

Resurgent QCD

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7 May, 2025

Outline

- ▶ QCD: analysis on function manifolds, functional PDEs, functional Fourier Integral Operators
- ▶ Holomorphic dynamics of classical and quantum n-body Coulomb problems
- ▶ Transcendental numbers in holomorphic dynamics and in QFT
- ▶ Riemann Hilbert problems
- ▶ Local renormalization group
- ▶ Functional Riemann Hilbert problems and non perturbative QCD

Who cares about axiomatic QFT?

- ▶ Proton mass
- ▶ Huge data from colliders, largely unexplained from first principles
- ▶ Alternative to lattice
- ▶ Next step to TQFT

Experimental characteristics

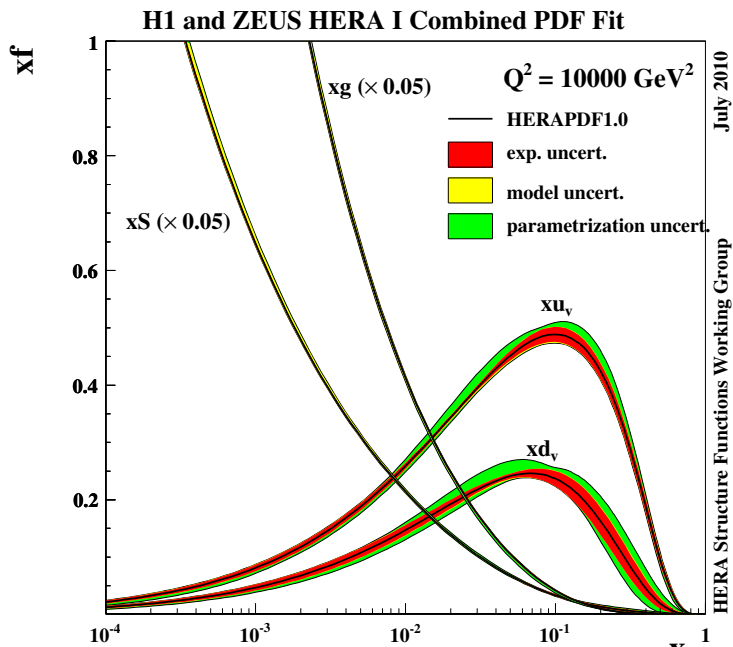
$$\int d\lambda \exp^{i\lambda x} \langle P' | \bar{\psi}(-\lambda n/2) \gamma^\mu \psi(\lambda n/2) | P \rangle = \quad (1)$$
$$= H(x, \xi, \Delta^2) \bar{U}(P') \gamma^\mu U(P) + E(x, \xi, \Delta^2) \bar{U}(P') \frac{i\sigma^{\mu\nu} \Delta_\nu}{2M} U(P) +$$

+higher twist terms

$$p = \Lambda(1, 0, 0, 1), n = \Lambda(1, 0, 0, -1), \bar{P}_\mu = p_\mu + 1/2 \bar{M}^2 n_\mu$$

- ▶ state vector $|P\rangle$: quantum wave functional
- ▶ integration along a Stokes line: analogy of Voros symbol
- ▶ $H(x, 0, 0) = q(x)$

Parton distribution functions



Part I: Formulations of non perturbative QCD

Formulations of QCD

$$Z_{fi}(x_1, \dots, x_n) = \int DA \exp\left(-\frac{1}{4g^2} \int dx \operatorname{tr}(F^2)\right) A(x_1) \dots A(x_n) \quad (2)$$

$$Z_{op}(x_1, \dots, x_n) = \langle 0 | T \exp\left(\int (dy) L_{int}(y) A(x_1) \dots A(x_n)\right) | 0 \rangle \quad (3)$$

Th: $Z_{fi} = Z_{op}$ for the canonical choice of field operators (quantization conditions) and perturbative expansion of the path integral.

Obs: This formulation does not include non perturbative bound states, such as the proton. Need to extend this formalism.

