# Hyperbolic components and iterated monodromy of polynomial skew-products of $\mathbb{C}^2$

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# Setting and results

# Setting

- Polynomial skew-products of degree $d \geq 2$ :

$$f_\lambda(z,w) = (p(z), w^d + a_{\lambda,d-2}(z)w^{d-2} + \cdots + a_{\lambda,0}(d))$$

- $q_{\lambda,z}(w) = w^d + a_{\lambda,d-2}(z)w^{d-2} + \cdots + a_{\lambda,0}(d)$ monic, centered

- Parametrization :

$$a_{\lambda,j}(z) = a_{j,d-j}z^{d-j} + \cdots + a_{j,0}$$

$$\lambda = (a_{j,k})_{\substack{0\leq j\leq d-2 \\ 0\leq k\leq d-j}}$$

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- Homogeneous parametrization (no loss of information) :

$$a_{\lambda,j}(z) = a_{j,d-j}^{d-j} z^{d-j} + \cdots + a_{j,0}^{d-j}$$

$$\lambda = (a_{j,k})_{\substack{0\leq j\leq d-2 \\ 0\leq k\leq d-j}} \in \mathbb{C}^{D_d}$$

- $(\lambda, w) \mapsto q_{\lambda,z}(w)$ homogeneous degree $d$

# Setting

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- $q_{\lambda,z}(w) = w^d + a_{\lambda,d-2}(z)w^{d-2} + \cdots + a_{\lambda,0}(d)$

- Homogeneous parametrization :

$$a_{\lambda,j}(z) = a^{d-j}_{j,d-j} z^{d-j} + \cdots + a^{d-j}_{j,0}$$

- Parameter space

$$\mathrm{Sk}(p,d) = \{f_\lambda = (p(z), q_{\lambda,z}(w))\ ;\ \lambda \in \mathbb{C}^D_d\} \simeq \mathbb{C}^{D_d}$$

# Setting

- Polynomial skew-products of degree $d \geq 2$ :

$$f(z,w) = (p(z), w^d + a_{d-2}(z)w^{d-2} + \cdots + a_0(d))$$

- $a_{\lambda,j}(z) = a_{j,d-j}^{d-j} z^{d-j} + \cdots + a_{j,0}^{d-j}$

- $f : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ is a endomorphism i.e. $\deg_z a_j \leq d - j$

- $J_p =$ Julia set of $p$ in $\mathbb{C}$

- $J(f_\lambda) =$ small Julia set $=$ support of the equilibrium measure $\mu_{f_\lambda}$

$$J(f_\lambda) = \overline{\bigcup_{z \in J_p} \{z\} \times J_z(f_\lambda)} \subset J_p \times \mathbb{C} \quad \text{(Jonsson)}$$

# Structural stability

**Theorem (Astorg-Bianchi 23')**

*For $f_{\lambda_0} \in \mathrm{Sk}(p,d)$, $\exists U$ neighborhood and a continuous family $(h_\lambda)_{\lambda \in U}$, where $h_\lambda$ is the unique $C^0$ map from $J(f_{\lambda_0})$ to $\mathbb{C}^2$ s.t.*

- $h_\lambda(z,w) = (z, g_{\lambda,z}(w))$ *respects product structure of* $J_p \times \mathbb{C}$
- $h_\lambda : J(f_{\lambda_0}) \to J(f_\lambda)$ *homeomorphism*
- $h_\lambda \circ f_{\lambda_0} = f_\lambda \circ h_{\lambda_0}$ *on* $J(f_{\lambda_0})$

**Theorem (Astorg-Bianchi 23')**

*Stability preserves hyperbolicity in* $\mathrm{Sk}(p,d)$

- The notion of hyperbolic components makes sense in $\mathrm{Sk}(p,d)$

# Structural stability

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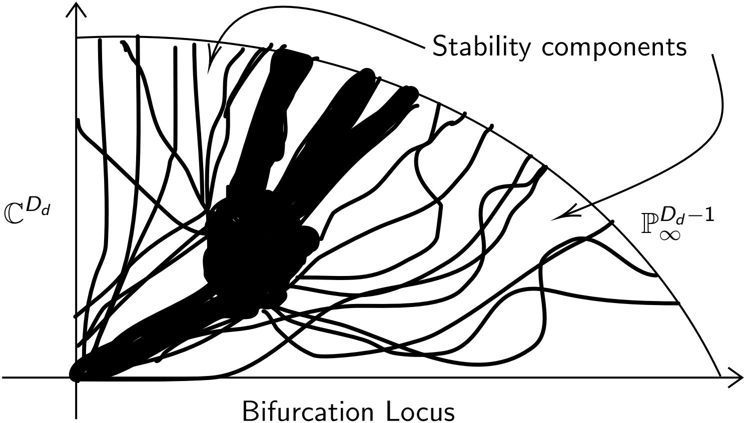
**Theorem (Astorg-Bianchi 23')**

*Stability preserves hyperbolicity in* $\mathrm{Sk}(p,d)$

- $f_\lambda$ hyperbolic if there exists $C > 0$, $K > 1$ s.t.

$$\forall (z,w) \in J(f_\lambda),\ ||df_\lambda^n(z,w)|| \geq CK^n,\ \forall n \geq 0$$

