DYNAMICS OF FUCHSIAN MEROMORPHIC CONNECTIONS WITH REAL PERIODS

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Transcendental dynamics from one to several complex variables
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MOTIVATION

A homogeneous vector field of degree $\nu + 1 \geq 2$ on $\mathbb{C}^n$ is given by

$$Q = Q^1 \frac{\partial}{\partial z^1} + \cdots + Q^n \frac{\partial}{\partial z^n}$$

where $Q^1, \ldots, Q^n$ are homogeneous polynomials in $z^1, \ldots, z^n$ of degree $\nu + 1$.

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A characteristic leaf is a Q-invariant line $L_v = \mathbb{C}v \subset \mathbb{C}^n$. The dynamics of Q inside a characteristic leaf is 1-dimensional and easy to study. We are interested in the dynamics outside characteristic leaves.

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This allows the use of geometrical tools (curvature, Gauss-Bonnet theorem, residue theorems, singular flat metrics, etc.).

Meromorphic connections

A meromorphic connection on the (holomorphic) tangent bundle TS of a Riemann surface S is a $\mathbb{C}$-linear map $\nabla\colon \mathcal{M}_{TS} \to \mathcal{M}^1_S \otimes \mathcal{M}_{TS}$ satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s$$

for all $s \in \mathcal{M}_{TS}$ and $f \in \mathcal{M}_S$, where $\mathcal{M}_S$ is the sheaf of meromorphic functions, $\mathcal{M}_{TS}$ is the sheaf of meromorphic vector fields and $\mathcal{M}^1_S$ is the sheaf of meromorphic 1-forms.

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If (U_α, z_α) is a local chart on S and $\partial_\alpha = \frac{\partial}{\partial z_\alpha}$ then we can write $\nabla\partial_\alpha = \eta_\alpha \otimes \partial_\alpha$, where $\eta_\alpha = k_\alpha\, dz_\alpha$ is a meromorphic 1-form (that is, k_α is a meromorphic function). η_α is the local representation of ∇.

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A pole of a meromorphic connection ∇ is a pole of any local representation. The residue $\mathrm{Res}_p\nabla$ of ∇ at a pole p is the residue of any local representation at p.

Geodesics

Let ∇ be a meromorphic connection on S and let $S^o = S \setminus \{\text{poles}\}$ be the subset of regular points.
A curve $\sigma\colon I \to S^o$ is a geodesic for ∇ if

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In local coordinates, write $\sigma' = \sigma'_\alpha \partial_\alpha$. Then σ is a geodesic if and only if

$$\sigma''_\alpha + (k_\alpha \circ \sigma)(\sigma'_\alpha)^2 \equiv 0 \,.$$

Homogeneous vector fields and geodesics

Theorem (A.-Tovena, 2011)

Let Q be a homogeneous vector field in $\mathbb{C}^2$ of degree $\nu + 1 \geq 2$ not proportional to the radial vector field. Let $[\cdot]\colon \mathbb{C}^2 \setminus \{O\} \to \mathbb{P}^1(\mathbb{C})$ be the canonical projection. Then there exists a meromorphic connection ∇^o on $\mathbb{P}^1(\mathbb{C})$ such that:

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The geodesic $\sigma(t) = [\gamma(t)]$ gives the complex line containing $\gamma(t)$; the speed $\sigma'(t)$ "gives" (after some manipulations depending also on ν) the position of $\gamma(t)$ in that line.

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The local representation of ∇^o in the standard chart centered at 0 is

$$\eta = -\left[\nu \frac{Q^1(1,\zeta)}{R(\zeta)} + \frac{R'(\zeta)}{R(\zeta)}\right] d\zeta \ ,$$

where $R(\zeta) = Q^2(1,\zeta) - \zeta Q^1(1,\zeta)$.

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There is an analogous statement in dimension $n \geq 3$ expressed in terms of a singular holomorphic foliation in Riemann surfaces of $\mathbb{P}^{n-1}(\mathbb{C})$ and of a partial meromorphic connection defined along the foliation.

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Thus to study real integral curves of homogeneous vector fields it suffices to study geodesics for (partial) meromorphic connections on (a singular holomorphic foliation in Riemann surfaces of) $\mathbb{P}^{n-1}(\mathbb{C})$.