- ▶ Functional differential equations
- ▶ Flat bundles on function manifolds

Flat bundles on function manifolds

$$\begin{aligned} & \frac{\delta}{\delta T(t, \vec{x})} F(\vec{p}_1, \dots, \vec{p}_n)[J_\mu^a(x)] = \quad (4) \\ & = \sum_m \int (dq) \Omega(t, \vec{x}; \vec{p}_1, \dots, \vec{p}_n; \vec{q}_1, \dots, \vec{q}_m)[T, J, \frac{\delta}{\delta J}] F(\vec{q}_1, \dots, \vec{q}_m)[J_\mu^a(x)] \end{aligned}$$

These equations generalize the *Texp* of perturbation theory.

$$\frac{\delta}{\delta T(t)} F_i[T] = \sum \frac{A_{i,j,l}[T]}{B_l[T]} F_j[T] \quad (5)$$

Def: Function manifolds are zero sets of polynomial integral functionals

$$\{f(t) : G_i[f] = 0\} \quad (6)$$

where

$$G_i[f] = \sum_n \int_{\zeta_1, \dots, \zeta_n} G_i(t_1, \dots, t_n) f(t_1) \dots f(t_n) dt_1 \dots dt_n \quad (7)$$

$G_i(t_1, \dots, t_n)$ are holomorphic vector valued functions with singularities on codimension 1 subvarieties.

Some results on the theory of functional flat bundles

- ▶ Examples of functional manifolds with fundamental group with continuum of generators
- ▶ Wilson lines(Voros symbols), brackets
- ▶ Moduli spaces of the flat connections
- ▶ Factorization theorems on behavior of flat connection near boundary components

Functional PDEs

$$\mathbb{L}_i[J, \frac{\delta}{\delta J}]F[J] = 0 \quad (8)$$

E.g. functional Laplacian

$$\int (dx) \frac{\delta^2}{\delta f(x)^2} F = 0 \quad (9)$$

To define Laplacian on function manifolds, need functional version of metric tensor

$$ds^2 = \int_{x,y} G(x,y)[f] \delta f(x) \delta f(y) \quad (10)$$

Analysis function manifolds

Th:(Godefroy,Hjorth,Todorcevic,...) There is analytic family of Banach subspaces of $C([0, 1])$. Need ordinal analysis to describe families of subspaces.

Th:(Milyutin) Suppose K is compact uncountable metric space. Then $C(K)$ is isomorphic to $C([0, 1])$.

Obs: The space of analytic functionals on function spaces is non separable. Submanifolds of loop spaces LG are interesting

Rk: Submanifolds of $Diff(D^d)$, e.g. $\mathbb{M}(D^d; N_1, \dots, N_k)$ that fixes submanifolds N_i . $f^{n_1}g_1f^{n_2}\dots = id$.

Resurgent functions as local models for function manifolds

$$\int G_i(x_1, \dots, x_n) f(x_1) \dots f(x_n) dx_1 \dots dx_n = 0 \quad (11)$$

Let $L_a(x_1, \dots, x_n)$ be singularity loci of $G_i(x_1, \dots, x_n)$.

Def: Elimination algebra is the algebra of subvarieties in $\{x\}$ that is closed on the operation of elimination

Th:(S) The set of singularities of the functions in the manifold $\{G_i[f] = 0\}$ are closed in the under elimination algebra of the singularities $L_a(t_1, \dots, t_n)$ of the functions G_i . The set of holomorphic solutions is exhausted by Puiseux series on the complement of the set of subvarieties

Ex: $x \in C, \int K(x, y) f(y) = \lambda f(x)$

Prop: The set of singularities is dense on C

Proposal: Constructive functional integration: integrate only on the space of functions on pro-finite spaces closed under elimination algebra, which are locally expandable into Puiseux series, i.e. resurgent functions with infinity of singularities.

Part II: Holomorphic dynamics, KAM, resurgence, transcendental number theory

Holomorphic dynamics in ODEs

- ▶ Constructive solutions are lacking
- ▶ Usually have many moving singularities of branch type
- ▶ ex: Lorenz, Henon-Heiles, 3-body, n-body Coulomb
- ▶ objects of interest: stable manifolds (Poincare tangle).
Symbolic dynamics w.r.t. the tangle. For each periodic point, we have n stable manifolds $W_i^s(t)$.

Stable manifolds in polynomial systems

Ex: Henon Heiles

$$\begin{aligned}d^2x/dt^2 &= -x - 2\lambda xy & (12) \\d^2y/dt^2 &= -y - \lambda(x^2 - y^2)\end{aligned}$$

$$dx_i/dt = \text{poly}_i(x) \quad (13)$$

Th: Stable manifolds are given by exponential sums

$$w_i(t) = \sum w_{i,n} \exp(n\kappa t) \quad (14)$$

Rk: Exponential sums have very deep and beautiful theory: Leontiev(functions in wedge), Karatsuba(number theory), Moreno (distribution of zeros),...