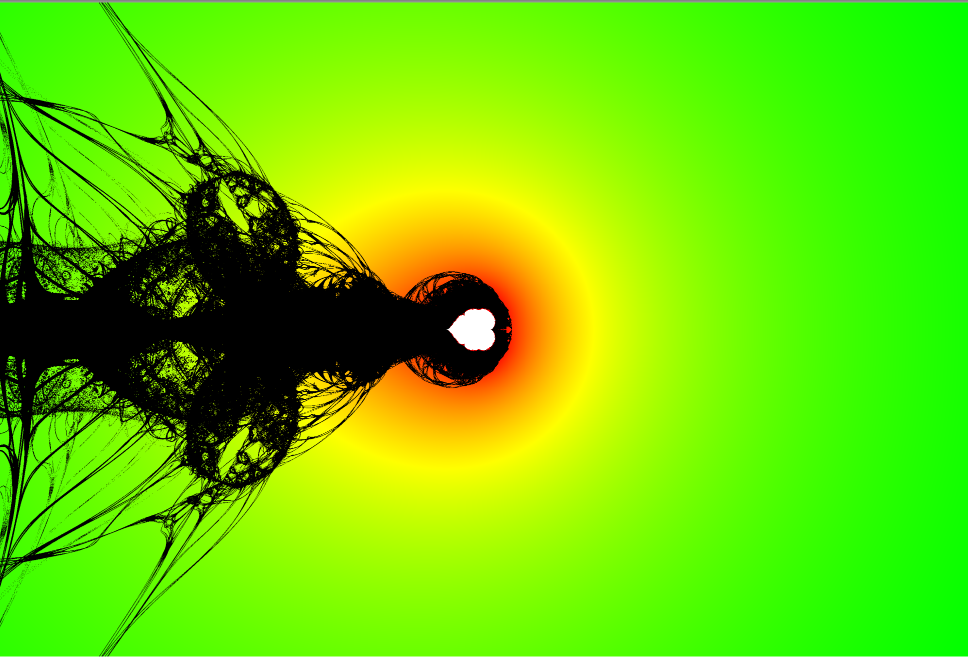
# Parameter space



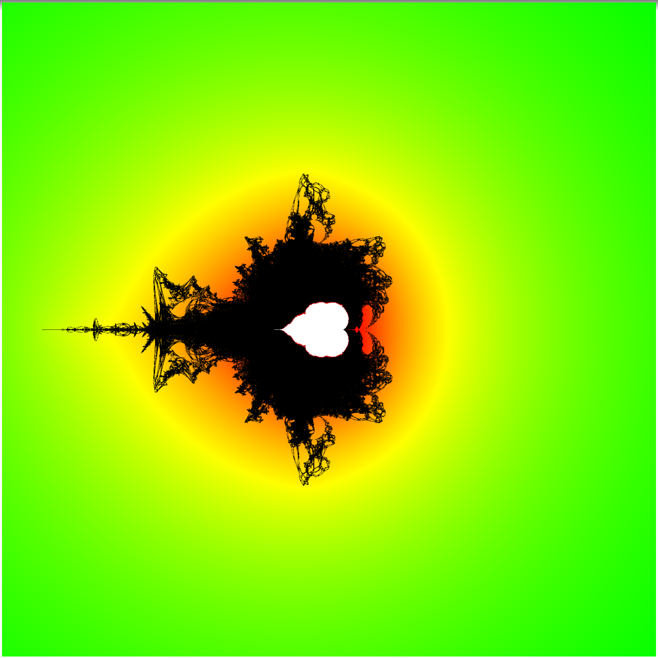

→ Non empty interior

→ Non compact in $\mathbb{C}^{D_d}$

# Slice $\{(z^2 - 2, w^2 + c(2 - z))\}_{c \in \mathbb{C}}$ in $\mathrm{Sk}(z^2 - 2, 2)$

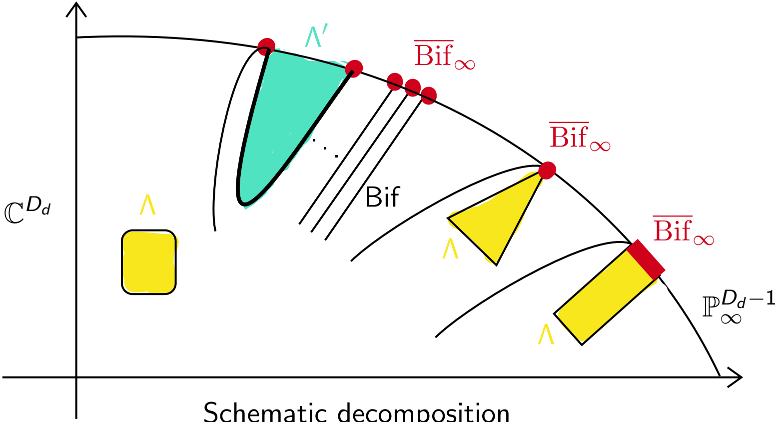


# Slice $\{(z^2 - 2, w^2 + c(4 - z))\}_{c \in \mathbb{C}}$ in $\mathrm{Sk}(z^2 - 2, 2)$



# Aim : Study $\overline{\mathrm{Bif}}_\infty$ & hyperbolic components $\Lambda'$

$$\{\text{Unbounded Hyperbolic components } \Lambda' \text{ s.t. } \overline{\Lambda'} \not\subset \overline{\mathrm{Bif}}_\infty\}$$




Schematic decomposition

# Topological Results

- $P(\lambda, z) := \mathrm{Discrimiant}(q_{\lambda,z})$ : homogeneous of degree $d(d-1)$
- $J_p$ = Julia set of $p$ in $\mathbb{C}$

**Theorem (Astorg-Bianchi $d = 2$, T. $\geq 2$)**

*For every $p$ we have*

$$\overline{\mathrm{Bif}}_\infty = \bigcup_{z \in J_p} \left\{ [\lambda] \in \mathbb{P}_\infty^{D_d} \ : \ P(\lambda, z) = 0 \right\}.$$

# Topological Results

**Theorem (Valid for every $p$)**

$$\overline{\mathrm{Bif}}_\infty = \bigcup_{z\in J_p} \left\{[\lambda] \in \mathbb{P}_\infty^{D_d} \ : \ P(\lambda, z) = 0\right\}.$$

- Let $\Lambda'$ Unbounded Component, and let $\lambda \in \Lambda'$

**Theorem (Assume $p(z) = z^d$)**

$\exists$ *finite homeomorphism $h$ of* $\{1, \cdots, d\}^{\mathbb{N}}$ *s.t. for all* $\lambda \in \Lambda'$

$$J(f_\lambda) \overset{\text{Homéo.}}{\xrightarrow[\simeq]{}} \frac{[0,1] \times \{1, \cdots, d\}^{\mathbb{N}}}{(0,x) \sim (1,y) \Leftrightarrow h(x) = y}$$

*Connected components of* $J(f_\lambda)$ = *continuous simple closed curves*

- $J(f_\lambda)$ = small Julia set = support of the equilibrium measure $\mu_{f_\lambda}$

# Topological Results

**Theorem (Assume $p(z) = z^d$)**

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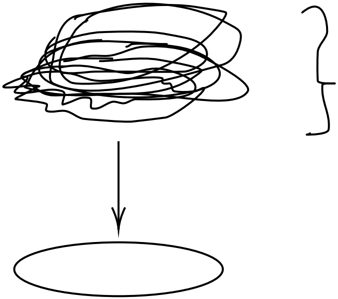
*Connected components of $J(f_\lambda)$ = continuous simple closed curves*

$J(f_\lambda) \subset \mathbb{S}^1 \times \mathbb{C}$



Accumulation of closed curves
"Cantor of curves"

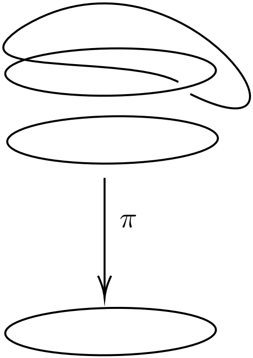
$J_p = \mathbb{S}^1$

# Results beyond topology

- Let $\lambda \in \Lambda'$ s.t. $[\lambda] \notin \overline{\mathrm{Bif}}_\infty$ and $||\lambda|| \gg 1$

