Topological expansion of unitary integrals, and maps

PIICQ online seminar

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Introduction

A tale of two matrices

Consider two $N \times N$ Hermitian random matrices X and Y that are unitarily invariant:

$$(X, Y) \sim (UXU^*, UYU^*) \quad \forall U \in \mathbb{U}(N).$$

What can we say about the singular values/eigenvalues of e.g.

$$XY \quad \text{or} \quad X + Y \quad \text{or even} \quad X^3YXY^2?$$

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In general, what can be said about the spectrum of any *non-commutative polynomial* P in X and Y?

- A non-commutative *monomial* is a word in the matrices X and Y, or the identity.
- A non-commutative polynomial is a $\mathbb{C}$-linear combination of n.c. monomials.

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Consider two $N \times N$ Hermitian random matrices X and Y that are unitarily invariant:

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What can we say about the moments of e.g.

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$$\rightarrow \text{Compute the large } N \text{ asymptotics of } \mathbb{E}\left[\frac{1}{N}\operatorname{Tr} P(X,Y)\right].$$

Two independent matrices

Assume X and Y are independent. How to compute:

$$\mathbb{E}\left[\frac{1}{N}\operatorname{Tr} X^{k_1} Y^{l_1} \cdots X^{k_d} Y^{k_d}\right]?$$

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Unitary invariance + independence:

$$(X, Y) \sim (UXU^*, Y) \quad \text{with } U \text{ Haar-distributed.}$$

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1. Free probability approach: by Voiculescu's theorem, UXU^* and Y are asymptotically free.

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1. Free probability approach: by Voiculescu's theorem, UXU^* and Y are asymptotically free.
2. Weingarten calculus approach: assume X, Y fixed, compute the expectation over U Haar-distributed.

Correlated matrices

Assume the joint law of (X, Y) is

$$\propto \mathrm{e}^{-N \operatorname{Tr} \mathcal{V}(X,Y)} \mathrm{d}X \mathrm{d}Y.$$

Diagonalize X and Y:

$$(X, Y) = (UAU^*, VBV^*) \quad \text{with } U, V \in \mathbb{U}(N) \text{ and } A, B \text{ diagonal}$$

The joint law of (U, V, A, B) is

$$\propto \mathrm{e}^{N \operatorname{Tr} W(U,U^*,V,V^*,A,B)} \mathrm{d}U \mathrm{d}V \mathrm{d}\mu(A) \mathrm{d}\nu(B).$$

$$\to \ \text{Compute } \mathbb{E}\left[\frac{1}{N} \operatorname{Tr} P(U, U^*, V, V^*, A, B) \mid A, B\right].$$

Unitary integrals

Measure:

$$\mu_{N,V} = \frac{1}{Z_{N,V}} \exp\big(N \operatorname{Tr} V(U_1, U_1^*, \dots, U_n, U_n^*, A_1, \dots, A_p)\big) \mathrm{d}U_1 \cdots \mathrm{d}U_n.$$

Assumptions:

1. V is a non-commutative polynomial.
2. $\sup_{N\geq 1} \max_{1\leq i\leq p} \|A_i\| < +\infty$.
3. $\operatorname{Tr} V$ is real-valued.

Unitary integrals

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Compute under $\mu_{N,V}$:

- the moments $\mathbb{E}\left[\frac{1}{N} \operatorname{Tr} P(U_1, U_1^*, \ldots, U_n, U_n^*, A_1, \ldots, A_p)\right]$
- the joint cumulants $\kappa_l\left(\operatorname{Tr} P_1, \ldots, \operatorname{Tr} P_l\right)$

Cumulants

$$\kappa_l(X_1, \ldots, X_l) = \partial_{t_1} \cdots \partial_{t_l}|_{t_1 = \cdots = t_l = 0} \ln \mathbb{E}\left[\exp\left(\sum_{i=1}^{l} t_i X_i\right)\right].$$

Overview

1. Introduction

2. Topological expansion and maps

3. First ingredient: Weingarten calculus

4. Second ingredient: Dyson-Schwinger equations

5. Monotone Hurwitz numbers

Topological expansion and maps

Topological expansion

- $\mu_{N,V} = \dfrac{1}{Z_{N,V}} \mathrm{e}^{N \operatorname{Tr} V} \mathrm{d}U_1 \cdots \mathrm{d}U_n \quad \text{with } V = \sum_{i=1}^{k} t_i \underbrace{Q_i}_{\text{monomial}}$