Th:(Sanjuan et.al). There are basins of Wada in the system W^s (i.e., indecomposable continua).

Problem Can we understand this from resurgence point of view?

Ex:(Ilyashenko lips). It is possible to construct ODE with exp-towers of arbitrary height.

KAM

- ▶ Tree expansion of Gavallotti-Eliason. Resurgence in the nonlinearity small parameter.
- ▶ Analyticity in frequencies: appearance of natural boundary at the circle $|\omega_j| = 1$.
- ▶ Resummation depending on the Diophantine properties of ω_j .
- ▶ How can we understand the foliations on KAM tori? The KAM series looks like a series in a C^* algebra.
- ▶ Koch: renormalizable Hamiltonians, associated with breakdown of KAM tori.

Normalization of parabolic maps

- ▶ Origin of resurgence(Ecalle).

$$F(w) = w + cw^2 + o(w^2) \quad (15)$$

$$f(z) = -1/cF(-1/cz) = z + 1 + a(z) \quad (16)$$

$$v \circ f = v + 1 \quad (17)$$

- ▶ Th(Dudko-Sauzin): The formal Borel transform of the solution $2\pi iZ$ resurgent. The singularities at $z_n = 2\pi in$ have the form $\log^k(z - z_n)$
- ▶ Shishikura-Inou on parabolic renormalization: horn maps
- ▶ Th(Yampolsky): There exist values of θ for which J_θ is not computable by any Turing machine with oracle access to θ .

Diophantine geometry and exponential predicates

- ▶ Every recursively enumerable set is exponential Diophantine
- ▶ Th(Matiyasevich-Robinson): every r.e. predicate is Diophantine
- ▶ Realization of Turing degrees using resurgent functions
- ▶ Analytic continuation of Ackermann function
- ▶ Consistency strength of segments of PA (Duque, Weiermann, Pakhomov,...)

Resurgent number theory

Def: Resurgent numbers of type I are intersection points of stable manifolds that correspond to maximally degenerate points in the parameter family.

Prop: There is countably many resurgent numbers.

Rk: Connections to global bifurcation theory of Ilyashenko

- ▶ Results: exp/log definability, complexity, measures of irrationality
- ▶ Importance: Cremer points in dynamics, renormalizability of quadratic maps,
- ▶ Inf. gen. Fuchsian groups, L-functions
- ▶ connections to model theory: Hrushovski, Zilber, ...

marriage of resurgence, logic, and holomorphic dynamics

Quantum resurgent number theory

Bifurcation problems for

$$\frac{\partial}{\partial t} f(t, x) = \int (dy) A(x, y) f(y) + \int (dy)(dz) B(x, y, z) f(y) f(z) \quad (18)$$

W^s stable manifolds:

$$\int (dy) A(x, y) f(y) = \lambda_i f(x) \quad (19)$$

Modular surfaces example(p.T. around modular forms)

$$\frac{\partial}{\partial t} f(t, x) = \Delta f(y) + \int (dy)(dz) B(x, y, z) f(y) f(z) \quad (20)$$

Def: quantum resurgent numbers are 0-dim intersections of complementary $W^s(\lambda_{i_1}, \dots, \lambda_{i_n}, \dots)$ after maximal degeneration.

Resurgence in n-body problems

Th:(Xia) 5-body Coulomb problem has non collisional singularity $t_n \rightarrow t^*$.

Th:(Gerver) Planar symmetric N-body problem has non collisional singularity.

Rk: Compare with the resurgence in Dudko-Sauzin. The position of singularities is essential data for the factorization problem (moving singularities).