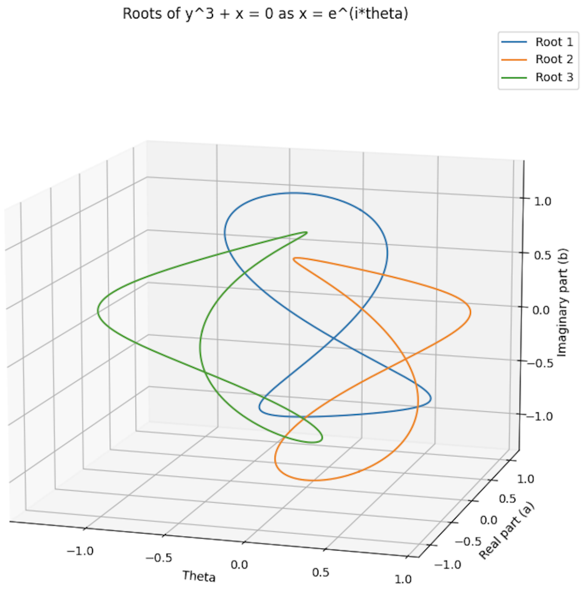
$$K_\lambda := f_\lambda^{-1}(\mathbb{S}^1 \times \{0\}) = \{(z,w) \in \mathbb{S}^1 \times \mathbb{C} \;:\; q_{\lambda,z}(w) = 0\}$$

- Isotopy class of $K_\lambda$ in $\mathbb{S}^1 \times \mathbb{C}$ = Algebraic braid of degree $d$




$$K_\lambda \subset \mathbb{S}^1 \times \mathbb{C} \xrightarrow{d\,:\,1} \mathbb{S}^1$$

# An algebraic braid of degree 3



# Results beyond topology

**Theorem (Assume p(z)=$z^d$)**

*For $\lambda$ has before :*

$$\text{Isotopy class of } K_\lambda = \text{Isotopy class of } \bigcup_{j=1}^{m} \mathcal{C}_j =: \mathrm{ab}(f_\lambda)$$

$(\mathcal{C}_j)_j$ = *connected components of fix points of $f_\lambda$ in $J(f_\lambda) \cap \{1\} \times \mathbb{C}$*

- $\mathrm{ab}(f_\lambda)$ well defined for each $\lambda \in \Lambda'$

**Theorem (Assume p(z)=$z^d$)**

*The following map is well defined :*

$$\mathrm{ab} : \begin{cases} \{\textit{Hyp. Comp. } \Lambda'\} & \longrightarrow & AB_d = \{\textit{Alg. braids degree } d\} \\ \Lambda' & \longmapsto & \mathrm{ab}(f_\lambda),\ \lambda \in \Lambda' \end{cases}$$

- In other words, $\mathrm{ab}(f_\lambda)$ only depends on $\Lambda'$.

# Results beyond topology

**Theorem (Assume p(z)=$z^d$)**

*The following map is well defined :*

$$\mathrm{ab} : \begin{cases} \{\textit{Hyp. Comp. } \Lambda'\} & \longrightarrow & AB_d = \{\textit{Alg. braids degree } d\} \\ \Lambda' & \longmapsto & \mathrm{ab}(f_\lambda),\ \lambda \in \Lambda' \end{cases}$$

- $\mathrm{ab}$ injective $\Longrightarrow$ classification of hyperbolic components $\Lambda'$

**Question**

*Is* $\mathrm{ab}(f_\lambda)$ *a complete invariant ?*

# Results beyond topology

**Theorem (Assume p(z)=z$^d$)**

*The following map is well defined :*

$$\mathrm{ab} : \begin{cases} \{\mathit{Hyp.\ Comp.}\ \Lambda'\} & \longrightarrow & AB_d = \{\mathit{Alg.\ braids\ degree}\ d\} \\ \Lambda' & \longmapsto & \mathrm{ab}(f_\lambda),\ \lambda \in \Lambda' \end{cases}$$

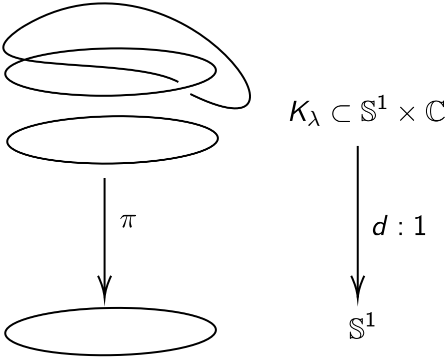
**Theorem (Assume p(z)=z$^d$)**

*The following map is surjective :*

$$\mathrm{cab} : \begin{cases} \{\mathit{Hyp.\ Comp.}\ \Lambda'\} & \xrightarrow{\mathrm{ab}} & AB_d & \xrightarrow{\mathrm{c}} & \mathrm{Conj}(\mathfrak{S}_d) \\ \Lambda' & \longmapsto & \mathrm{ab}(f_\lambda) & \longmapsto & \mathrm{c} \circ \mathrm{ab}(f_\lambda) \end{cases}$$

# Results beyond topology

- $\mathrm{cab}(f_\lambda)$ = perm. given by the monodromy of $K_\lambda$ above $z = 1$



Recall from dimension 1

# Moduli space $\mathcal{P}_d$ of polynomial mappings in one variable

- $d \geq 2$ integer
- $\lambda = (a_0, \cdots, a_{d-2}) \in \mathbb{C}^{d-1}$

$$p_\lambda(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0$$

- $p_\lambda^{\circ n} = p_\lambda \circ \cdots \circ p_\lambda$ $n$ times

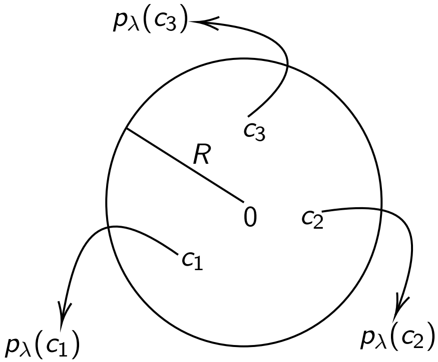
**Definition (Shift locus)**

$$\mathcal{S}_d := \{\lambda \in \mathbb{C}^{d-1} \ : \ \forall c \in \mathrm{Crit}(p_\lambda), \ \lim_{n\to+\infty} p_\lambda^{\circ n}(c) = \infty\}.$$

$\mathcal{S}_d$ is an hyperbolic component of $\mathcal{P}_d$

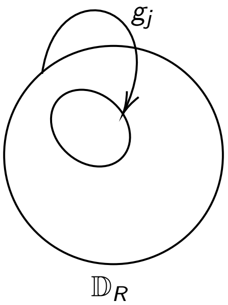
# Escape radius for $p_\lambda \in \mathcal{S}_d$

- Post-critical set : $P_{p_\lambda} = \bigcup_{n\geq 1} p_\lambda(\mathrm{Crit}\ p_\lambda)$
- $\exists R > 0$ such that $p_\lambda^{-1}(\mathbb{D}_R) \Subset \mathbb{D}_R$ and $\mathbb{D}_R \cap \overline{P}_{p_\lambda} = \emptyset$