Theorem (Guionnet, Novak '15; B. '24)

Under our assumptions, there exists $\epsilon > 0$ such that if

$$\max_{1 \leq i \leq k} |t_i| < \epsilon$$

then for all $g \geq 0$, $l \geq 1$, $P_1, \ldots, P_l$ n.c. polynomials:

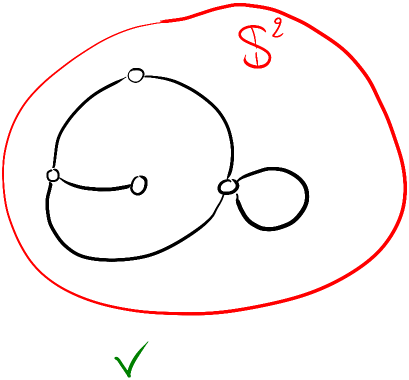
$$\kappa_l(\operatorname{Tr} P_1, \ldots, \operatorname{Tr} P_l) = N^{2-l} \sum_{h=0}^{g} \frac{1}{N^{2h}} \mathcal{M}_{N,V,l}^{(h)}(P_1, \ldots, P_l) + \mathcal{O}\left(N^{-2g-2}\right),$$

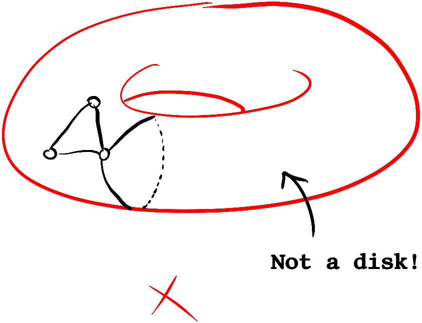
where $\mathcal{M}_{N,V,l}^{(h)}(P_1, \ldots, P_l)$ are generating series of maps of unitary type of genus h.

Maps

Map

A map is a graph drawn on a compact connected surface such that faces are polygons, up to deformation.





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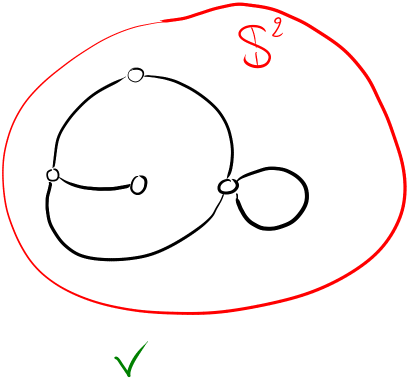
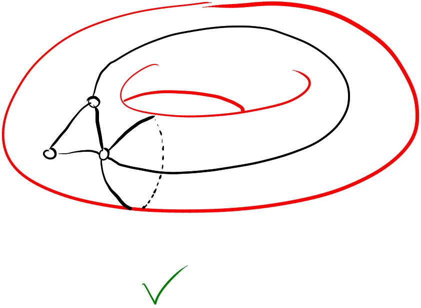
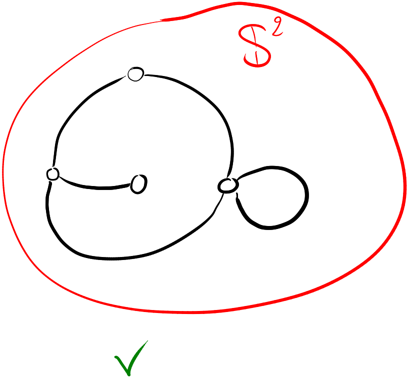


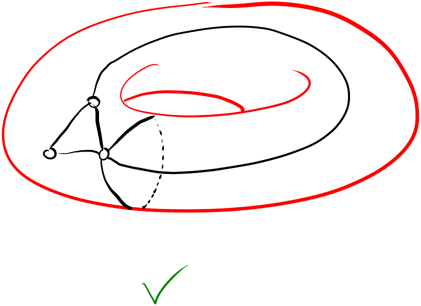


Maps

Map

A map is a 2-cellular embedding of a graph in a compact connected surface, up to orientation-preserving homeomorphism.