Ideas from function theory

- ▶ Bishop method of folding: realization of exp growth classes with prescribed dynamics
- ▶ Rempe work: pseudoarcs in Julia sets; tracts
- ▶ approximation theory toolbox: Arakelyan, Mergelyan theorems used to construct functions with specified dynamics outside fractal sets

Q: how these emerge in RH problems

Part III: Riemann surfaces of infinite genus

Riemann surfaces of infinite genus: history

- ▶ Govorov, Heyman, Goldberg, Ostrovskii; Ecalle, Voronin; Pham, Voros; Kontsevich, Soibelman
- ▶ Explosion of results in dynamical systems: Calegari, Rafi, Mann, Vlamiš, Valdez,...
- ▶ Classification by end structure
- ▶ C^* bundles on R_∞ , representations of $\pi_1(R_\infty)$, and the package (curve complex, Goldman bracket, Fenchel-Nielsen coordinates, Teichmüller space, accessory parameters and uniformization)

Example: Resurgent functions and billiards in holomorphic wave equation

Consider wave equation $(\partial_t^2 - \Delta)u = 0$ with boundary values defined on algebraic surfaces P, F . Suppose that these boundary values are algebraic with singularities S_j .

Th: The solution is resurgent on the complement of the elimination algebra of S_j, P, F and the light cone.

□

Proved by propagation of singularities technique(explicit formula using Kirchoff with reflection).

Seems new: asked M. Zworski. The first application of resurgence to wave eqn(but, see Balser,...)

Ample connections with ergodic billiards. See the work of Chaika, Weiss... Connections to IEMs and Kontsevich Zorich cocycle. Hard problem : see McMullen's talk at IHES.

Example: integral equations with algebraic singularities

$x, y \in C$ -Riemann surface

$$\int_{\zeta} K_{a,b}(x, y) f_b(y) dy = \lambda f_a(x) \quad (21)$$

$\zeta \in H_1(C - \text{singularities of } f, K, \{p_0, p_1\})$

Prop: The solution is resurgent. The singularity set is defined iteratively from the singularity set of K and p_0, p_1 . The fundamental group is infinitely generated. The bundle is infinite dimensional. It realizes a representation of $\pi_1(C - \text{singularities of } f)$

Rk: There are irregular versions of this equation.

R_∞ in KAM

Prop: KAM tori are noncommutative tori with AF C^* algebra (Brjuno condition). The C^* algebra of the torus acts on the space of series that are the solutions of the KAM iteration scheme.

Th: Breakdown of KAM tori corresponds to bifurcation into different solution of the ODE.

R_∞ and singularities in n-body Problem

Siegel theory : singularities = collisions. Puiseux solutions with infinity of moving singularities(complex time)

Th: The solution corresponds to fixed point of a map in the Banach space of Puiseux series with moving singularities.

Th: The time surface wraps onto the Devaney manifold.

Q: What about Xia's family of non collisional singularities? Conj: they are irregular resurgent with moving singularities.

Th: Time has 1-ended topology. The singularities have branch type.

Rk: Physical time in n-body problem is a Riemannian surface of infinite type.

Riemann surfaces of infinite genus and Non Standard analysis

- ▶ Suslin tree and Cantor surfaces
- ▶ Results of Weiermann et.al. on Ackermann function and slow consistency
- ▶ Ordinal analysis
- ▶ Continuum of generators in MCG
- ▶ Unclassifiability results.
- ▶ There are explosions in both fields: 1) Montova, Berarducci,...
2) Vlamis, Valdez,... Both ready to use in QCD

Part IV: Local Renormalization Group

Wilsonian Renormalization Group

$$Z = \int Df \exp\left(- \int_p f(p)(p^2 + m^2)f(-p) + \right. \quad (22)$$
$$+ \int_{p_1, p_2, p_3} u(p_1, p_2, p_3, -p_1 - p_2 - p_3) f(p_1) f(p_2) f(p_3) f(-p_1 - p_2 - p_3) +$$
$$\left. + \dots \right)$$

Cut and rescale

Generating functionals

$$Z\{S\} = \int \exp(S[f]) Df \quad (23)$$

$$Z[A, B] = \int \exp\left(\int_{x,y} A(x, y) f(x) f(y) + \int B(x, y, z) f(x) f(y) f(z)\right) \quad (24)$$

RG transform

$$Z[A]\{F\} = \int \exp\left(\int A(x)f(x)f(x)\right) F[f] \quad (25)$$

$$Z[A]\{F\} = \sum \int F_n(x_1, \dots, x_{2n}) \int \exp\left(\int A_1 + \int A_2\right) \times \quad (26) \\ \times (f_1 + f_2)(x_1) \dots (f_1 + f_2)(x_{2n})$$

$$Z[A]\{F\} = \sum \int_{x \in X, y \in Y} F_n(x_1, \dots, y_{2l}) C_s \sum \int \exp\left(\int A_1\right) \times \quad (27)$$

$$f_1(x_1) \dots f_1(x_{2s})(A_2)^{-1}(y_1, y_2) \dots (A_2)^{-1}(y_{2l-1}, y_{2l})$$

