Small-N expansion in the $O(N)$ model: constructive field theory and transseries

Dario Benedetti

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with R. Gurau, H. Keppler, D. Lettera

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Outline

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Perturbative series in QFT

In Quantum Field Theory we typically compute observables in a power series expansion in coupling g:

$$f(g) \simeq \sum_{n\geq 0} a_n g^n$$

Perturbative series in QFT are generically asymptotic series with zero radius of convergence

[argument: Dyson (1952); proof for specific models: Jaffe (1965); de Calan, Rivasseau (1982); ...]

- General perturbative reason: **factorial growth of a_n**

  a_n from from sum of $\sim n!$ contributions (Feynman diagrams)

  Even single diagrams can result in contribution of order $n!$ to a_n ("renormalons")

- Nonperturbative reason: **stability**

  $\Rightarrow$ if perturbative series was convergent, it would converge in disk around origin

  but for $g < 0$, the action $S[\phi]$ is typically unbounded from below
  $\Rightarrow$ Euclidean path integral $\int[d\phi]e^{-S[\phi]}$ is ill defined

Reconstruction problem

If we knew the whole series, could we reconstruct $f(g)$, e.g. by Borel summation?

There are theorems giving conditions for when Borel sum of the perturbative series reconstructs the full function $f(g)$.
(e.g. Nevanlinna-Sokal theorem [Sokal (1979)]: conditions on rest term and on analyticity of $f(z)$)

However:

- Hypothesis are hard to check in QFT.
- They might also be violated in physically interesting cases.

$\Rightarrow$ two roads to this problem, with very different methods:

- Constructive QFT
- Transseries and resurgence

Constructive QFT

Constructive quantum field theory: $\sim$ make sense of formal functional integrals

rigorously prove nonperturbative existence of specific examples of interacting models that satisfy QFT axioms (Osterwalder-Schrader axioms)

Some classic milestones:

- $\mathcal{P}(\phi)_2$ and ϕ_3^4 theories [Glimm, Jaffe, Spencer (1973)]
- Gross-Neveu ($d = 2$), Yukawa ($d = 3$), sine-Gordon and Thirring model ($d = 2$), ...
  ... triviality of ϕ_4^4

Borel summability has been also a central question, resulting in particular in:
proof of Borel summability for $\mathcal{P}(\phi)_2$ and ϕ_3^4 with positive squared mass
[Eckmann, Magnen, Seneor (1975); Magnen, Seneor (1977)]
and for massive Gross-Neveu [Feldman, Magnen, Rivasseau, Seneor (1986)]

Borel summability remains open question in many physically interesting models
(e.g. in $d = 4$ or in broken phase)

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Borel summability remains open question in many physically interesting models
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And what if not Borel summable?

Transseries and resurgence

⇒ More general hope:

Principle of semiclassical decoding [Mariño]

$f(z)$ admits a semiclassical decoding if perturbative series $h_0(z)$ can be promoted to a **transseries**

$$\Phi(z;C) = h_0(z) + \sum_{\ell \geq 1} C^\ell e^{-\ell A/z} h_\ell(z)$$

whose Borel-Écalle summation $s(.)$ reconstructs $f(z)$:

$$f(z) = s(h_0)(z) + \sum_{\ell \geq 1} C^\ell e^{-\ell A/z} s(h_\ell)(z)$$

if $s(h_\ell)(z)$ exists for all $\ell \in \mathbb{N}_0$ and the series over ℓ converges for some range of z and C

⇒ Large amount of research on transseries expansion and resurgence
[Dunne; Ünsal; Mariño; Aniceto; Basar; Schiappa; Dorigoni; Serone; Bajnok; Balog; ...]

In general, proving/testing semiclassical decoding in a non-trivial QFT is rather difficult (but ∃ results for Chern-Simons or models with integrability, supersymmetry, large-N, ...)

A constructive approach to transseries?