Combinatorial maps

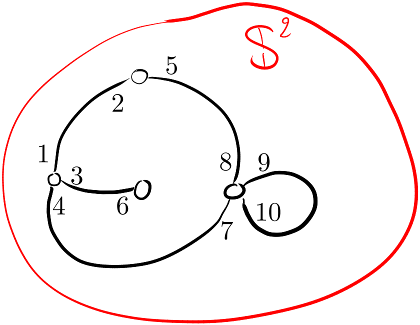
Combinatorial map

A combinatorial map is a pair $(\sigma, \alpha) \in \mathfrak{S}_n \times \mathcal{I}_n$, where

$\mathcal{I}_n = \left\{\pi \in \mathfrak{S}_n \colon \pi^2 = \pi, \forall i, \pi(i) \neq i\right\}$ involutions without fixed point

Example

$$\sigma = (1\,3\,4)(2\,5)(6)(7\,8\,9\,10) \quad \alpha = (1\,2)(3\,6)(4\,7)(5\,8)(9\,10).$$

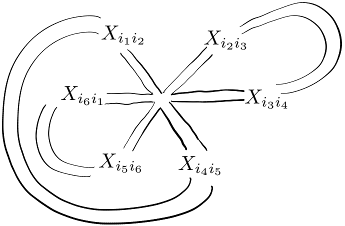


Diagrams in physics and in RMT

- ’t Hooft ’74
- Brézin, Itzykson, Parisi, and Zuber ’78
- Bessis, Itzykson, and Zuber ’80
- Ercolani and McLaughlin ’03
- … and many others …

Wick-Isserlis formula: GUE moments → Topological expansion

$$\mathbb{E}[\frac{1}{N}\mathrm{Tr}X^6]$$



First ingredient: Weingarten calculus

Weingarten calculus

To start: one random unitary matrix, no potential ($n = 1$, $V = 0$).

Theorem (Weingarten formula, Samuel '80, Collins '03)

Let $m \geq 1$ *and* $i, j, i', j' : [m] \to [N]$,

$$\int_{\mathbb{U}(N)} U_{i(1)j(1)} \cdots U_{i(m)j(m)} \overline{U}_{i'(1)j'(1)} \cdots \overline{U}_{i'(m)j'(m)} = \sum_{\rho,\sigma \in \mathfrak{S}_m} \delta_{i,i' \circ \rho} \delta_{j,j' \circ \sigma} \mathrm{Wg}_N(\sigma \rho^{-1}).$$

- Wg_N: expressed in terms of characters of the symmetric group
- Novak '10: Large N expansion of Wg_N

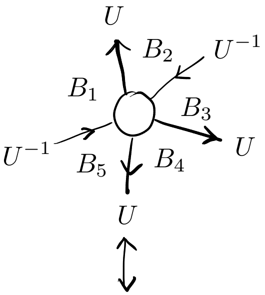
Turn computation of Haar integrals into a combinatorial problem.

Combinatorial description of monomials

- One monomial

$P = B_1 U^{\epsilon_1} B_2 U^{\epsilon_2} \cdots B_d U^{\epsilon_d}$ where

- $B_i =$ product of elements in (A_i)
- $\epsilon_i \in \{-1, +1\}$
- $d \in \mathbf{N}^*$



$B_1 U B_2 U^{-1} B_3 U B_4 U B_5 U^{-1}$

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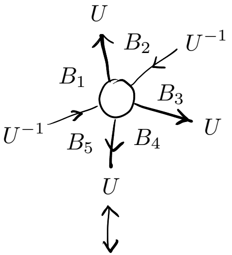


$$B_1 U B_2 U^{-1} B_3 U B_4 U B_5 U^{-1}$$

- Several monomials

$$P_1 = B_1 U^{\epsilon_1} \cdots B_{d_1} U^{\epsilon_{d_1}} \quad P_2 = B_{d_1+1} U^{\epsilon_{d_1+1}} \cdots B_{d_1+d_2} U^{\epsilon_{d_1+d_2}} \quad \ldots$$

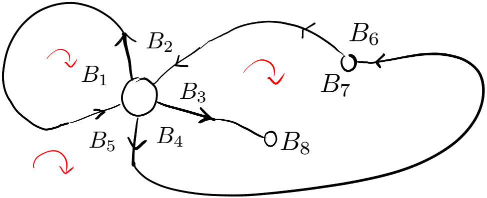
- $\circ$ $\gamma = (1 \ldots d_1)(d_1 + 1 \ldots d_1 + d_2) \cdots$
- $\circ$ (B_i)
- $\circ$ (ϵ_i).