A_2^{-1} is the inverse operator. In our case

$$A_2^{-1}(y, z) = 1/A_2(y)\delta(y, z).$$

Compensating diffeomorphisms

$$\int_X d\xi A_1(\xi) f(\xi)^2 = \int_D A_1(d(x)) f(d(x))^2 J(d(x)) dx \quad (28)$$

Compensating diffeomorphism

$$d : D \rightarrow X \quad (29)$$

Change of function variable $f(d(x)) = g(x)$

$$G_n(x_1, \dots, x_{2n}) = \text{Jac}(d) \prod \sqrt{\frac{A_1(d(x_i)) J(d(x))}{A(x_i)}} \sum \int (dy) F_{n+l}(d(x_1), \dots) \quad (30)$$

Wilsonian RG is an infinitesimal version of this transform, for a particular family of diffeomorphisms.

Need for resurgent S_{eff} , see later slides.

Self similar spaces

Def: Complex manifold is self similar if it is biholomorphic to its own subset.

Ex: complement to a countable set or a Cantor set in \mathbb{C}

Ex: certain complements to countable sets of algebraic varieties in \mathbb{C}^n .

Ex: complements to Cantor tree Riemannian surfaces in \mathbb{C}^2 .

Q: how to construct such domains where singularities are themselves self similar?

RG and isomonodromy

$I(p^2)$ = bundle of master integrals.

$$\frac{dI(p^2)}{dp} = \sum \frac{A_i(\epsilon)}{p - r_i m(\epsilon)} I(p^2) \quad (31)$$

Th: RG is the restriction of isomonodromy equations on the space $a_i = r_i m$.

Resurgent effective action

$$\int Df \exp(S_{\text{eff}}[f]) \quad (32)$$

E.g.

$$\int Df E[S_1[f], \dots, S_n[f], \dots], \quad S_i[f] = \int_M (dx) s_i(x) \prod \partial^{\alpha_i} f(x) \quad (33)$$

Part V: Riemann Hilbert problems

RH factorization problem

Given a set of curves C_i on the plane, find $F_a(z)$ s.t.

$$F_+(z) = M_i(z)F_-(z) \quad (34)$$

for known functions $M_i(z)$.

Factorization problems on complements

- ▶ $dim = 1$:

$$X = \mathbb{C} - \{x_n\}, f_a(x) = \sum_k f_{a,n,k}(x - x_n)^{\alpha_{n,a}+k} + reg(x) \quad (35)$$

- ▶ $dim > 1$:

$$X = \mathbb{C}^d - \{S_n\}, f_a(x) = \sum_k f_{a,n,k}(x_{||}) S_n^{\alpha_{n,a}+k} + reg(x) \quad (36)$$

- ▶ nested: $dim > 1$: S_n itself solution to a factorization problem in $d - 1$
- ▶ irregular problems:

$$f_a(x) = \sum_k f_{a,n,k} \exp((x - x_n)^{\beta_n}) (x - x_n)^{\alpha_{n,a}+k} + reg(x) \quad (37)$$

Structure of solutions $d = 1$

- ▶ Fix first few coefficients $f_{a,n,k}$
- ▶ Solution depends on representation $\pi_1(\mathbb{C} - \{x_b\})$ (a inf gen group)
- ▶ The powers $\alpha_{a,n,g}$ can depend on $g \in \pi_1$
- ▶ The fiber is infinite dimensional $C^*(\mathbb{Z}\pi_1)$
- ▶ Can there be moduli?

High dimensional RH[tNT]

Case $d=2$. Given S_i - curves in \mathbb{C}^2 , 3-manifolds C_i that can branch on S_i , find $F(z)$ s.t.

$$F_+(z) = M_i(z)F_-(z) \quad (38)$$

- ▶ S_i curves of exp type (e.g. $W^s(t)$)
- ▶ The bundle can be infinite dimensional e.g. $C^*(\pi(\mathbb{C}^2 - \cup S_i))$

Conj1: For any C^* factorization data, there exists a solution for the $RH[tNT]$ factorization problem

Rk: Note relations to eqns in group C^* algebras(work of Kharlampovich-Myasnikov)

Conj2: $d=2$ For any nested factorization data, there exists a solution.