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Idea of the proof: there is a canonical ν-to-1 holomorphic covering map $\chi\colon \mathbb{C}^n \setminus \{O\} \to N_S^{\otimes\nu} \setminus S$, where N_S is the normal line bundle of the exceptional divisor $S = \mathbb{P}^{n-1}(\mathbb{C})$ in the blow-up of the origin in $\mathbb{C}^n$.

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Idea of the proof, cont.: Using Q one builds a bundle map $X\colon N_S^{\otimes \nu} \to TS$, that is a singular holomorphic foliation in Riemann surfaces of S, and partial meromorphic connections both on $N_S^{\otimes \nu}$ and on S. The push-forward of Q via χ and X coincides with the geodesic field of the connection(s), giving the correspondence between integral curves and geodesics.

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THEOREM (A.-TOVENA-BIANCHI-RAKHIMOV, 2011, 2016, 2024)

Let $\sigma\colon [0,\varepsilon)\to S^o$ *be a maximal geodesic for a meromorphic connection* ∇ *on a compact Riemann surface S. Then:*

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Closed in general does not mean periodic.

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A saddle connection is a geodesic connecting two poles.

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A transversally Cantor-like geodesic set is a connected, compact set with empty interior such that its intersection with transversal geodesics is a Cantor set.

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All cases can occur (A.-Tovena, A.-Bianchi, Duryev-Fougeron-Ghazouani, Novikov-Shapiro-Tahar).

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Case (1) is generic; cases (2), (3), (5) and (7) can happen only if some necessary conditions expressed in terms of the residues of ∇ are satisfied.

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We do not know what is the ω-limit set of geodesics self-intersecting infinitely many times, except in particular cases (Rakhimov, 2020).

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We have a less precise statement for non-compact Riemann surfaces.

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Main tools for the proof: ∇ is flat; a Gauss-Bonnet theorem relating geodesics and residues; a Poincaré-Bendixson theorem for smooth flows.

Fuchsian connections and periods

Special case: *Fuchsian connections with real periods.* They have a particularly strong geometric structure; we would like to see how it affects the asymptotic behaviour of geodesics. Also suggested by the work of Duryev-Fougeron-Ghazouani, Novikov-Shapiro-Tahar and others on dilation surfaces and k-differentials.

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Special case: *Fuchsian connections with real periods.* They have a particularly strong geometric structure; we would like to see how it affects the asymptotic behaviour of geodesics. Also suggested by the work of Duryev-Fougeron-Ghazouani, Novikov-Shapiro-Tahar and others on dilation surfaces and k-differentials.

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If S^o is the set of regular points of ∇, it is possible to define a monodromy representation $\rho\colon \pi_1(S^o) \to \mathbb{C}^*$. If γ is a small loop around a pole, then

$$\rho([\gamma]) = \exp\bigl(2\pi i \,\mathrm{Res}_p \nabla\bigr) \;.$$

We say that ∇ has real periods if the image of ρ is contained in $\mathbb{S}^1$ (in particular, all residues are real).

SINGULAR FLAT METRICS

Let S be a Riemann surface and $\Sigma \subset S$ a discrete set. A singular flat metric on S with singular set Σ is a flat Hermitian metric g on $S \setminus \Sigma$ such that, if we write $g^{1/2} = e^u |dz|$ with respect to a local coordinate (U, z) centered at $p \in \Sigma$, there is $\rho_p \in \mathbb{R}$, the residue of g at p, such that

$$\lim_{z \to 0} \frac{e^u}{|z|^{\rho_p}} \in (0, +\infty) \, .$$

Notice that g flat implies u harmonic.

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THEOREM (RAKHIMOV, 2020)

If ∇ is a Fuchsian meromorphic connection with real periods on S then there exists a unique (up to a positive constant multiple) singular flat metric on S with the same singular set Σ and the same geodesics as ∇.
Conversely, if g is a singular flat metric then there is a unique meromorphic connection ∇ with the same singular set and the same geodesics as g.
Furthermore, ∇ is Fuchsian with real periods.
Finally, if ρ_p is the residue of g at $p \in \Sigma$ then $\rho_p = \mathrm{Res}_p \nabla$ and conversely.

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We say that ∇ and g are adapted to each other.

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If ∇ and g are adapted to each other, we can use ∇ to build local isometries between g and the Euclidean metric of the plane sending geodesics into Euclidean segments.