# Contraction for inverse branches

- $p_\lambda : p_\lambda^{-1}(\mathbb{D}_R) \to \mathbb{D}_R$ is a $d-$fold covering map
- $g_j : \mathbb{D}_R \to p_\lambda^{-1}(\mathbb{D}_R)$, $1 \leq j \leq d$, inverse branches

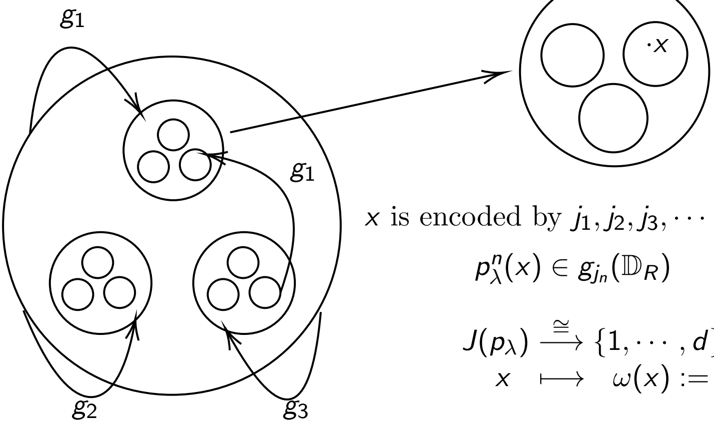



$$g_j^* \left( \frac{2R|dz|}{R^2 - |z|^2} \right) < \frac{2R|dz|}{R^2 - |z|^2}$$

$g_j$ contracts the Poincaré metric (Schwarz-Pick)

# Contracting IFS (Iterated Function System)

$$J(p_\lambda) = \bigcap_{n=1}^{+\infty} \bigcup_{1\le j_1,\cdots,j_n\le d} g_{j_1}\circ\cdots\circ g_{j_n}(\mathbb{D}_R)$$

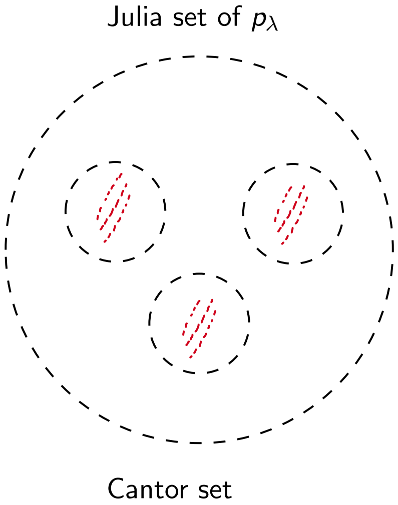



$x$ is encoded by $j_1, j_2, j_3, \cdots, j_n, \cdots$

$$p_\lambda^n(x) \in g_{j_n}(\mathbb{D}_R)$$

$$\begin{array}{ccc} J(p_\lambda) & \xrightarrow{\cong} & \{1,\cdots,d\}^{\mathbb{N}} \\ x & \longmapsto & \omega(x) := (j_n)_{n\ge 1} \end{array}$$

# Cantor set



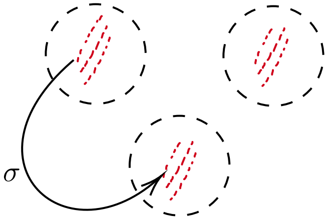
# Symbolic dynamical system

**Theorem**

*If* $p_\lambda \in \mathcal{S}_d$ *then* $\exists \omega : J(p_\lambda) \to \{1, \cdots, d\}^{\mathbb{N}}$ *homeomorphism s.t.*

$$\omega \circ p_\lambda = \sigma \circ \omega,$$

$$\sigma(j_1 \cdots j_n \cdots) = (j_2 \cdots j_{n+1} \cdots).$$



# Polynomial skew-products of $\mathbb{C}^2$

# Setting

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- Homogeneous parametrization (no loss of information) :

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$$\lambda = (a_{j,k})_{\substack{0 \leq j \leq d-2 \\ 0 \leq k \leq d-j}} \in \mathbb{C}^{D_d}$$

- $(\lambda, w) \mapsto q_{\lambda,z}(w)$ homogeneous degree $d$

# Moduli space $\mathrm{Sk}(p, d)$ of skew-products with a fixed base

- Non autonomous iterations $f_\lambda^{\circ n} = (p^{\circ n}(z), Q_{\lambda,z}^{\circ n}(w))$ with

$$Q_{\lambda,z}^{\circ n}(w) := q_{\lambda,p^{\circ(n-1)}(z)} \circ \cdots \circ q_{\lambda,z}(w)$$

**Definition (Higher dimensional analogue of the Shift locus $\mathcal{S}_d$)**

$$\mathcal{D} := \{\lambda \in \mathbb{C}^{D_d} : \forall z \in J_p, \forall c \in \mathrm{Crit}(q_{\lambda,z}),\ \lim_n |Q_{\lambda,z}^{\circ n}(c)| = +\infty\}$$

- $\mathcal{D}$ has infinitely many connected components in general

- $p$ hyperbolic : con. comp. $\Lambda$ of $\mathcal{D}$ are hyperbolic components

# Bifurcation locus

**Theorem (Astorg-Bianchi)**

$$\mathrm{Bif} = \overline{\bigcup_{z\in J_p} \mathrm{Bif}_z},$$

$$\mathrm{Bif}_z = \partial\{\lambda \in \mathbb{C}^{D_d}, \exists c \in \mathrm{Crit}(q_{\lambda,z}) \ s.t. \ \sup_n |Q^{\circ n}_{\lambda,z}(c)| < +\infty\}$$

- $\mathcal{D} \subset \mathbb{C}^{D_d} \backslash \mathrm{Bif}$ open set of the stability locus

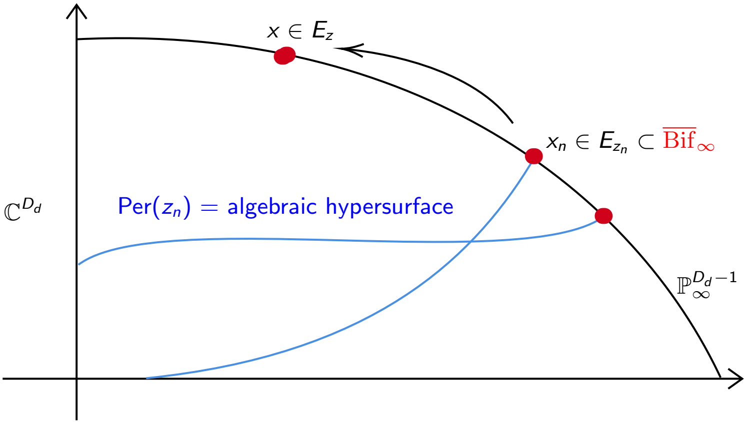
**Theorem (Astorg-Bianchi $d = 2$, T. $d \geq 2$)**

$$\overline{\mathrm{Bif}}_\infty = \bigcup_{z\in J_p} \left\{[\lambda] \in \mathbb{P}^{D_d}_\infty \ : \ P(\lambda, z) = 0\right\}.$$