Rk: Nested = the singularity set is itself solution to RH factorization problem, in $d-1$.

Th:(Calderon-Farah) It is relatively consistent with ZFC that there exists a counterexample to Naimark's problem while $\diamond_{\mathbb{N}_1}$ fails.

Mixed Riemann Hilbert problems

- ▶ $x \in \mathbb{C}^n$

$$L_i(x_j, \frac{\partial}{\partial x_j})f(x) = 0, i = 1, \dots, k \leq n \quad (39)$$

- ▶ If $k = n$ - holonomic D-module. If $k < n$, need additional conditions
- ▶ $F_+(x) = M_i(x)F_-(x)$
- ▶ Factorization near a $\text{codim} = n - k - l$ complete intersection $Y = \cap Y_s$

$$f_a(x) = \sum f_{a,l_s}(x_{||}) Y_s^{\alpha_s + l_s} + \text{reg}(x) \quad (40)$$

- ▶ Compare with RH for Painleve(Kapaev-Its): factorization in sectors.

Nested RH problems and non linear RH problems

- ▶ $e^{e^{t^\alpha}}$
- ▶ Type 1: algebraic divisor + exponential behavior near components
- ▶ Type 2: analytic divisor + exponential behavior near components
- ▶ Ex: divisor of exponential type (Ilyashenko, Rempe,...)

Part VI: Resurgent QCD

Functional equations: examples

- ▶ Self consistent vertex

$$V_{p,q,r} = \lambda_{p,q,r} \int_{u,v,w} V_{p,u,v} V_{q,v,w} V_{r,u,w} \quad (41)$$

- ▶ Bound state equations

$$\begin{aligned} \hat{L}_n \psi_n(x_1, \dots, x_n) = & \quad (42) \\ = \int & K_{n;n_1, \dots, n_k}(x_1, \dots, x_n; y_{1,1}, \dots, y_{1,n_1}; \dots; y_{k,1}, \dots, y_{k,n_k}) \times \\ & \times \psi_1(y_{1,1}, \dots, y_{1,n_1}) \dots \psi_k(y_{k,1}, \dots, y_{k,n_k})(dy) \end{aligned}$$

Th:(S) If the coefficients $K_{n;n_1, \dots, n_k}$ are resurgent (in either of two senses) then there is an infinity of constructible resurgent solutions to this hierarchy.

Covariant functional PDEs and resurgence of their solutions

- ▶ Forms and tensors on function spaces

$$F_{\alpha_1, \dots, \alpha_n}(x_1, \dots, x_n)[\bar{\eta}, J_\mu, \eta]$$

- ▶ Top forms $F(x_1, \dots, x_n, \dots)[f]$ for a dense set x_i
- ▶ Functional PDEs: FPDEs

$$(\hat{\mathbb{L}}[f, \frac{\delta}{\delta f}]F)(x_1, \dots, x_n)[f] = 0 \quad (43)$$

- ▶ **Th:(S)** If
 - 1) The coefficients of the FPDE
 - 2) The coefficients of boundary conditionsare resurgent, then the solution is also resurgent.

Functional Fourier Integral Operators (FFIOs)

- ▶ For the classic wave eqn $(\partial_t^2 - \Delta)w = 0$ with some Boundary Conditions we seek solution

$$w(t, x) = \int (d\xi) \exp(\phi(x, \xi)) a(x, \xi) \hat{f}(\xi) \quad (44)$$

- ▶ Functional equations $\hat{\mathbb{L}}W = 0$ seek solutions

$$W[F] = \int Dg \exp(\Phi[f, g]) A[f, g] \hat{F}[g] \quad (45)$$

- ▶ **Th:(S)** Solutions to fPDEs with resurgent coefficients can be sought as FFIOs with functional resurgent argument over adapted function spaces.