Singular flat metrics

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Furthermore, ∇ is Fuchsian with real periods.
Finally, if ρ_p is the residue of g at $p \in \Sigma$ then $\rho_p = \mathrm{Res}_p \nabla$ and conversely.

Corollary

Every closed geodesic of a Fuchsian connection with real periods is necessarily periodic.

Local isometries

If ∇ is a meromorphic connection and $U \subseteq S^o$ is simply connected, it is *always* possible to find a (regular) metric g_U adapted to ∇ on U and a holomorphic map $J \colon U \to J(U) \subseteq \mathbb{C}$ so that J is an isometry between g_U and the Euclidean metric; so J sends geodesics into Euclidean segments.

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REMARK

The change of coordinates obtained in this way are affine; so the meromorphic connection induces a structure of *(singular) affine surface* on S. Thus the dynamics of homogeneous vector fields can be studied by examining the dynamics of straight lines in affine surfaces; this approach is currently being actively pursued by, for instance, Buff, Chéritat, Raissy and others.

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This is an important tool in the proofs, because it allows to study the behaviour of a non self-intersecting geodesic in a neighbourhood of a regular point of the ω-limit set by studying families of disjoint Euclidean segments.

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This is an important tool in the proofs, because it allows to study the behaviour of a non self-intersecting geodesic in a neighbourhood of a regular point of the ω-limit set by studying families of disjoint Euclidean segments.

However, in general, g_U depends on U and thus it does not provide a canonical parametrization of the Euclidean segments. When ∇ is Fuchsian with real periods then we have a *global* metric on S^o adapted to ∇ and this allows to give a canonical parametrization of the Euclidean segments by arc-length.

Local behaviour around Fuchsian poles

We have a simple holomorphic normal form around Fuchsian poles.

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Proposition

Let $p_0 \in S$ be a Fuchsian pole of a meromorphic connection ∇ with residue $\rho = \mathrm{Res}_p \nabla \in \mathbb{C}$. Assume that $\rho \notin \{-2, -3, \ldots\}$. Then ∇ can be locally represented around p by

$$\eta = \frac{\rho}{w}\, dw \; .$$

If, instead, $\rho \in \{-2, -3, \ldots\}$, then ∇ can be locally represented around p by

$$\eta = \left(\frac{\rho}{w} + a w^{n-1} \right) dw \; ,$$

where $n = |\rho| - 1 > 0$ and $a \in \mathbb{C}$ is a holomorphic invariant.

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We also have holomorphic normal forms for apparent singularities and formal normal forms for irregular singularities.

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Recall: our aim is to understand the asymptotic behaviour, i.e., the ω-limit set, of geodesics.

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Proposition

If a geodesic σ accumulates a Fuchsian pole p_0 with $\mathrm{Re}(\mathrm{Res}_{p_0} \nabla) \leq -1$ then σ converges to p_0.

FUCHSIAN POLES WITH REAL RESIDUES

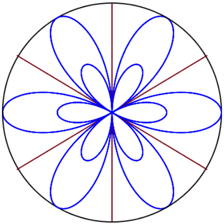
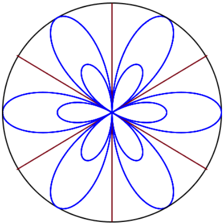


FIGURE 1: Residue < -1

FUCHSIAN POLES WITH REAL RESIDUES



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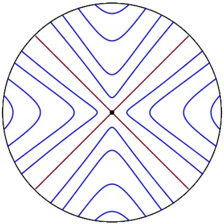

FIGURE 2: Residue > -1

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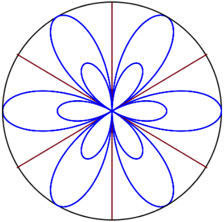

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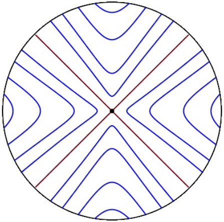

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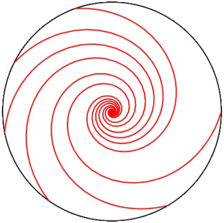

FIGURE 3: Residue $= -1$

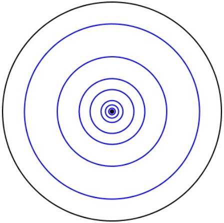

FIGURE 4: Residue $= -1$

Metrics, residues and geodesics

Let ∇ be a Fuchsian connection with real periods and g an adapted singular flat metric.