- $P(\lambda, z) =$ (discriminant of $q_{\lambda,z}$) ; Homogeneous in $\lambda$

# Proof

- $\mathrm{Bif} = \overline{\bigcup_{z\in J_p} \mathrm{Bif}_z} \Rightarrow \overline{\mathrm{Bif}}_\infty \subset \bigcup_{z\in J_p} \left\{[\lambda] \in \mathbb{P}^{D_d}_\infty \ : \ P(\lambda, z) = 0\right\} = \bigcup_{z\in J_p} E_z$
- If $x = [\lambda] \in E_z, z \in J_p$, then $x \in \overline{\bigcup_{n\geq 0} E_{z_n}} \subset \overline{\mathrm{Bif}}_\infty$




- $\overline{\mathrm{Per}(z_n)}_\infty = E_{z_n} \subset \overline{\mathrm{Bif}}_\infty$ ; $z_n \in J_p \to z$ and $p^{N_n}(z_n) = z_n$ periodic

# Hyperbolic components $\overline{\Lambda'} \not\subset \overline{\mathrm{Bif}_\infty}$

- $\mathcal{D} = \{\lambda \; : \; \forall z \in J_p, \forall c \in \mathrm{Crit}(q_{\lambda,z}), \; \lim_n |Q_{\lambda,z}^{\circ n}(c)| = +\infty\}$

**Definition**

$\mathcal{D}' := \{\lambda \in \mathcal{D}, \; \exists \text{continuous path inside } \mathcal{D} \text{ joining } \lambda \text{ to } \mathbb{P}_\infty^{D_d - 1} \backslash \overline{\mathrm{Bif}_\infty}\}$

- Components $\Lambda' \subset \mathcal{D}'$ are hyperbolic (if $p$ hyperbolic)

# Link with $\mathcal{S}_d$ : periodic fibers

- $p^N(z) = z$ , $z \in J_p$, $p \in \mathcal{P}_d$ hyperbolic

- $\lambda \in \mathcal{D}'$ of large norm, $[\lambda] \notin \overline{\mathrm{Bif}}_\infty$ and $R \gg 1$

$$J_z(f_\lambda^N) = \bigcap_{n=1}^{+\infty} Q_{\lambda,z}^{-Nn}(\mathbb{D}_R) : \ \mathrm{IFS}$$

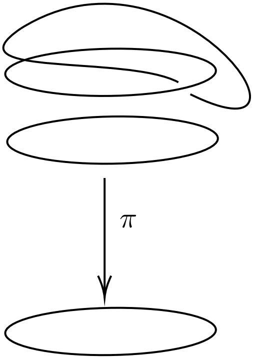
- $Q_{\lambda,z}^N \in \mathcal{S}_{d^N}$

# Algebraic braids

# Closed Braids of degree $d$

**Definition**

*$K \Subset \mathbb{S}^1 \times \mathbb{C}$ topological $1-$manifold with unramified first projection $\pi : K \to \mathbb{S}^1$ of degree $d$*




$$\begin{array}{c} K \subset \mathbb{S}^1 \times \mathbb{C} \\ \big\downarrow \; d:1 \\ \mathbb{S}^1 \end{array}$$

# Algebraic braids

- A closed braid $K$ is algebraic if $K = \{q_z(w) = 0\} \cap (\mathbb{S}^1 \times \mathbb{C})$ with

$$q_z(w) = a_d(z)w^d + a_{d-1}(z)w^{d-1} + \cdots + a_0(z) \in \mathbb{C}[z][w].$$

- There exists a bijective map

$$\frac{\{\text{algebraic braids of degree } d\}}{\text{Ambient isotopies of } \mathbb{S}^1 \times \mathbb{C}} \xrightarrow{\cong} AB_d,$$

where $AB_d \subset \mathrm{Conj}(B_d)$, and where $B_d$ is the braid group of Artin :

$$B_d = \{\sigma_1, \cdots, \sigma_{d-1} | \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\ ;\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2\}$$

# Classification of Astorg-Bianchi for $\mathrm{Sk}(z^2, 2)$

- $\mathbb{C}^3 \ni \lambda = (a, b, c) \in \mathcal{D}'$ s.t. $[\lambda] \notin \overline{\mathrm{Bif}}_\infty$ ; $K := \{q_\lambda = 0\} \cap (\mathbb{S}^1 \times \mathbb{C})$.
- If $||(a, b, c)|| >> 1$, $K$ is an algebraic braid of degree 2
- $s(\lambda) := \{z \in \mathbb{D} : az^2 + bz + c = 0\}$ constant on components of $\mathcal{D}'$

| $s(\lambda)$ | $K = \{q_\lambda = 0\} \cap (\mathbb{S}^1 \times \mathbb{C})$ |
|---|---|
| 0 | 2 connected components which do not interlace and wind 1 time above $\mathbb{S}^1$ |
| 1 | 1 connected component winding 2 times above $\mathbb{S}^1$ |
| 2 | 2 connected components which interlace 1 time and wind 1 time above $\mathbb{S}^1$ |

**Theorem** ($d = 2$, $p(z) = z^2$)

*The map* $s : \{\Lambda' \subset \mathcal{D}'\} \to \{0, 1, 2\}$ *is bijective*

- Astorg-Bianchi classification extends for $\mathrm{Sk}(p, 2)$ for each $p$

# Classification of Astorg-Bianchi for $\mathrm{Sk}(z^2, 2)$

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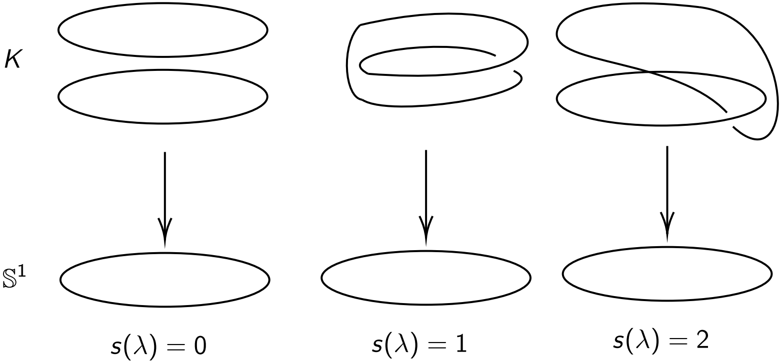