This talk:
a first step in trying to bridge resurgent transseries research to constructive QFT

Just a proof of concept:

- Trivial toy model ($d = 0$)
- One single specific method of constructive QFT ("LVE")

Hope:
Method designed for proper QFT $\Rightarrow$ it should be applicable in higher d

Outline

The toy model

Zero dimensions: functional integral = standard integral

Partition function of the $0d$ $O(N)$ model:

$$Z(g,N) = \int_{\mathbb{R}^N} \left(\prod_{a=1}^{N} [d\phi_a] \right) e^{-S[\phi;g]} \,, \qquad S[\phi;g] = \frac{1}{2} \sum_{a=1}^{N} \phi_a^2 + \frac{g}{4!} \left(\sum_{a=1}^{N} \phi_a^2 \right)^2$$

- No UV or IR divergences, no renormalization
- Combinatorics of perturbation theory still the same: factorial growth
- Amplitudes of Feynman diagrams are of course trivial: combinatorial models (vector, matrix, and tensor models $\Rightarrow$ many applications)
- Useful playground for instantons (but no renormalons)

The $N = 1$ case has been extensively studied as a toy model of resurgence (e.g. reviewed in [Aniceto, Basar, Schiappa (2018)])

General N case seems less so, but $Z(g,N) \propto$ Tricomi confluent hypergeometric function

Leitmotiv: we could do it differently, but we look for method not specific to $0d$

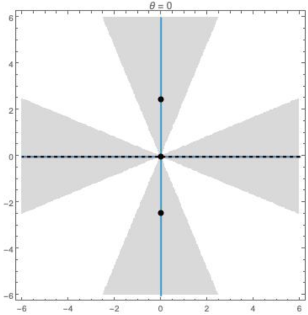
Lefschetz thimbles and Stokes phenomenon ($N = 1$ case)

$$Z(g) = \int_{\mathcal{C}} [d\phi]\, e^{-\frac{1}{g}S[\phi;1]} = \sum_i \int_{\mathcal{J}_i} [d\phi]\, e^{-\frac{1}{g}S[\phi;1]} \simeq \sum_i e^{-\frac{1}{g}S[\phi_i^*;1]} h_i(g)$$

$\mathcal{J}_i$: Lefschetz thimble (or steepest-descent contour)

- Downward flows of $\mathrm{Re}(-S[\phi;g])$ originating at saddle point ϕ_i^* (i.e. $\partial_\phi S_{|\phi=\phi_i^*} = 0$)
- The imaginary part of $S[\phi;g]$ is constant along $\mathcal{J}_i$ ($\Rightarrow$ Laplace method)

Analytic continuation in $g = |g|e^{\imath\theta}$: rotate ϕ for convergence, then deform



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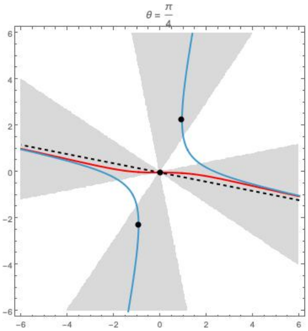

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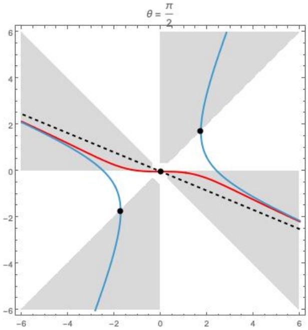

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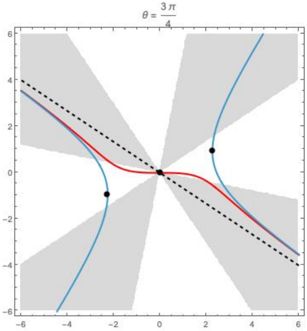

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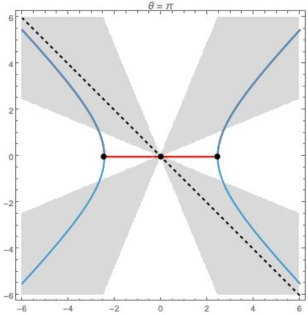

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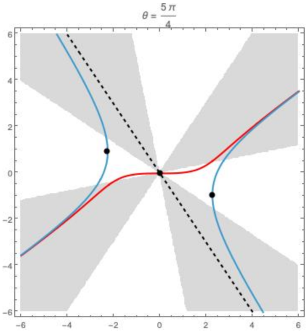


Hubbard-Stratonovich trick

The $N > 1$ case is more involved (contours $\Rightarrow$ higher-dimensional thimbles )

$\Rightarrow$ Use the Hubbard-Stratonovich trick:

$$e^{-\frac{g}{4!}(\phi^2)^2} = \int_{-\infty}^{+\infty} [d\sigma]\ e^{-\frac{1}{2}\sigma^2 + \imath\sqrt{\frac{g}{12}}\sigma\phi^2} \ ,$$

