv) *U is a* ***Baker domain****, that is, $f^n(z) \to \infty$ for all $z \in U$ as $n \to \infty$, but $f(\infty)$ is not well-defined. Only t.e.f. can have Baker domains.*

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Classification in one dimension:

- Simply-connected wandering domains are classified in **[Benini, Evdoridou, Fagella, Rippon, and Stallard, 2022]**.
- Multiply-connected wandering domains are described in **[Bergweiler, Rippon, and Stallard, 2013]** and also in **[Ferreira and Rempe, 2024]**.

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- Space of all holomorphic functions too large.
  $\to$ Consider a subclass of functions.

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→ Easy iteration of first coordinate on invariant fibers.

Skew-Products

Definition

An **invariant fiber** of the skew-product F is a set of the form $\{w = c\} := \{(z, w) \in \mathbb{C}^2 \mid w = c\}$ for some $c \in \mathbb{C}$ such that $F(\{w = c\}) \subset \{w = c\}$, or equivalently, $g(c) = c$.

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- Without loss of generality $c = 0$.
- **Attracting fiber** = the fixed point of g is attracting.
- On fiber, reduction to $f_0(z) := f(z, 0)$ with one-dimensional dynamics:

$$\pi_z F^n(z, 0) = f_0^n(z)$$

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What do we mean by “reconstruct”?

Reconstruct = Bulging

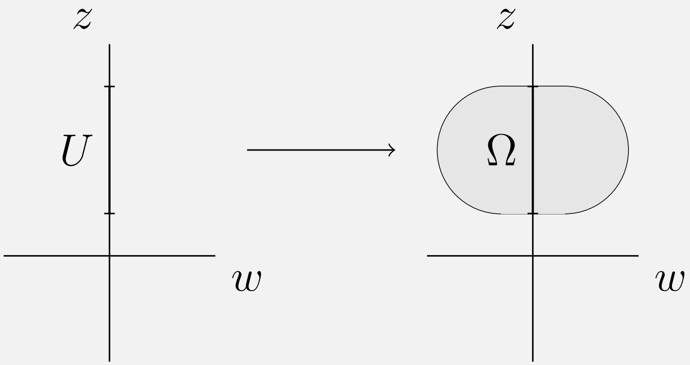
Definition

A Fatou component $U \subset \mathbb{C}$ of f_0 is said to **bulge** if there exists a Fatou component $\Omega \subset \mathbb{C}^2$ of F such that

$$U \times \{0\} \subset \Omega.$$

Ω is also called the **bulged component** of U.

Bulging



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- Rotation domains bulge (by **[Lilov, 2004]**).

Proof Ideas

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Other Fatou components: Adjust the conjugacy to work in a small neighborhood of the fiber. E.g., for rotation domains, rotation like behavior with small errors.

Questions for the Rest of the Talk

1. What happens for wandering domains?
2. How to define normality? What are the differences between different definitions?

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Common Approach

Extend $f : \mathbb{C}^n \to \mathbb{C}^n$ to $f : X \to X$ with X some compactification of $\mathbb{C}^n$, e.g., projective space $\mathbf{P}_n\mathbb{C}$.

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Example

The function $p : \mathbb{C}^2 \to \mathbb{C}^2, (z, w) \mapsto (z^2, w^2)$ can be extended via

$$p : \mathbf{P}_2\mathbb{C} \to \mathbf{P}_2\mathbb{C}, [z : w : t] \mapsto [z^2 : w^2 : t^2]$$

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$\rightarrow$ In general, not possible for transcendental entire functions. Thus, extend only the image space.

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- $\rightarrow$ $\widehat{\mathbb{C}^n}$ in general not a complex manifold.
- $\rightarrow$ No equivalence to equicontinuity.
- $\rightarrow$ Not the only approach to measure convergence to infinity.

Alternatives

Using other compactifications (as in **[Arosio, Benini, Fornæss, and Peters, 2019]**):

Definition

A family $\mathcal{F} \subset \mathrm{Hol}(X, \mathbb{C}^n)$ is called

- $\mathbf{P}_n\mathbb{C}$**-normal** if every sequence in $\mathcal{F}$ has a locally uniformly convergent subsequence converging to some function in $\mathrm{Hol}(X, \mathbf{P}_n\mathbb{C})$, and
- $(\widehat{\mathbb{C}})^n$**-normal** if every sequence in $\mathcal{F}$ has a locally uniformly convergent subsequence converging to some function in $\mathrm{Hol}(X, \widehat{\mathbb{C}} \times \cdots \times \widehat{\mathbb{C}})$.

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$\rightarrow$ Equivalence to equicontinuity.

$\rightarrow$ Different notions of escaping to infinity.