Differential topology on Banach and function spaces

- ▶ The spaces of 1) Puiseux series 2) irregular series on R_∞
- ▶ Reflexivity: James spaces, Argyros-Hayden spaces, Bourgain-Delbaen, Tsirelson
- ▶ Homological algebra on Banach manifolds
- ▶ Hereditary indecomposable spaces
- ▶ Essentially non-Noetherian: need limits
- ▶ Non uniqueness of tensor product: Connes embedding problem

Infinite dimensional bundles

- ▶ C^* -valued bundles: ex: $C^*(\pi_1)$, self absorbing (Rordam, Toms,...), C^* of minimal dynamical systems, unclassifiable (Hjorth, Farah,). Relationship with non commutative topology.
- ▶ For our purposes, we need bundles with structure group generated by integral operators with resurgent kernels $K(\vec{p}, \vec{q})$.
- ▶ The structure group of the bundle is an operator space where the composition is convolution of resurgent functions. Note potential analogies to Hereditary Indecomposable spaces.
- ▶ Analogy of Stiefel-Whitney and Pontrjagin theory(the space of "dimensions" is larger: e.g. dimension theory of nuclear C^* algebras.

Functional Riemann Hilbert problems

- ▶ In the finite dimensional case we have factorization problem

$$F_i^{(+)}(z) = M_{i,j}(z)F_j^{(-)}(z), \quad z \in \mathbb{C}^n \quad (46)$$

for two sides of a contour C , $M(z)$ known (note the hierarchy of the problems, by dimension; $M(z)$ can itself be a solution to RH problem in dimension 1 lower).

- ▶ In the functional case $F(x_1, \dots, x_n)[f]$

$$F^{(+)}(x_1, \dots, x_n)[f] = \int (dy) M(x_1, \dots; y_1, \dots, y_n)[f] F^{(-)}(y_1, \dots, y_n)[f] \quad (47)$$

Note that in contrast to the fin dim case we can (and often do) have an infinite descending chain of *RH* factorization problems.

- ▶ The singularity locus can be resurgent (e.g. exponentially irregular).

Resurgent QCD and functional Riemann Hilbert problems

- ▶ RH factorization problem for functional bundle with regular singularities on a set of functional submanifolds $L_I[f] = 0$.
- ▶ Intuition: how uniqueness emerges out of infinite dimensional dynamics: seek solutions to RH problems with rigid intersection theory.
- ▶ RH-symmetric solutions to FPDEs and physical states.
Conjecture: proton is such a solution.

Bifurcation theory in functional dynamical systems

► Ex:

$$F : \mathbb{M} \rightarrow \mathbb{M} \quad (48)$$

where \mathbb{M} is a functional algebraic manifold. F - integral polynomial.

► **Prop:** There are 2^ω solutions of $F^{\circ n}[f] = f$. The solutions are resurgent.

The necessity of Transcendental Number Theory in resurgent QCD

- ▶ Intersection theory of rigid manifolds in the space of classical fields. The Functional RH problem that results from this intersection.
- ▶ Quantum bifurcation theory resulting from factorization ansatzes in n-body Coulomb problem
- ▶ Role of classical W^s in n-body problems in WKB

The role of C^* algebras, set theoretic issues, non separable function spaces

- ▶ C^* and definition of functional integration;
- ▶ tensor product, non reflexive Banach spaces, analysis on function manifolds; Indecomposable Banach spaces
- ▶ Riemann integration, definable polyhedra, infinitary combinatorics, Ramsey theory
- ▶ Intrinsic use of ordinal analysis and Non standard analysis

Experimental tests

- ▶ There are unsolvability effects already in Quantum Mechanics, in quantum chaos, in high quantum numbers and wave functions (compare arithmetic chaos in Maass forms, Lindenstrauss' work, non unique ergodicity)
- ▶ Very hard to see experimentally because very sensitive equipment is necessary for this
- ▶ QCD gives us a chance(if we believe that QFT evolution equations are the same). But we need better tools for computation.
- ▶ This tooling is what Resurgent QCD will provide.
- ▶ Conj: Proton mass is the standard part of a non standard number. Path to solve: Riemann surfaces of inf genus, non standard analysis, analysis on Banach spaces.

Resurgent Space Time

- ▶ Example: time in Sitnikov problem is R_∞
- ▶ Example: YM on cardioid(deg=4) with algebraic boundary conditions.
- ▶ Physical time is Riemannian surface of infinite genus.
- ▶ Correspondingly, for spacetime. (Note ample connections to geometry: Fornaess, Forstneric, ...)

Conclusion

- ▶ Topologically infinite spaces in particle dynamics
- ▶ Functional PDEs and methods for their solution. Functional Riemann Hilbert problems.
- ▶ Transcendental numbers and relations to recursion theory and holomorphic dynamics
- ▶ Various set theoretic and recursion theoretic issues

Part VII: Kontsevich Soibelman resurgence conjecture

Kontsevich Soibelman conjectures in "Holomorphic Floer Theory.."