Iterated monodromy : a tool for higher degrees

# Expanding covering maps

- Let $\Lambda' \subset \mathcal{D}'$, and let $\lambda \in \Lambda'$ s.t.
  - $\star$ $\exists R > 0$ s.t. $f_\lambda^n : f_\lambda^{-n}(\mathbb{S}^1 \times \mathbb{D}_R) \to \mathbb{S}^1 \times \mathbb{D}_R$ is a $d^{2n}$–fold covering map
  - $\star$ $J(f_\lambda) =$ limit of the nested sequence $f_\lambda^{-(n+1)}(\mathbb{S}^1 \times \mathbb{D}_R) \Subset f^{-n}(\mathbb{S}^1 \times \mathbb{D}_R)$ :

$$J(f_\lambda) = \bigcap_{n=0}^{\infty} f_\lambda^{-n}(\mathbb{S}^1 \times \mathbb{D}_R)$$

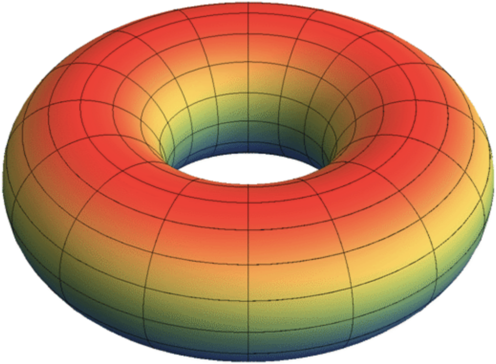
**Definition**

*$\lambda$ is a "good" parameter of $\Lambda'$ and $f_\lambda$ is an expanding covering map*

- Good parameters always exists in a component $\Lambda' \subset \mathcal{D}'$

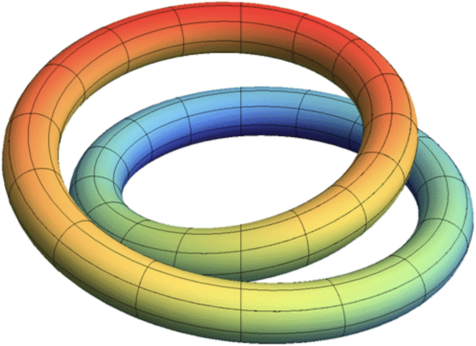
# Expanding covering maps

$$\mathbb{S}^1 \times \mathbb{D}_R$$



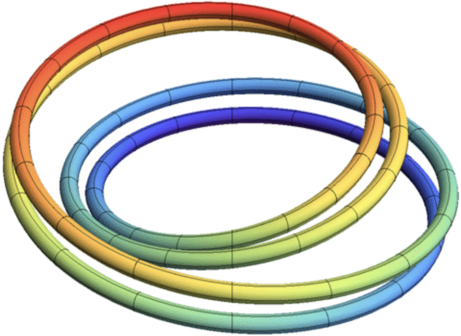
# Expanding covering maps

$$f_\lambda^{-1}(\mathbb{S}^1 \times \mathbb{D}_R)$$



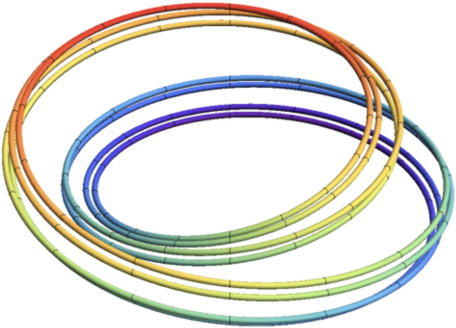
# Expanding covering maps

$$f_\lambda^{-2}(\mathbb{S}^1 \times \mathbb{D}_R)$$



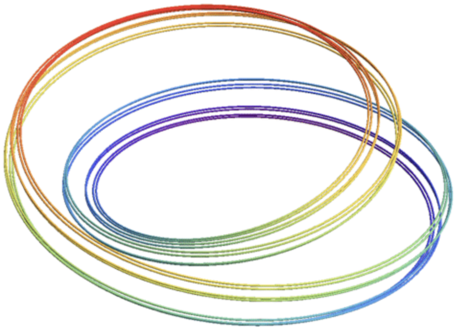
# Expanding covering maps

$$f_\lambda^{-3}(\mathbb{S}^1 \times \mathbb{D}_R)$$



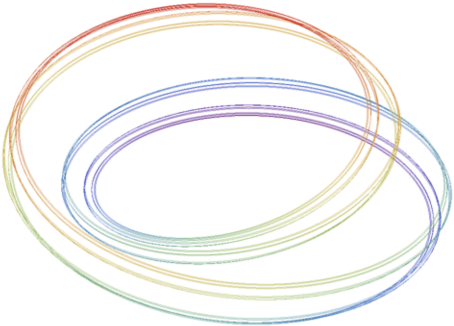
# Expanding covering maps

$$f_\lambda^{-4}(\mathbb{S}^1 \times \mathbb{D}_R)$$



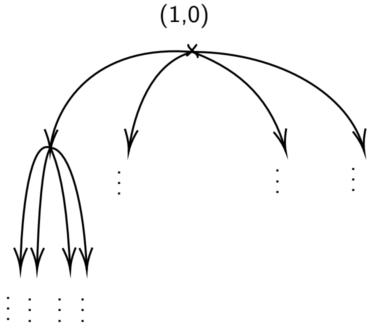
# Expanding covering maps

$$J(f_\lambda) = \bigcap_{n=1}^{\infty} f_\lambda^{-n}(\mathbb{S}^1 \times \mathbb{D}_R)$$

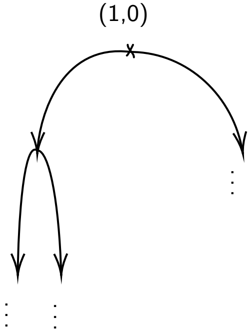


- We introduce two pre-image trees $\mathcal{X} \supset X$ with base point $(1, 0)$
- $X$ is contained in the fiber $z = 1$

$$\mathcal{X} = \bigcup_{n=0}^{+\infty} f^{-n}(1,0) \supset X = \bigcup_{n=0}^{+\infty} f^{-n}(1,0) \cap \{1\} \times \mathbb{C}$$

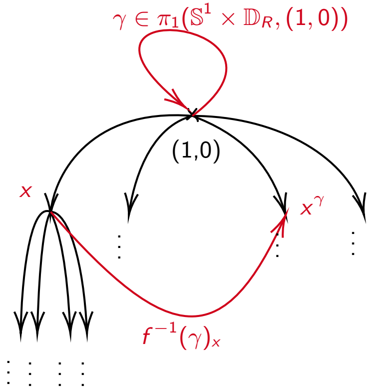



$\mathcal{X}$




$X$

- $\pi_1(\mathbb{S}^1 \times \mathbb{D}_R, (1,0)) \cong \mathbb{Z}$ acts on $\mathcal{X}$ by monodromy




$$x \mapsto x^\gamma := f^{-1}(\gamma)_x(1)$$

- The action of $\pi_1(\mathbb{S}^1 \times \mathbb{D}_R, (1,0))$ on $\mathcal{X}$ does not preserve the subtree $X$

**Lemma**

*The subgroups of $\pi_1(\mathbb{S}^1 \times \mathbb{C}, (1,0))$ acting on $X$ are the subgroups $\mathbb{A}_{d^n}$, $n \geq 0$, defined by*