The integral over ϕ becomes Gaussian and can be easily performed for $g > 0$, leading to:

$$Z(g, N) = \int_{-\infty}^{+\infty} [d\sigma]\ e^{-\frac{1}{2}\sigma^2} \frac{1}{\left(1 - \imath\sqrt{\frac{g}{3}}\sigma\right)^{N/2}}$$

Notice: now we can continue to $N \in \mathbb{C}$

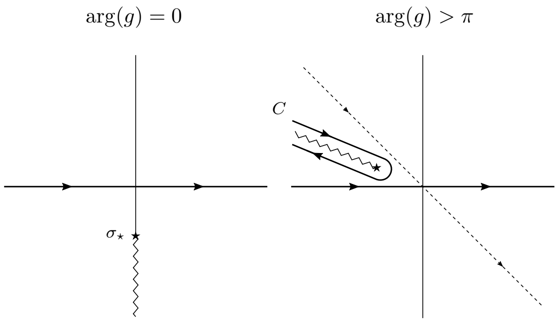
We could also use differential equation (in g) satisfied by $Z(g, N)$ and apply known ODE methods, but again those methods do not work in higher d (no ODE for $Z(g, N)$).

HS is instead very much used in QFT

Instantons vs branch cut

In HS representation, the Lefshetz thimble is always the real axis, irrespective of g.

$\Rightarrow$ Different origin of Stokes phenomenon:



- As $\arg(g)$ increases, the branch cut moves clockwise in the complex σ-plane.
- When g crosses the negative real axis the contour needs to be tilted.
- Moving it back to the thimble on real line, we pick also a Hankel contour C.

Results for the partition function

With HS, we traded ϕ^4 which dominates over the Gaussian at large field with $\ln(1 - \imath\sqrt{\frac{g}{3}}\sigma)$ which does not! $\Rightarrow$ Useful when bounding integrals.

Theorem (Properties of $Z(g, N)$)

Let $N \in \mathbb{R}$ be a fixed parameter. The partition function $Z(g, N)$ satisfies the following properties:

1) $Z(g, N)$ is analytic in $\mathbb{C}_\pi = \mathbb{C} \setminus \mathbb{R}_-$.

2) For $g \in \mathbb{C}_\pi$, the partition function has the asymptotic perturbative expansion:

$$Z(g, N) \simeq \sum_{n=0}^{\infty} \frac{\Gamma(2n + N/2)}{2^{2n} n! \, \Gamma(N/2)} \left(-\frac{2g}{3}\right)^n \ .$$

3) The function $Z(g, N)$ is Borel summable along all the directions in $\mathbb{C}_\pi$.

Results for the partition function

Theorem (Properties of $Z(g,N)$)

Let $N \in \mathbb{R}$ be a fixed parameter. The partition function $Z(g,N)$ satisfies the following properties:

4) $Z(g,N)$ can be continued past the cut, on the entire Riemann surface; $\mathbb{R}_-$ is a Stokes line ($Z(g,N)$ ceases to be Borel summable at $\mathbb{R}_-$); a second Stokes line is found at $\mathbb{R}_+$ on the second sheet. Analytic continuation:

$$2k\pi < |\arg(g)| < (2k+1)\pi :$$

$$Z(g,N) = \omega_{2k}\ Z^{\mathbb{R}}(g,N) + \eta_{2k}\ \frac{\sqrt{2\pi}}{\Gamma(N/2)}\ e^{\imath\tau\frac{\pi}{2}}\ e^{\frac{3}{2g}}\ \left(e^{\imath(2k+1)\tau\pi}\frac{g}{3}\right)^{\frac{1-N}{2}}\ Z^{\mathbb{R}}(-g, 2-N)\ ,$$

$$(2k+1)\pi < |\arg(g)| < (2k+2)\pi : \quad \textit{similar}$$

where $\tau = -\mathrm{sgn}(\arg(g))$ and the Stokes parameters (ω,η) are:

$$(\omega_{2k},\eta_{2k}) = \begin{cases} e^{\imath\tau\pi N\frac{k}{2}}\ (1,0) & ,\ k \textit{ even} \\ e^{\imath\tau\pi N\frac{k+1}{2}}\ (1,-1) & ,\ k \textit{ odd} \end{cases} .$$

The monodromy group of $Z(g,N)$ is of order 4 if N is odd, and of order 2 if N is even. More generally, we have a monodromy group of finite order if N is a rational number, and an infinite monodromy otherwise.