Conjecture 7.1.1:

- a) Voros coefficients $V_{\delta_i}(\epsilon_i)$ are resurgent in \hbar
- b) Each series $Tr_{\gamma}(\epsilon, \mu)$ are resurgent in \hbar

Exp integrals

$$Z = \int_{\Gamma \in X} \exp(S/h) dx \quad (49)$$

$$J(p_{i,\omega}) = \int \exp(\prod P_i^\alpha) dx \quad (50)$$

Prop:

$$\frac{\partial J}{\partial p_\omega} = \sum_{N=(n_i)} \frac{A_{\omega,N}}{\prod L_i^{n_i}} J \quad (51)$$

- ▶ "Hankel" contour at infinity on the complement of discriminant
- ▶ L_i contain the discriminant at infinity, but may also contain spurious poles

An approach using flat bundles

Th1: The D-module pulls back to Kapranov uniformization of discriminant

Th2: There exist coordinate system

$$\frac{\partial J}{\partial z_i} = \left(\sum \frac{A_{i,N}}{\prod z_i^{n_i}} + \text{reg} \right) J \quad (52)$$

and the solution has Balser-Jurkat-Birkhoff normal form

$$J = \prod \exp(Q_i(1/z_i)) z_i^{\alpha_i} \text{reg}(z) \quad (53)$$

Th3: The solution is resurgent in z_i .

Rk: The solution is "uniformly" resurgent, in contrast to dynamic situations.

Rk: similar is true for $\int_X \exp(P^\alpha) A(x) (dx)$ where $A(x)$ is algebraic.

Quantitative version of the conjecture

Conj: The Stokes constants are given by the factorized form of Xu-Alexeev-Nietzke T_{exp} factors. These latter factors can be expanded in series where terms have interpretation using generalized Gelfand-Tsetlin patterns.

Rk: Xu's results are direct consequence of the T_{exp} representation of monodromy around infinity.

Q: What about the state integrals in CS, which involve Faddeev dilogs?

Isomonodromy problem

Th: The irregular isomonodromy problem satisfies generalized Schlesinger equations.

Rk: These equations are formulated on the variety that satisfies consistency conditions.

Th: The Stokes data for these isomonodromy problem satisfies multidimensional version of Its-Kapaev RH factorization problem.

Rk: There exists Its-Kapaev-Fokas complex of cells that connect 0-dim intersections of components of singularity divisor. There exists irregular RH factorization problem that corresponds to these cells and it gives solution to the isomonodromy eqns.

Part VIII: Kontsevich resurgence conjecture

$$J(m_i) = \int Dx_i(t) \exp(S_{n\text{-body}}[x]) \quad (54)$$

is resurgent in m_i .

Riemannian path integral

$$\int_{C^{\mathbb{L}}(I), \sup|f(t)| < a} Dxx(t_1) \dots x(t_k) = f(\mathbb{L}, a; t_1, \dots, t_k) \quad (55)$$

Integration over definable polyhedra.

Polyhedra with continua of vertices. Definable polyhedra.

$$\int Dx \exp\left(-\int x(t_1)A(t_1, t_2)x(t_2)\right) = (\det(A))^{f(\mathbb{L})} c(\mathbb{L}) \quad (56)$$

$$\int Dx \exp\left(-\sum_n \int dt p_n x^n(t)\right) = \prod_{t \in I} \left(\int dx \exp\left(-\sum p_n x^n\right)\right) = c^{2\omega} \quad (57)$$

$$\sum_{i \in 2^\omega} a_i^{b_i^\omega c_i}, \quad a_i, b_i, c_i \in \mathbb{R}$$

Resurgent Non Standard Analysis

$$Z[\mathbb{L}, R_\infty](m_i) = \int_{C^{\mathbb{L}}(R_\infty)} Dx \exp(S_{n\text{-body}}[x]) \quad (58)$$

where the moving singularities in R_∞ and the factorization near singularities are adapted to the analyticity class of n-body problem (in particular, respect the analytic model for KAM tori and for $W^s(t)$, e.g. have finite nested iterates of exp function.)

Rk: non standard numbers embed into R_∞ with suitable chosen moving singularities.

Prop: Z takes values in resurgent non standard number field.