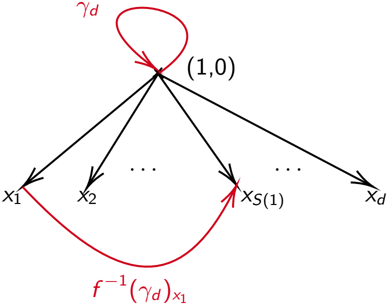
$$\mathbb{A}_{d^n} := \langle [\gamma_{d^n}] \rangle = \text{subgroup generating by the class } [\gamma_{d^n}]$$

*where $\gamma_{d^n}$ is the loop defined by*

$$\gamma_{d^n}(t) := (e^{2i\pi t d^n}, 0), \ \ t \in [0,1]$$

- We say that the collection $\mathbb{A} := \{\mathbb{A}_{d^n}, \ \ n \geq 0\}$ acts on $X$

- Let $\{x_1, \cdots, x_d\} := f^{-1}(1,0)$ be the first level of the tree $X$
- The action of $\mathbb{A}_{d^1}$ on $\{x_1, \cdots, x_d\}$ induces a permutation $S \in \mathfrak{S}_d$




$S(j)$ is defined implicitly by $x_{S(j)} = f^{-1}(\gamma_d)[x_j] = x_j^{\gamma_d}$

# Iterated monodromy : encoding information for good parameters

- Let $\mathcal{A} := \{1, \cdots, d\}$ be an alphabet ; $a \in \mathcal{A}$ is the label of $x_a \in f^{-1}(1,0)$

- Let $\mathcal{A}_n := \{a_n \cdots a_1\}$ be the set of words of length $n$ in this alphabet (written from right to left). $\mathcal{A}_0 = \{\emptyset\}$ is the empty word

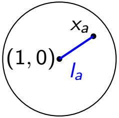
- Let $\mathcal{A}^* = \bigcup \mathcal{A}_n$ be the set of all possible words

**Proposition**

*The following map defined by induction is bijective*

$$L : \begin{cases} \mathcal{A}^* & \xrightarrow{\simeq} & X \\ a_n \cdots a_1 & \longmapsto & f^{-(n-1)}(l_{a_n})[L(a_{n-1} \cdots a_1)] \text{ (endpoint of the lift)} \end{cases}$$

- $l_a$ is a smooth path joining $(1,0)$ to $x_a$ inside $\{1\} \times \mathbb{D}_R$




- The action of $\mathbb{A}$ on $X$ is transferred on $\mathcal{A}^*$

$$(a_n \cdots a_1)^\gamma := L^{-1}(L(a_n \cdots a_1)^\gamma), \ [\gamma] \in \mathbb{A}_{d^n}$$

- The action of $\mathbb{A}$ on $X$ is transferred on $\mathcal{A}^*$

$$(a_n \cdots a_1)^{\gamma} := L^{-1}(L(a_n \cdots a_1)^{\gamma}), \ [\gamma] \in \mathbb{A}_{d^n}$$

- Recall $\gamma_{d^n}(t) = (e^{2i\pi t d^n}, 0)$, $[\gamma_{d^n}] \in \mathbb{A}_{d^n}$
- Recall $S \in \mathfrak{S}_d$ is the permutation induced by $\mathbb{A}_{d^1}$ on $\mathcal{A}_1 = \{1, \cdots, d\}$

**Proposition (Action of $\mathbb{A}$ on $\mathcal{A}^*$)**

$$(a_n \cdots a_1)^{\gamma_{d^n}} = S^{d^{n-1}}(a_n) \cdots S^{d^0}(a_1)$$

# Iterated monodromy groups

- Group morphism $\phi : \pi_1(\mathbb{S}^1 \times \mathbb{D}_R, (1,0)) \to \mathrm{Aut}(\mathcal{X})$

- The Iterated Monodromy Group of $(f,(1,0))$ is

$$\mathrm{IMG}(f,(1,0)) := \frac{\pi_1(\mathbb{S}^1 \times \mathbb{D}_R, (1,0))}{\mathrm{Ker}(\phi)},$$

this is the set of $\mathrm{elements\ acting\ non-trivialy\ on}\ \mathcal{X}$

- Initially introduced through specific examples by various authors, the concept was later formalized by Volodymyr Nekrashevych, who also provided applications to complex dynamics.

- For the sub tree $X$, the collection $\mathbb{A} = \{\mathbb{A}_{d^n},\ n \geq 0\}$ can be interpreted as the restriction of $\mathrm{IMG}(f,(1,0))$ on $X$

Application of the iterated monodromy action of $\mathbb{A}$ on $X$

- Let $p \in J(f)$ in the fiber $z=1$
- $p=(1,w)$ and $w \in J(q_1)$ ($z=1$ is a fixed point). Thus $w$ is in the limit of the IFS generated by the inverse branches of $q_1 \in \mathcal{S}_d$

$$w = \lim_n g_{a_1} \circ \cdots g_{a_n}(0),$$

- So, we deduce $p = \lim_n L(a_n \cdots a_1)$ and

**Proposition**

*The connected component of $p$ in $J(f)$ is a simple closed curve $\mathcal{P}_p$ that winds at least once above $\mathbb{S}^1$ in $\mathbb{S}^1 \times \mathbb{C}$.*

*Proof.* Let $m_S$ be the order of $S$. The curve $\mathcal{P}_p$ is the uniform limit in $n$ of the curves

$$\mathcal{P}_{p,n}(t) = f^{-n}(\gamma_{d^n m_S})_{L(a_n \cdots a_1)}(t), \ \ t \in [0,1],$$

it is a closed curve since for $k \in \{0, \cdots, m_S\}$

$$\mathcal{P}_p\left(\frac{k}{m_S}\right) = \lim_n L\left(S^{d^{n-1}k}(a_n) \cdots S^{d^0 k}(a_1)\right),$$

thus $\mathcal{P}_p(0) = \mathcal{P}_p(1)$. □

# Connected components of $J(f)$ : the general description

**Proposition**

*The connected component of each $p = (1, w) \in J(f)$ is a simple closed curve $\mathcal{P}_p$ that winds at least once above $\mathbb{S}^1$ in $\mathbb{S}^1 \times \mathbb{C}$.*

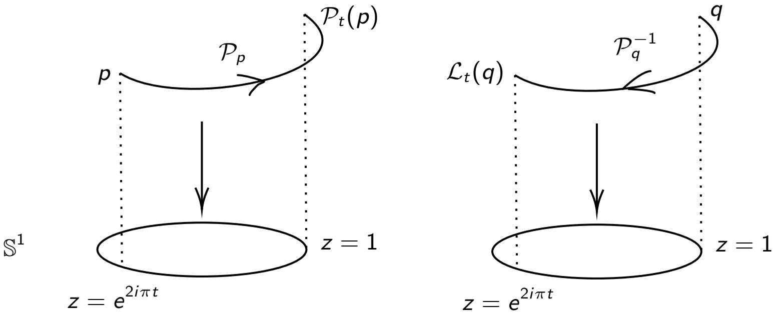
**Corollary**

*The connected components of $J(f)$ are simple loops in $\mathbb{S}^1 \times \mathbb{C}$ winding at least once above $\mathbb{S}^1$*