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Consider the function $F = f \times g : \mathbb{C}^2 \to \mathbb{C}^2, (z, w) \mapsto (z^2, w^2)$:

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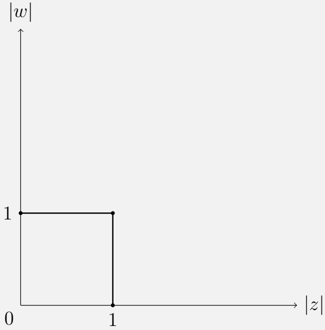

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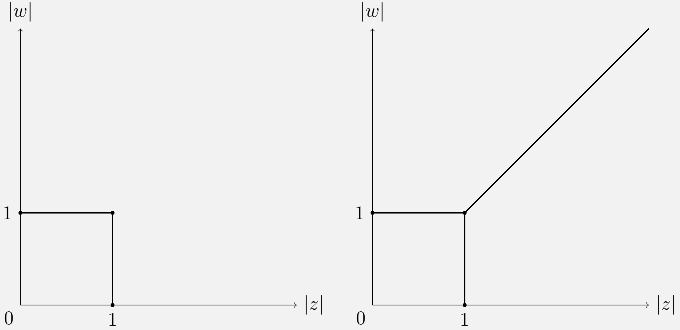

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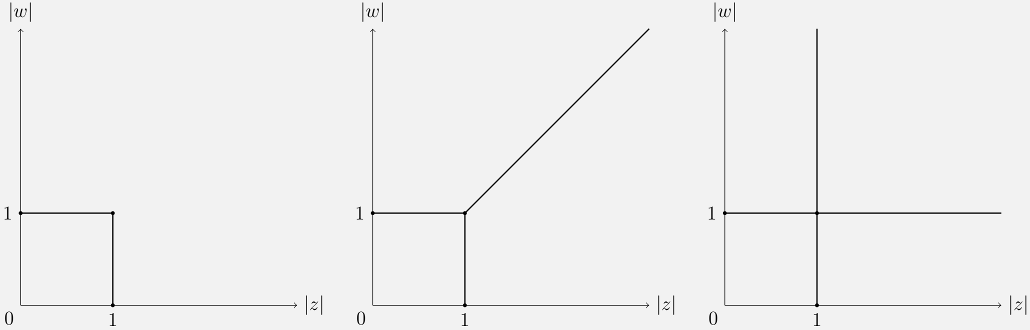

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(b) As $F : \mathbb{C}^2 \to \mathbf{P}_2\mathbb{C}$.

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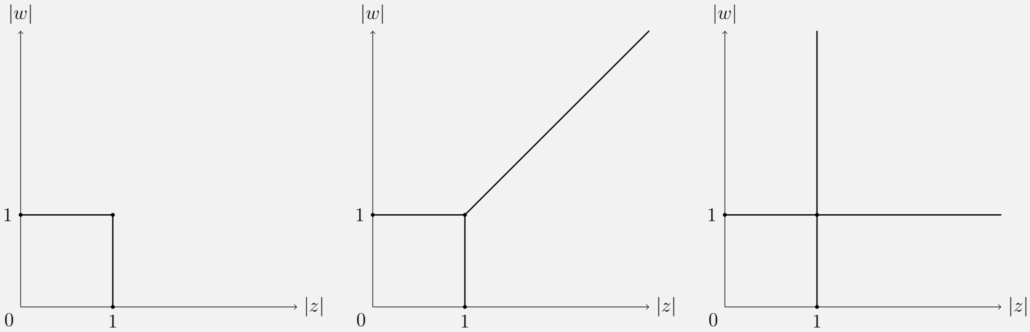

(a) As $F : \mathbb{C}^2 \to \widehat{\mathbb{C}^2}$.

(b) As $F : \mathbb{C}^2 \to \mathbf{P}_2\mathbb{C}$ or as $F : \mathbf{P}_2\mathbb{C} \to \mathbf{P}_2\mathbb{C}$.

(c) As $F : \mathbb{C}^2 \to \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$.

Relevance

- Bulging is a local property near the fiber. In the attracting case, we go towards the fiber and at most one coordinate goes to infinity.
- Relevant for global structural questions, e.g., whether bulged components are still distinct.

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Setting

Skew-product

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where

- $r(z, 0) = 0$ for all $z \in \mathbb{C}$, i.e., $\pi_z F|_{\{w=0\}} = f_0$
- $g(0) = 0$ and $|g'(0)| < 1$
- f_0 has wandering domains $(U_n)_{n\in\mathbb{N}_0}$ such that $f_0(U_n) \subset U_{n+1}$

Questions

1. Can wandering domains bulge in non-trivial settings?
2. Under which conditions do wandering domains bulge?
3. Do they always bulge?
4. What is the structure of the bulged domains?