Proposition

Let $p \in S$ be a Fuchsian pole with real residue $\rho \in \mathbb{R}$.

1. *if $\rho > -1$ then p is at finite g-distance from S^o.*

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2. *if $\rho > -1$ then geodesics starting close enough (to p and to themselves) with close enough directions intersect nearby p.*

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1. *if $\rho > -1$ then p is at finite g-distance from S^o.*
2. *if $\rho > -1$ then geodesics starting close enough (to p and to themselves) with close enough directions intersect nearby p.*
3. *if $-1 < \rho < -\frac{1}{2}$ then geodesics passing close enough to p either converge to p or self-intersect at least once nearby p; in particular, simple geodesics accumulating p must converge to p.*

GENERALIZED TEICHMÜLLER FORMULA

Another important tool is a generalized Techmüller formula.

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THEOREM (A.-RAKHIMOV, 2024)

Let $\gamma \subset S$ be a Fuchsian multicurve with $m_f \geq 1$ free components bounding a part $P \subset S$. Assume that the vertices $q_1, \ldots, q_s$ of the free components are either regular or have real residues $\rho_j = \mathrm{Res}_{q_j}\nabla > -1$. Let $v_j \in (0, 2\pi)$ be the internal angle of γ at q_j. Let $\{p_1, \ldots, p_g\}$ be the poles contained in the interior of P and $g_{\widetilde{P}}$ the genus of the filling of P. Then

$$\sum_{j=1}^{s}\bigl(\pi - (\rho_j + 1)v_j\bigr) = 2\pi \left(2 - m_f - 2g_{\widetilde{P}} + \sum_{j=1}^{g} \mathrm{Re}\bigl(\mathrm{Res}_{p_j}\nabla\bigr)\right) .$$

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A geodesic cycle is a closed piecewise geodesic simple curve. The breaking points are the vertices of the geodesic cycle; they can be regular points or poles.

Generalized Teichmüller formula

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A (geodesic) multicurve is the union of a finite number of disjoint geodesic cycles. A multicurve is Fuchsian if the vertices are regular points or Fuchsian poles.

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The external angle $\epsilon \in (-\pi, \pi)$ at a vertex is the angle between the tangent vectors to the geodesics meeting at the vertex; the internal angle is $v = \pi - \epsilon$.

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A part $P \subseteq S$ is the closure of a connected open subset whose boundary is a multicurve γ. A component of γ is free if the interior of P does not accumulate it from both sides.

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The filling $\widetilde{P}$ of a part P is the compact surface obtained by glueing a disk along any free component of ∂P.

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This generalizes a classical formula proved by Teichmüller for quadratic differentials as well as previous formulas proved by A., Tovena and Bianchi for regular multicurves.

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$$\sum_{j=1}^{s} \bigl(\pi - (\rho_j + 1) v_j\bigr) = 2\pi \left(2 - m_f - 2 g_{\widetilde{P}} + \sum_{j=1}^{g} \mathrm{Re}\bigl(\mathrm{Res}_{p_j} \nabla\bigr) \right) .$$

The proof when the vertices are all regular follows from the Gauss-Bonnet theorem, using the fact that the local metrics associated to ∇ are flat. The general case is obtained by a delicate argument allowing to replace the singular vertices by regular vertices keeping track of the internal angles.

PARTICULAR CASES

COROLLARY

Let $\sigma \subset S$ be a simple closed geodesic freely bounding a part $P \subset S$. Let $\{p_1, \ldots, p_g\}$ be the poles contained in the interior of P and $g_{\tilde{P}}$ the genus of the filling of P. Then

$$\sum_{j=1}^{g} \mathrm{Re}(\mathrm{Res}_{p_j} \nabla) = 2g_{\tilde{P}} - 1 \ .$$

In particular, P contains at least one pole in its interior; moreover, if P is simply connected then the sum in the left-hand side is equal to -1.