Results for the partition function

Theorem (Properties of $Z(g,N)$)

Let $N \in \mathbb{R}$ be a fixed parameter. The partition function $Z(g,N)$ satisfies the following properties:

5) In the sector $k\pi < |\arg(g)| < (k+1)\pi$ of the Riemann surface $Z(g,N)$ has the following **transseries expansion***:*

$$Z(g,N) \simeq \omega_k \sum_{n=0}^{\infty} \frac{\Gamma(2n+N/2)}{2^{2n} n!\,\Gamma(N/2)} \left(-\frac{2g}{3}\right)^n + \eta_k\, e^{\imath\tau\pi(1-\frac{N}{2})} \sqrt{2\pi} \left(\frac{g}{3}\right)^{\frac{1-N}{2}} e^{\frac{3}{2g}} \sum_{q\geq 0} \frac{1}{2^{2q} q!\,\Gamma(\frac{N}{2}-2q)} \left(\frac{2g}{3}\right)^q$$

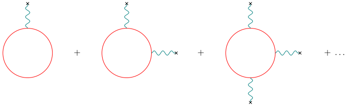
The transseries is resurgent: the instanton series is obtained from the perturbative one by substituting $N \to 2-N$ and $g \to -g$ and vice versa.

Small-N expansion

$$Z(g,N) = \sum_{n\geq 0} \frac{1}{n!}\left(-\frac{N}{2}\right)^n Z_n(g)\ , \quad Z_n(g) = \int_{-\infty}^{+\infty} [d\sigma] e^{-\frac{1}{2}\sigma^2} \left(\ln(1-\imath\sqrt{\frac{g}{3}}\sigma)\right)^n$$

$Z_n(g)$ has n "loop vertices" (more evident in $d>0$), hence "loop vertex expansion" (later): [Rivasseau (2007)]

$$\ln(1-\imath\sqrt{\frac{g}{3}}\sigma) = -\sum_{k\geq 1}\frac{1}{k}\left(\imath\sqrt{\frac{g}{3}}\right)^k \sigma^k =$$



Unlike the expansions in g, this is a convergent expansion:

Lemma

$Z_n(g)$ is analytic in the cut plane $\mathbb{C}_\pi$. Indeed, for every $g\in\mathbb{C}_\pi$, the integral is absolutely convergent and bounded from above by:

$$|Z_n(g)| \leq K^n \frac{\left(|\ln(\cos\frac{\varphi}{2})|+1\right)^n}{\varepsilon^n}\left(1+|g|^{\frac{n\varepsilon}{2}}\Gamma(\tfrac{n\varepsilon+1}{2})\right)\ , \quad \forall\varepsilon>0\ .$$

Using this bound with some fixed $\varepsilon<2$ shows that, $\forall g\in\mathbb{C}_\pi$ the small-N series has infinite radius of convergence in N.

A more challenging object: the free energy

Also the free energy has a small-N expansion:

$$W(g,N) = \ln(Z(g,N)) \equiv \sum_{n\geq 1} \frac{1}{n!}\left(-\frac{N}{2}\right)^n W_n(g)$$

$Z_n(g)$: moments of the random variable $\ln(1 - \imath\sqrt{g/3}\,\sigma)$

$\Rightarrow W_n$: cumulants of the same variable

$\Rightarrow W_n$ in terms of $Z_n(g)$ by using the Möbius inversion formula

$$\boxed{Z_n(g) = \sum_{\pi}\prod_{b\in\pi} W_{|b|}(g) \quad \Rightarrow \quad W_n(g) = \sum_{\pi}\lambda_\pi \prod_{b\in\pi} Z_{|b|}(g)}$$

where π a partition of the set $\{1,\dots n\}$, and $\lambda_\pi = (-1)^{|\pi|-1}(|\pi|-1)!$

More explicitly:

$$W_n(g) = \sum_{k=1}^{n}(-1)^{k-1}(k-1)! \sum_{\substack{n_1,\dots,n_{n-k+1}\geq 0\\ \sum i n_i = n,\, \sum n_i = k}} \frac{n!}{\prod_i n_i!(i!)^{n_i}} \prod_{i=1}^{n-k+1} Z_i(g)^{n_i}$$

$\Rightarrow$ relation between $W(g,N)$ and $Z(g,N)$ which holds in the sense of formal power series in N: analyticity properties of $W(g,N)$ are not obvious

Loop Vertex Expansion

Let $\mathcal{T} \in T_n$, with T_n the set of combinatorial trees with n vertices labeled $1, \dots n$.