*Proof.* If $p \in J(f)$, the projection onto $\mathbb{S}^1$ of the connected component $\mathcal{C}$ containing $p$ is the whole of $\mathbb{S}^1$, so $\mathcal{C}$ is also the connected component of a point in $J(f)$ lying in the fiber $z = 1$, and the result follows from the previous proposition. □

# Return maps and the topological model

- Let $p = (e^{2i\pi t}, w_p) \in J(f)$, and let $\mathcal{P}_p$ be a path parameterizing its connected component
- Let $q = (1, w_q) \in J(f)$, and let $\mathcal{P}_q$ be a path parameterizing its connected component



# Return maps and the topological model

- If $p = \lim_n L(a_n \cdots a_1)$ then $\omega(p) = (a_1 \cdots a_n \cdots) \in \mathcal{A}^{\mathbb{N}}$ (coding of $J(q_1)$)
- Let $h : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ homeomorphism defined by

$$h(a_1 \cdots a_n \cdots) = (S^{d^0}(a_1) \cdots S^{d^{n-1}}(a_n) \cdots)$$

- The order of $h$ divides $m_S$
- Let $\sim_h$ defined by $(0, x) \sim (1, y) \Leftrightarrow h(x) = y$

**Theorem**

*The two maps are well defined and are inverses of each other*

$$\mathcal{P} : \begin{cases} J(f) & \longrightarrow & ([0,1] \times \mathcal{A}^{\mathbb{N}})/ \sim_h \\ p = (e^{2i\pi t}, w_p) & \longmapsto & (t, \omega \circ \mathcal{P}_t(p)) \end{cases}$$

$$\mathcal{L} : \begin{cases} ([0,1] \times \mathcal{A}^{\mathbb{N}})/ \sim_h & \longrightarrow & J(f) \\ (t, a_1 \cdots a_n \cdots) & \longmapsto & \mathcal{L}_t(\lim_n L(a_n \cdots a_1)) \end{cases}$$

**Corollary (Topological model)**

*For all $\lambda'$ in the same hyperbolic component than $\lambda$, $J(f_\lambda) \simeq ([0,1] \times \mathcal{A}^{\mathbb{N}})/ \sim_h$*

# Algebraic braids

**Theorem**

*Let $\Lambda' \subset \mathcal{D}$ and let $\lambda \in \Lambda'$. Let $\mathcal{C}_1, \cdots, \mathcal{C}_m$ be the different connected components in $J(f_\lambda)$ of the fixed points of $f_\lambda$ in the fiber $z = 1$.*

- *$\bigcup_{j=1}^m \mathcal{C}_j$ is a closed braid, let $\mathrm{ab}(f_\lambda)$ be its ambient isotopy class in $\mathbb{S}^1 \times \mathbb{C}$.*
- *If $\lambda$ is good, then $\mathrm{ab}(f_\lambda)$ is the isotopy class of the algebraic braid $f_\lambda^{-1}(\mathbb{S}^1 \times \{0\})$.*
- *For all other parameter $\lambda' \in \Lambda'$, $\mathrm{ab}(f_\lambda) = \mathrm{ab}(f_{\lambda'})$.*

*In particular, $\lambda \in \Lambda' \mapsto \mathrm{ab}(f_\lambda)$ is constant and is an algebraic braid*

**Corollary**

*The following map is well defined*

$$\mathrm{ab} : \begin{cases} \{\textit{Hyp. Comp. } \Lambda' \subset \mathcal{D}'\} & \longrightarrow \quad AB_d = \{\textit{Algebraic braids degree } d\} \\ \Lambda' & \longmapsto \quad \mathrm{ab}(f_\lambda),\ \lambda \in \Lambda' \end{cases}$$

**Corollary**

*The following map is surjective :*

$$\mathrm{cab} : \begin{cases} \{Hyp.\ Comp.\ \Lambda'\} & \xrightarrow{\mathrm{ab}} & AB_d & \xrightarrow{\mathrm{c}} & \mathrm{Conj}(\mathfrak{S}_d) \\ \Lambda' & \longmapsto & \mathrm{ab}(f_\lambda) & \longmapsto & \mathrm{c} \circ \mathrm{ab}(f_\lambda) \end{cases}$$

*Proof.* If $\lambda$ is good, then $\mathrm{cab}(f_\lambda) = S$ modulo conjugacy.
If $d = \delta + d_1 + \cdots + d_m$ is a partition of $d$ with $d_i \geq 2$, then for $R \gg 1$, the parameter $\lambda_R$, defined implicitly by (for $a_0, \ldots, a_m \in \mathbb{C}^*$ chosen so that the roots are simple)

$$q_{\lambda_R, z}(w) = \left(w^\delta - a_0 R^\delta z^\delta\right) \prod_{j=1}^{m} \left(w^{d_j} - a_j R^{d_j} z\right),$$

belongs to $\mathcal{D}'$ and is a good parameter. Then $\mathrm{cab}(f_{\lambda_R}) = C_1 \cdots C_m$, where $C_1, \ldots, C_m$ are cycles with disjoint supports in $\mathfrak{S}_d$, with $C_j$ of length $d_j$. Thus $\mathrm{cab}$ is surjective. □

# Quadratic algebraic braids and Astorg-Bianchi classification

| $s(\lambda)$ | $K = \{q_\lambda = 0\} \cap (\mathbb{S}^1 \times \mathbb{C})$ | $\mathrm{ab}(f_\lambda)$ | $\mathrm{cab}(f_\lambda)$ |
|---|---|---|---|
| 0 | 2 connected components which do not interlace and wind 1 time above $\mathbb{S}^1$ | $\mathrm{Id}$ | $\mathrm{Id}$ |
| 1 | 1 connected component winding 2 times above $\mathbb{S}^1$ | $\sigma_1$ | $(1,2)$ |
| 2 | 2 connected components which interlace 1 time and wind 1 time above $\mathbb{S}^1$ | $\sigma_1^2$ | $\mathrm{Id}$ |

**Corollary (Classification for $\mathrm{Sk}(z^2, 2)$)**

*For each* $\lambda \in \mathcal{D}'$, $\mathrm{ab}(f_\lambda) = \sigma_1^{s(\lambda)}$

- $\lambda \mapsto s(\lambda)$ and $\lambda \mapsto \mathrm{ab}(f_\lambda)$ carry exactly the same information and give the same classification of hyperbolic components in $\mathcal{D}'$.

# Acknowledgements

Thank you for your attention !

Thanks to Matthieu for the pictures !