Example of Wandering Domains

Consider function

$$f_0(z) = z + 2\pi i + 1 - \exp(z).$$

This kind of function has been studied, e.g., in **[Bergweiler, 1995]** and **[Fagella and Henriksen, 2009]**.

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→ Structurally periodic function combined with a translation.

→ Exhibits highly regular wandering domains.

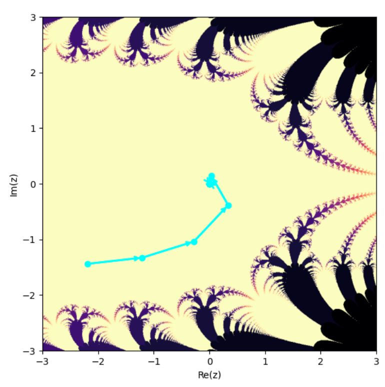
Understanding This Function

If we define $h(z) = z + 1 - \exp(z)$, then $f_0(z) = h(z) + 2\pi i$. Then:

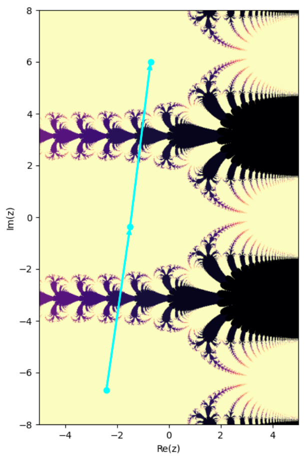
- $h(0) = 0$ and $h'(0) = 0$ and the same at $z_k := k2\pi i$, $k \in \mathbb{Z}$ with basins of attraction U_k.
- $h(z + 2\pi i) = h(z) + 2\pi i$
  $\rightarrow$ $f_0^n(z) = h^n(z) + n2\pi i$
- $\mathcal{F}(f_0) = \mathcal{F}(h)$ and thus U_k are also Fatou components of f_0.
- $f_0(z_k) = z_{k+1}$ yields $f_0(U_k) \subset U_{k+1}$.
  $\rightarrow$ U_k are wandering domains of f_0.

Wandering Domains

This structure yields wandering domains:



(a) h



(b) f_0

Two Mostly Equivalent Descriptions

Larger class of functions:

- $H(Z) = \lambda Z^d e^{\mu Z}$

  or

- $h(z) = a_1 z + a_2 T + p(z)$ for some $a_1, a_2 \in \mathbb{Z}$, $T \in \mathbb{C}^*$, and $p : \mathbb{C} \to \mathbb{C}$ holomorphic and T-periodic.

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More complicated behavior resulting in translations

$$\tau_{l_0,n} := l_0 a_1^n + a_2 \sum_{k=0}^{n-1} a_1^k.$$

Bulging Wandering Domains

Theorem (P.-Raissy)

Take f_0 as above. Under some growth conditions on r, the wandering domains of f_0 bulge.

Bulging Wandering Domains

Consider the setting of a function

$$F(z,w) := (z + 2\pi i + 1 - \exp(z) + r(z,w), g(w))$$

such that

- g holomorphic such that $g(0) = 0$ and $|g'(0)| < 1$;
- Given $\delta_1, \delta_2 > 0$ and $k \in \mathbb{N}_0$ there exist constants C_k such that

  $$|r(z,w)| \leq C_k |w|$$

  for all $z \in D(k2\pi i, \delta_1)$ and $w \in D(0, \delta_2)$.

Theorem (P.-Raissy)

Assume that there exist $\delta_1, \delta_2 > 0$ *such that for all* $k \in \mathbb{N}_0$ *we have*

$$\sum_{j=0}^{\infty} C_{k+j} |\lambda|^j < \infty, \tag{1}$$

or

$$\sum_{j=0}^{\infty} C_{k+j} |w|^{d^j - 1} < \infty, \tag{2}$$

for $C_k = C_k(\delta_1, \delta_2)$ *and for all* $w \in D(0, \delta_2)$*, depending on whether* $g'(0) = \lambda \neq 0$ *or* $g'(0) = 0$ *with a zero of order* d*.*

Then the wandering domains of f_0 *all bulge.*

Remarks

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- Also works for some $g(z, w)$ (i.e. non-skew products) with $g(z, 0) = 0$. .
- The constants C_k are measuring the growth of r close to the disks $D(z_k, \delta)$. Race between the errors terms and the attraction on and towards the fiber.