If $S = \mathbb{P}^1(\mathbb{C})$ then at every turn the velocity vector is multiplied by

$$\exp\left(2\pi \sum_{j=1}^{g} \mathrm{Im}(\mathrm{Res}_{p_j} \nabla)\right) \ .$$

Particular cases

Corollary

Let $\sigma \subset S$ be a simple geodesic connecting a Fuchsian pole p_0 with itself, with an internal angle $v_0 \in (0, 2\pi)$. Assume p_0 has a real residue $\rho_0 > -1$ and that σ freely bounds a part $P \subset S$. Let $\{p_1, \ldots, p_g\}$ be the poles contained in the interior of P and $g_{\tilde{P}}$ the genus of the filling of P. Then

$$\sum_{j=1}^{g} \mathrm{Re}(\mathrm{Res}_{p_j} \nabla) = 2g_{\tilde{P}} - \frac{1}{2} - (1 + \rho_0)\frac{v_0}{2\pi} \in \left(2g_{\tilde{P}} - \frac{3}{2} - \rho_0, 2g_{\tilde{P}} - \frac{1}{2}\right) .$$

In particular, if P is simply connected, or if $g_{\tilde{P}} > 0$ and $\rho < 2g_{\tilde{P}} - \frac{3}{2}$, then the interior of P must contain a pole.

PARTICULAR CASES

COROLLARY

Let $\gamma \subset S$ be a Fuchsian geodesic cycle with $s = 2$ vertices, freely bounding a part $P \subset S$. Assume that the vertices q_1, q_2 of γ are either regular or have real residues $\rho_j = \mathrm{Res}_{q_j}\nabla > -1$. Let $v_j \in (0, 2\pi)$ be the internal angle of γ at q_j. Let $\{p_1, \ldots, p_g\}$ be the poles contained in the interior of P and $g_{\tilde{P}}$ the genus of the filling of P. Then

$$\sum_{j=1}^{g} \mathrm{Re}(\mathrm{Res}_{p_j}\nabla) = 2g_{\tilde{P}} - \sum_{j=1}^{2}(\rho_j + 1)\frac{v_j}{2\pi} \in (2g_{\tilde{P}} - 2 - \rho_1 - \rho_2, 2g_{\tilde{P}}) \ .$$

In particular, if P is simply connected, or if $g_{\tilde{P}} > 0$ and $\rho_1 + \rho_2 < 2(g_{\tilde{P}} - 1)$, then the interior of P must contain a pole.

Particular cases

Corollary

Let ∇ be a Fuchsian meromorphic connection with non-negative residues on a compact Riemann surface S. Let σ_1 and σ_2 be simple geodesics having the same starting and ending points. If σ_1 and σ_2 are homotopic in S then $\mathrm{supp}(\sigma_1) = \mathrm{supp}(\sigma_2)$.

POINCARÉ-BENDIXSON THEOREM REVISITED

THEOREM (A.-TOVENA-BIANCHI-RAKHIMOV, 2011, 2016, 2024)

Let $\sigma\colon [0,\varepsilon) \to S^o$ *be a maximal geodesic for a meromorphic connection* ∇ *on a compact Riemann surface S. Then either:*

1. σ *tends to a pole* p_0 *of* ∇*; or*
2. σ *is closed or accumulates the support of a closed geodesic; or*
3. σ *accumulates a boundary graph of saddle connections; or*
4. *the* ω*-limit set of* σ *is a transversally Cantor-like geodesic set; or*
5. *the* ω*-limit set of* σ *has non-empty interior and non-empty boundary consisting of boundary graphs of saddle connections; or*
6. σ *is dense in S; or*
7. σ *self-intersects infinitely many times.*

POINCARÉ-BENDIXSON THEOREM REVISITED

THEOREM (A.-RAKHIMOV, 2024)

Let $\sigma\colon [0,\varepsilon) \to S^o$ be a maximal geodesic for a Fuchsian meromorphic connection ∇ with real periods on a compact Riemann surface S. Then either:

1. *σ tends to a pole p_0 of ∇; or*
2. *~~σ is closed or accumulates the support of a closed geodesic~~ σ is a periodic geodesic; or*
3. *~~σ accumulates a boundary graph of saddle connections; or~~*
4. *the ω-limit set of σ is a transversally Cantor-like geodesic set; or*
5. *the ω-limit set of σ has non-empty interior and non-empty boundary consisting of boundary graphs of saddle connections; or*
6. *σ is dense in S; or*
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Poincaré-Bendixson theorem revisited

A ring domain $R \subset S^o$ is a domain foliated by simple periodic geodesics.