Consequence of Weingarten calculus

- $P_1, \ldots, P_l \quad \to \quad (B_i), (\epsilon_i), \gamma.$

Proposition

$$\begin{aligned}
&\kappa_l(\mathrm{Tr}\, P_1, \ldots, \mathrm{Tr}\, P_l) \\
&\quad = N^{2-l} \sum_{g \geq 0} \frac{1}{N^{2g}} \sum_{\mathfrak{m}} (-1)^{\# Faces(\mathfrak{m})} \prod_{f \in Faces(\mathfrak{m})} \frac{1}{N} \mathrm{Tr} \left(\prod_{i \in f} B_i \right) \\
&\quad = N^{2-l} \sum_{g \geq 0} \frac{1}{N^{2g}} \mathcal{M}^{(g)}_{N,0,l}(P_1, \ldots, P_l).
\end{aligned}$$



$$\frac{1}{N}\mathrm{Tr}(B_1)\frac{1}{N}\mathrm{Tr}(B_2B_5B_6)\frac{1}{N}\mathrm{Tr}(B_3B_7B_4B_8)$$

A formal computation

Take $V \neq 0$:

$$\begin{aligned}
\mathbb{E}_{N,V}\left[\frac{1}{N}\operatorname{Tr} P\right] &= \partial_\epsilon|_{\epsilon=0} \ln \mathbb{E}_{N,0}\left[e^{\frac{\epsilon}{N}\operatorname{Tr} P + N \operatorname{Tr} V}\right] \\
&= \frac{1}{N}\sum_{n\geq 1}\frac{N^n}{n!}\kappa_{n+1}\left(\operatorname{Tr} P, \operatorname{Tr} V, \ldots, \operatorname{Tr} V\right) \\
&= \frac{1}{N}\sum_{n\geq 1}\frac{N^n}{n!}N^{2-n-1}\sum_{g\geq 0}\frac{1}{N^{2g}}\mathcal{M}^{(g)}_{N,0,n+1}(P, V, \ldots, V) \\
&\stackrel{?}{=} \sum_{g\geq 0}\frac{1}{N^{2g}}\sum_{n\geq 1}\frac{1}{n!}\mathcal{M}^{(g)}_{N,0,n+1}(P, V, \ldots, V).
\end{aligned}$$

Second ingredient: Dyson-Schwinger equations

Dyson-Schwinger equations

- Invariance under the Haar measure: $U \sim \mathrm{e}^{i\epsilon M}U$, $M \in \mathcal{H}(N)$.
- Use invariance and differentiate

$$\int f(U, U^*)\mathrm{d}U = \int f(\mathrm{e}^{i\epsilon M}U, U^*\mathrm{e}^{-i\epsilon M})\mathrm{d}U$$

$$0 = \int \left(d_U f \cdot iM - d_{U^*} f \cdot M\right) \mathrm{d}U.$$

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- For a n.c. polynomial:

$$\partial_\epsilon\big|_{\epsilon=0} \operatorname{Tr} P\left(\mathrm{e}^{i\epsilon M} U, U^* \mathrm{e}^{-i\epsilon M}\right) = \sum_{P=QuR} \operatorname{Tr} QMUR - \sum_{P=Qu^* R} \operatorname{Tr} QU^* MR.$$

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- Motivates the introduction of the logarithmic n.c. derivatives

$$\partial P = \sum_{P=QuR} Q \otimes UR - \sum_{P=Qu^*R} QU^* \otimes R$$

$$\mathcal{D}P = \sum_{P=QuR} URQ - \sum_{P=Qu^*R} RQU^*.$$

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There is a whole family of equations. The first one: under $\mu_{N,V}$, we have

$$\frac{1}{N}\mathbb{E}\left[\mathrm{Tr}\,\cdot\right] \otimes \frac{1}{N}\mathbb{E}\left[\mathrm{Tr}\,\cdot\right](\partial P) + \frac{1}{N^2}\kappa_2(\mathrm{Tr} \otimes \mathrm{Tr}(\partial P)) = \frac{1}{N}\mathbb{E}\left[\mathrm{Tr}(\mathcal{D}V)P\right].$$

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Series of maps

Introduce the series of maps

$$\mathcal{M}^{(g)}_{N,V,l}(P_1,\dots,P_l) = \sum_{n\geq 1} \frac{1}{n!}\mathcal{M}^{(g)}_{N,0,l+n}(P_1,\dots,P_l,V,\dots,V)\,.$$

By a combinatorial argument, we can show:

$$\mathcal{M}^{(0)}_{N,V,1}\otimes\mathcal{M}^{(0)}_{N,V,1}(\partial P) = \mathcal{M}^{(0)}_{N,V,1}(\mathrm{Tr}(\mathcal{D}V)P)\,.$$

- The Dyson-Schwinger equations have a unique solution when V is small.
- Proof: Inversion of a particular “master operator”.