Some Examples

Example

In the superattracting case, any function

$$r(z,w) = \sum_{k=0}^{n} a_k(w) z^k$$

with $a_k : \mathbb{C} \to \mathbb{C}$ holomorphic and $a_0(0) = \cdots = a_k(0) = 0$, satisfies the growth conditions.

Proof Idea

Main ingredients:

- Very uniform wandering domains with (Euclidean) attraction at every iteration.
- Points close to a disk in U_0 get pulled along to ∞.
- The error conditions arise from the inequalities that are required.

Back to the Questions

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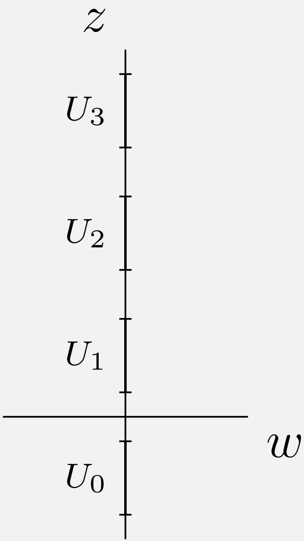
- Does the notion of bulging have to be adapted to the case of wandering domains? Behavior like partial bulging or non-bulging.
- Global structure: Are bulged wandering domains still wandering?

Small Excursion on the Global Structure



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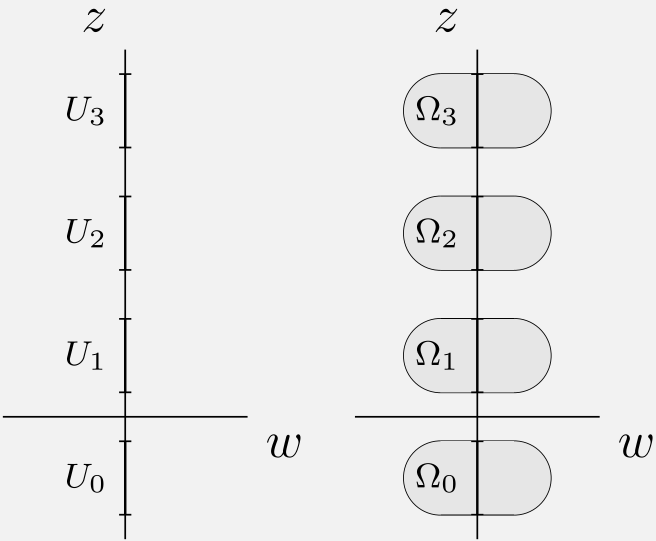

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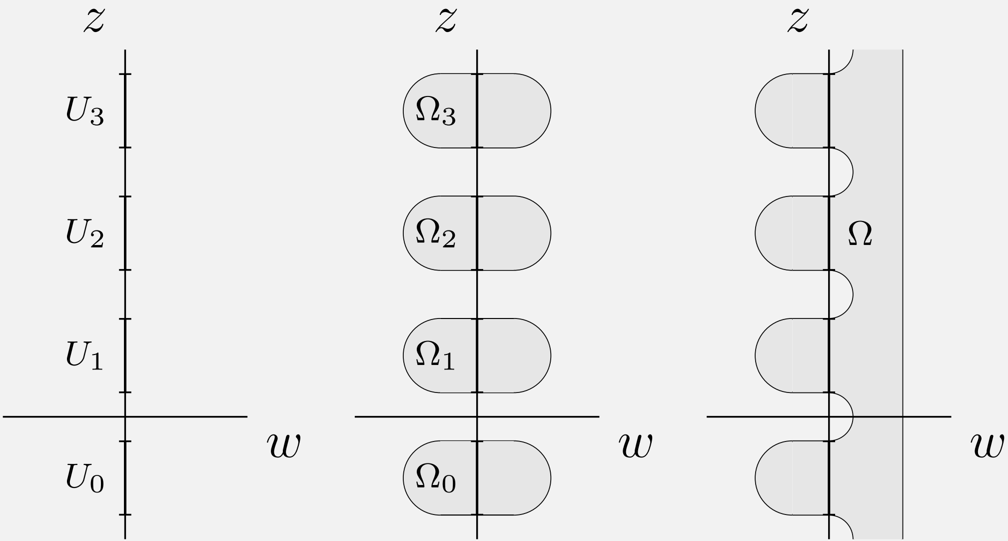

Small Excursion on the Global Structure

Example

Consider a direct product $F = f \times g$ where f has wandering domains. For $g(w) = w^2$ and f with wandering domains going to infinity (quickly enough), we have that

- the bulged domains w.r.t. to $\widehat{\mathbb{C}^2}$ form one large component, and
- the bulged domains w.r.t. to $\mathbf{P}_2\mathbb{C}$ or $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ are disjoint.

Thank you for your attention.

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