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A ring domain $R \subset S^o$ is a domain foliated by simple periodic geodesics.

Proposition

Let ∇ be a Fuchsian meromorphic connection with real periods and denote by g an associated singular flat metric. Let $R \subset S^o$ be a ring domain. Then:

1. *all leaves of R have the same g-length;*
2. *a maximal connected component of ∂R is either a pole with residue -1 or the support of a periodic geodesic or a graph of saddle connections;*
3. *every simple periodic geodesic is a boundary component of a ring domain;*
4. *every boundary graph of saddle connections that is the ω-limit set of a simple geodesic is a boundary component of a ring domain.*

Poincaré-Bendixson theorem revisited

A ring domain $R \subset S^o$ is a domain foliated by simple periodic geodesics.

Idea of proof of the theorem: if a simple geodesic σ accumulates the support of a (different) periodic geodesic γ, using a local isometry we can reduce to the case of a sequence of disjoint Euclidean segments of the same length accumulating a given Euclidean segment without intersecting it. But then they must be parallel; since γ is inside a ring domain, this implies that σ is a periodic geodesic and thus it cannot accumulate γ. A similar argument works for boundary graphs of saddle connections.

HOMOGENEOUS VECTOR FIELDS

Coming back to homogeneous vector fields in dimension 2:

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3. for some classes of (quadratic) homogeneous vector fields, the global results yield enough information to qualitatively describe the global dynamics;
4. in all cases, this approach provides a more meaningful visualization of integral curves.

EXAMPLES

$Q(z,w) = -\rho z^2 \frac{\partial}{\partial z} + (1-\rho) zw \frac{\partial}{\partial w}$

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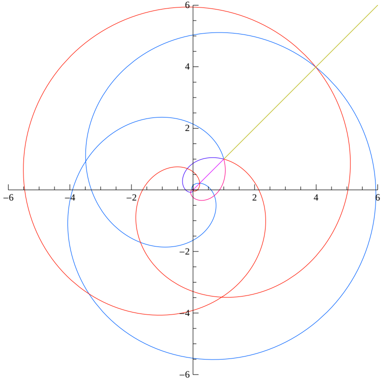

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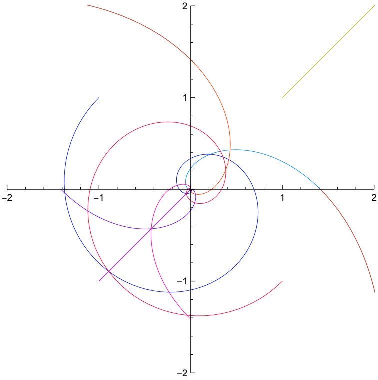

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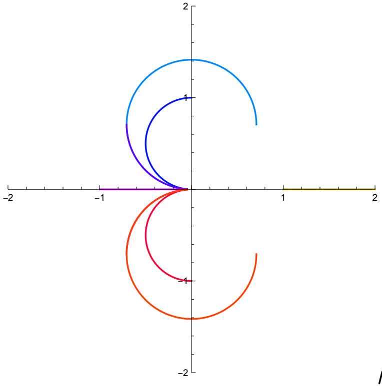

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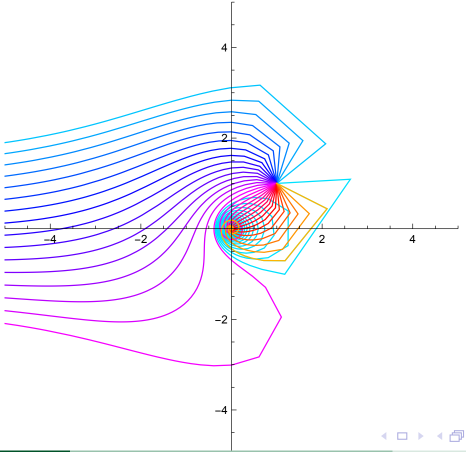

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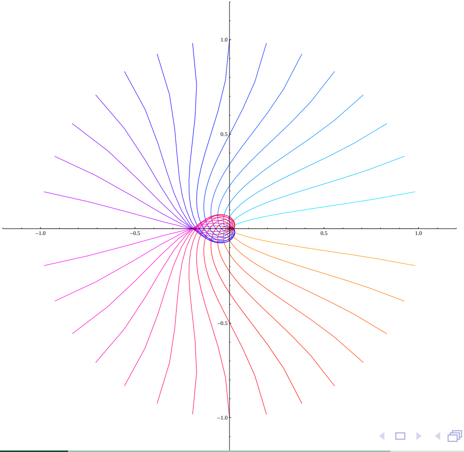