Associate to each edge $(k,l) \in \mathcal{T}$ a variable $u_{kl} \in [0,1]$, and define the $n \times n$ matrix $w^{\mathcal{T}}$:

$$w^{\mathcal{T}}_{kl} \equiv \begin{cases} 1 \, , & \text{if } \; k = l \\ \inf_{(i,j) \in P^{\mathcal{T}}_{k-l}} \{u_{ij}\} \, , & \text{else} \end{cases}$$

The matrix $w^{\mathcal{T}}$ is a positive matrix for any choice of u parameters, and is stirctly positive outside a set of measure 0.

Loop Vertex Expansion of $W(g,N)$

$$W(g,N) = \ln(Z(g,N)) = \sum_{n \geq 1} \frac{1}{n!} \left(-\frac{N}{2} \right)^n W_n(g)$$

The cumulants $W_n(g)$ can be written as:

$$W_1(g) = Z_1(g) = \int_{-\infty}^{+\infty} [d\sigma] \; e^{-\frac{1}{2}\sigma^2} \ln \left[1 - \imath \sqrt{\frac{g}{3}} \sigma \right] \, ,$$

$$W_n(g) = - \left(\frac{g}{3} \right)^{n-1} \sum_{\mathcal{T} \in T_n} \int_0^1 \prod_{(i,j) \in \mathcal{T}} du_{ij} \int_{-\infty}^{+\infty} \frac{\prod_i [d\sigma_i]}{\sqrt{\det w^{\mathcal{T}}}} \; e^{-\frac{1}{2} \sum_{i,j} \sigma_i (w^{\mathcal{T}})^{-1}_{ij} \sigma_j} \; \prod_i \frac{(d_i - 1)!}{\left(1 - \imath \sqrt{\frac{g}{3}} \sigma_i \right)^{d_i}} \, ,$$

where d_i is coordination of the vertex $i \in \mathcal{T}$.

LVE from BKAR forest formula (sketch)

- Rewrite Gaussian integral as differential operator (introducing $V(\sigma) = \ln(1 - \imath\sqrt{g/3}\,\sigma)$):

$$Z_n(g) = \int [d\sigma]\; e^{-\frac{1}{2}\sigma^2} V(\sigma)^n \equiv \left[e^{\frac{1}{2}\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma}} V(\sigma)^n \right]_{\sigma=0}$$

- Introduce copies with degenerate covariance and we introduce fictitious interpolating link parameters $x_{kl} = x_{lk} = 1$:

$$Z_n(g) = \left[e^{\frac{1}{2}\sum_{k,l=1}^{n} \frac{\partial}{\partial\sigma_k}\frac{\partial}{\partial\sigma_l}} \prod_{i=1}^{n} V(\sigma_i) \right]_{\sigma_i=0} = \left[e^{\frac{1}{2}\sum_{k,l=1}^{n} x_{kl}\frac{\partial}{\partial\sigma_k}\frac{\partial}{\partial\sigma_l}} \prod_{i=1}^{n} V(\sigma_i) \right]_{\sigma_i=0,\,x_{ij}=1}$$

- Use the Brydges-Kennedy-Abdesselam-Rivasseau (BKAR) forest formula (a Taylor formula for functions of several variables)

$$f(1,\ldots,1) = \sum_{\mathcal{F}} \underbrace{\int_0^1 \cdots \int_0^1}_{|\mathcal{F}|\ \text{times}} \left(\prod_{e\in\mathcal{F}} du_e \right) \left[\left(\prod_{e\in\mathcal{F}} \frac{\partial}{\partial x_e} \right) f \right] \left(w_{kl}^{\mathcal{F}}(u_{\mathcal{F}}) \right)$$

where $f : [0,1]^{n(n-1)/2} \to \mathbb{R}$, and F_n is the set of all the forests over n labeled vertices

LVE from BKAR forest formula (sketch)

- Obtain

$Z(g,N) =$

$$\sum_{n\geq 0} \frac{\left(-\frac{N}{2}\right)^n}{n!} \sum_{\mathcal{F}\in F_n} \int_0^1 \prod_{(i,j)\in\mathcal{F}} du_{ij} \left[e^{\frac{1}{2}\sum_{k,l} w^{\mathcal{F}}_{kl} \frac{\partial}{\partial\sigma_k}\frac{\partial}{\partial\sigma_l}} \left(\prod_{(i,j)\in\mathcal{F}} \frac{\partial}{\partial\sigma_i}\frac{\partial}{\partial\sigma_j} \right) \prod_{i=1}^n V(\sigma_i) \right]_{\sigma_i=0}$$