Series of maps

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Proposition

Assuming the coefficients of V are small enough, we have for all n.c. polynomial P:

$$\frac{1}{N}\mathbb{E}_{N,V}\left[\mathrm{Tr}\, P\right] - \mathcal{M}^{(0)}_{N,V,1}(P) = \mathcal{O}\left(\frac{1}{N^2}\right).$$

Ending the proof

- Higher order Dyson-Schwinger equations are satisfied by higher cumulants and generating series of maps of higher genus
- The error between the two can be shown to be small using the higher DS equations.

Monotone Hurwitz numbers

Large N expansion of the Weingarten function

Theorem (Novak '10)

Let $m \leq N$. We have for any $\pi \in \mathfrak{S}_m$:

$$\mathrm{Wg}_N(\pi) = \frac{1}{N^{m-\#\pi}} \sum_{r \geq 0} \frac{\overrightarrow{w}^r(\pi)}{N^r},$$

where

$$\overrightarrow{w}^r(\pi) = \#\left\{ (\tau_1, \dots, \tau_r) \in \mathfrak{S}_m^r : \begin{array}{l} \tau_i = (a_i\, b_i), a_i < b_i \\ \tau_r \cdots \tau_1 = \pi \\ b_1 \leq b_2 \leq \cdots \leq b_r \end{array} \right\}.$$

See also Goulden, Guay-Paquet, Novak '14

Triple monotone Hurwitz numbers

- Given a permutation $\sigma \in \mathfrak{S}_m$ write

$$\#\sigma = \#\{\text{ cycles of } \sigma\},$$

and

$$\mathrm{tr}_\sigma(\boldsymbol{A}) = \frac{1}{N^{\#\sigma}} \sum_{i\colon [m]\to[N]} \prod_{p=1}^{m} (A_p)_{i(p),i(\sigma(p))}.$$

e.g. if $\sigma = (1\,3\,4)(2\,5)$

$$\mathrm{tr}_\sigma(\boldsymbol{A}) = \frac{1}{N}\,\mathrm{Tr}\,(A_1A_3A_4) \times \frac{1}{N}\,\mathrm{Tr}\,(A_2A_5)\,.$$

- A monomial is balanced if it can be written as

$$P = A_1UB_1U^*A_2UB_2U^* \cdots A_dUB_dU^*.$$

Triple monotone Hurwitz numbers

Proposition

Let $P_1, \dots, P_l$ be balanced monomials and let m be the total number of letter U and U^* in the P_i's. We have under the Haar measure:

$$\kappa_l(\operatorname{Tr} P_1, \dots, \operatorname{Tr} P_l) \\ = N^{2-l} \sum_{g \geq 0} \frac{1}{N^{2g}} \sum_{\rho, \sigma \in \mathfrak{S}_m} (-1)^{\#\sigma + \#\rho} \operatorname{tr}_\rho(\boldsymbol{A}) \operatorname{tr}_\sigma(\boldsymbol{B}) \overrightarrow{h}_g(\rho, \gamma, \sigma),$$

where

$$\overrightarrow{h}_g(\rho, \gamma, \sigma) = \left\{ (\tau_1, \dots, \tau_r) \in \mathfrak{S}_m^r : \begin{array}{l} \tau_i = (a_i\, b_i), a_i < b_i \\ \tau_r \cdots \tau_1 = \pi \\ b_1 \leq b_2 \leq \cdots \leq b_r \\ \textit{the group generated by } \rho, \gamma, \sigma \\ \textit{ acts transitively on } \{1, \dots, m\} \end{array} \right\}$$

- Obtain a topological expansion of unitary integrals in the *perturbative regime.*
- Necessary bound on V is uniform in the order of expansion.
- Description of the expansion in terms of a family of maps
- These maps generalize the monotone Hurwitz numbers.
- Example of the link between RMT and the combinatorics of maps.

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Thank you!