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$$Q(z,w) = \left(-\rho z^2 + (1-\tau)zw\right)\frac{\partial}{\partial z} + \left((1-\rho)zw - \tau w^2\right)\frac{\partial}{\partial w}$$

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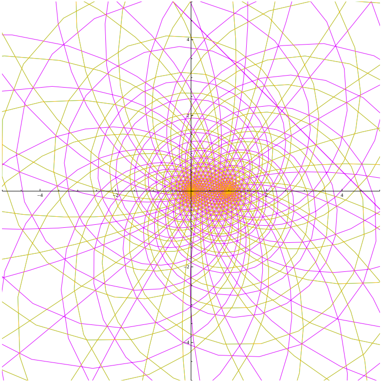

$\rho = \frac{1}{3}, \tau = \frac{1}{3}, \mathrm{Res}_0\nabla = \mathrm{Res}_1\nabla = \mathrm{Res}_\infty\nabla = -\frac{2}{3}.$

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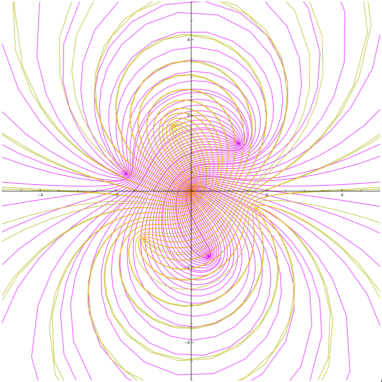

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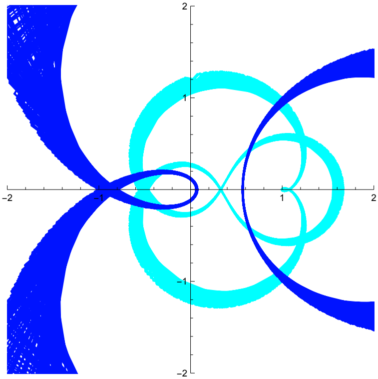

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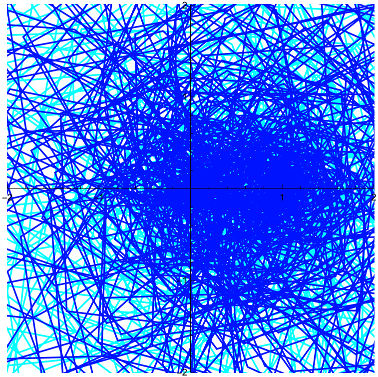

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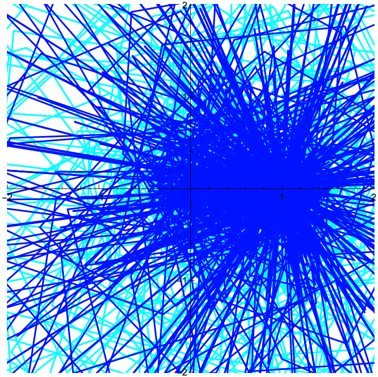

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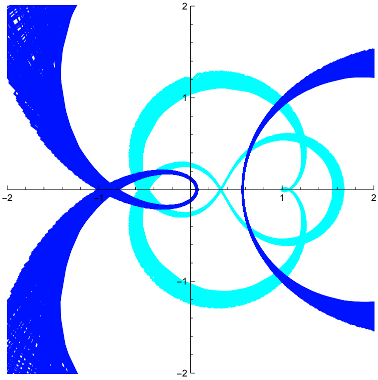

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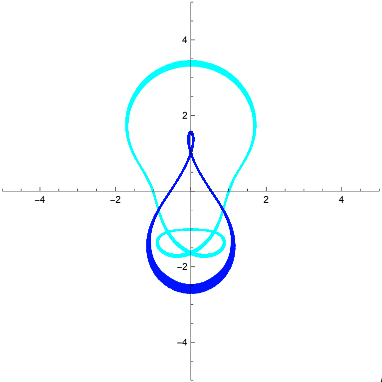

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