- Contribution of a forest factors over the trees (connected components) in the forest $\Rightarrow$ logarithm restricts the sum above to trees over n vertices:

$W(g,N) =$

$$\sum_{n\geq 1} \frac{\left(-\frac{N}{2}\right)^n}{n!} \sum_{\mathcal{T}\in T_n} \int_0^1 \prod_{(i,j)\in\mathcal{T}} du_{ij} \left[e^{\frac{1}{2}\sum_{k,l} w^{\mathcal{T}}_{kl} \frac{\partial}{\partial\sigma_k}\frac{\partial}{\partial\sigma_l}} \left(\prod_{(i,j)\in\mathcal{T}} \frac{\partial}{\partial\sigma_i}\frac{\partial}{\partial\sigma_j} \right) \prod_{i=1}^n V(\sigma_i) \right]_{\sigma_i=0}$$

- Use

$$\frac{\partial^d}{\partial\sigma^d} \ln\left(1 - \imath\sqrt{\tfrac{g}{3}}\sigma\right) = (-1)\, \frac{(d-1)!\left(\imath\sqrt{\frac{g}{3}}\right)^d}{\left(1-\imath\sqrt{\frac{g}{3}}\sigma\right)^d}$$

and get back to Gaussian integral form.

□

Results for the free energy

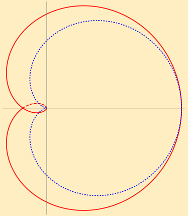
The LVE allows to prove:

Theorem (Properties of $W(g,N)$)

Let $N \in \mathbb{C}$ be fixed and let us denote $g = |g|e^{\imath\varphi}$.

1) The cumulants $W_n(g), n \geq 2$ are analytic in the cut plane $\mathbb{C}_\pi$.

2) By tilting the integration contours of the σ_i, the series $W(g,N) = \sum_{n\geq 1} \frac{1}{n!}\left(-\frac{N}{2}\right)^n W_n(g)$ becomes absolutely convergent in a cardioid domain $\mathbb{D}$ that protrudes beyond the negative axis:



$\Rightarrow$ *Borel summability in $\mathbb{C}_\pi$*

Transseries expansion of $W(g,N)$

Convergence of small-N series of $W(g,N)$ in cardioid domain $\mathbb{D}$
$\Rightarrow$ we can apply the steepest descent method term by term to

$$W(g,N) = \sum_{n\geq 1} \frac{1}{n!}\left(-\frac{N}{2}\right)^n W_n(g)$$

with $W_n(g)$ given in terms of $Z_n(g)$ by the moments-cumulants formula

For $|\arg(g)| > \pi$ we obtain a linear combination of products involving contributions from the thimble (real line) and the Hankel contour, and as a result:

The free energy $W(g,N)$ has the transseries expansion:

$$\begin{aligned} W(g,N) &= \sum_{n\geq 1} \frac{1}{n!}\left(-\frac{N}{2}\right)^n W_n(g) \\ &= \sum_{p\geq 0} e^{\frac{3}{2g}p}\left(\eta\sqrt{2\pi}e^{\imath\tau\frac{\pi}{2}}\left(\frac{e^{\imath\tau\pi}g}{3}\right)^{\frac{1-N}{2}}\right)^p \sum_{l\geq 0}\left(-\frac{2g}{3}\right)^l W_l^{(p)}(N)\,, \end{aligned}$$

where $\mathbb{R}_-$ is a Stokes line, $\tau = -\mathrm{sgn}(\arg(g))$ and η is a transseries parameter which is zero on the principal Riemann sheet and is one when $|\arg(g)| > \pi$. Bulky expression for $W_l^{(p)}(N)$.

Outline

Summary and outlook

LVE:

- Hubbard-Stratonovich trick $\Rightarrow$ instantons replaced by singularities
- logarithmic interaction has good large field properties $\Rightarrow$ better for bounding the integrals
- analyticity of $W(g, N)$ in cardioid domain allows to go beyond the Stokes line on negative axis and derive the corresponding transseries

Main point: constructive tools that in principle could be used in higher dimensions

For the future:

- Increase d gradually: $d = 1$, then $d = 2$
- More combinatorial models? E.g. tensor/matrix models [Rivasseau (2023): cumulants]