



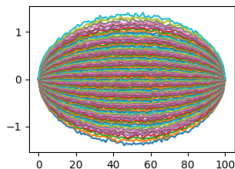
Asymptotics for deformed unitary ensembles associated to higher order Painlevé equations

Carla - joint work with Mattia Cafasso

University of São Paulo - Brazil

(Based on Arxiv: 2504.20721)
PLICQ Seminars - 30th June 2025

1 Applications



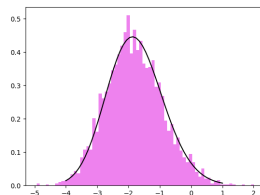
2 RMT

- RHP
- Example
- Deformed kernels
 - Examples of deformations



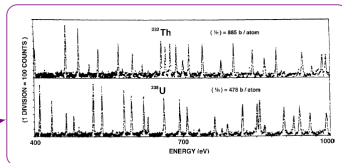
3 Our results

- Integrable systems
- Asymptotic results



Conjecture (Carla's conjecture)

Everything is at most seven steps (definitions) away from Random Matrix Theory.



Energies
distributed as
eigenvalues of RM.

Wigner
(1950)

Songs

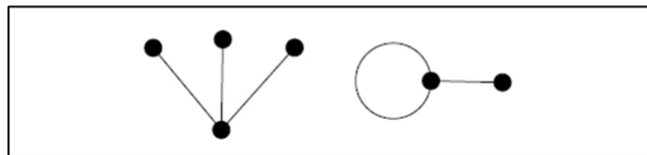
Dm Gm
 You went back to what you know
 Bb
 So far removed
 A7
 From all that we went through
 Dm Gm
 And I tread a troubled track
 Bb
 My odds are stacked
 A7
 I'll go back to black



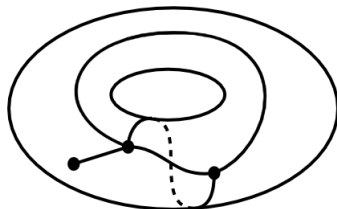
Figure: Back to Black - Amy Winehouse

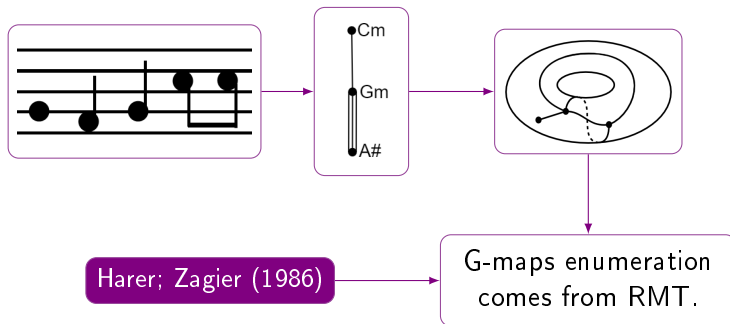


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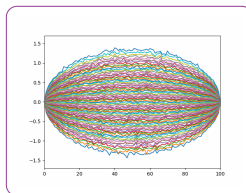


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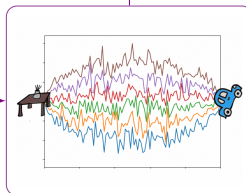


Happy Hour



Behaves as
eigenvalues
of RM.

Dyson (1962)



In the natural habitat of mathematicians...



Their intersection: coffee stains.



Random matrices could be
anywhere. Maybe in your
house. In your lab.
Maybe right here. Right now.
Look behind you...

1 Applications

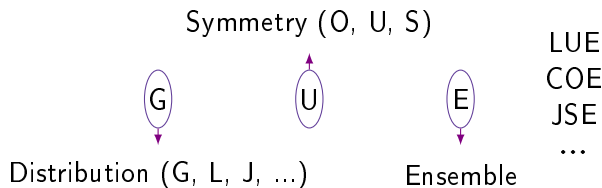
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3 Our results

Random Matrix Theory

Anatomy of the Random Matrix Ensembles



- **Weyl formulas:** allows us to recover the distribution in the space of eigenvalues from the distribution in the space of matrices;
- Eigenvalues can be connected to stochastic models (DPP, Pfaffian);

Formally...

- Start with $\mathcal{H}_n$ hermitian matrices $(\{\lambda_j\} \subset \mathbb{R})$ and

$$\mu(H) dH := \frac{1}{Z_\mu} e^{-\text{tr } V(H)} dH.$$

- In unitary ensembles, the eigenvalues form a DPP with m-th correlation point function

$$\rho_m(x_1, \dots, x_m) = \det[K_m(x_i, x_j)]_{i,j=1}^m.$$

Examples:

- $\mathbb{E}[\#\{\lambda_j\} \cap (a, b)] = \int_a^b K(\lambda, \lambda) d\lambda.$
- Gap probability

$$\mathbb{E}\left[\prod (1 - \chi_{(s, \infty]}(\lambda_j))\right] = \det(I - K).$$

- Multiplicative statistics

$$\mathbb{E}\left[\prod (1 - \sigma(\lambda_j))\right] = \det(I - K_\sigma).$$

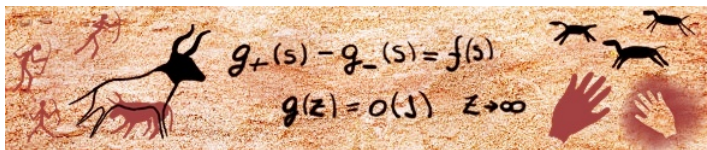
A little bit of RHP ♥

- Cauchy transform

$$Cf(z) := \frac{1}{2\pi i} \int_{\gamma} \frac{f(s)}{s-z} ds \quad z \in \gamma.$$

- Plemelj formula

$$C_+f(s) - C_-f(s) = f(s).$$



- For a complicated RHP ψ :

$$\psi(z) = g(z) \left(1 + \underbrace{o(1)}_{\sum_s \frac{g_s}{n^s}} \right)$$

GUE

- Matrix with entries $\{h_{jk}\}_{j,k=1}^n$ such that

$$\frac{h_{jk}}{\sqrt{n}} \approx \mathcal{N}(0, 1) + i(1 - \delta_{jk})\mathcal{N}(0, 1).$$

- It is equivalent to $\mathcal{H}_n$ under the probability density

$$\mu(H) dH := \frac{1}{Z_n} e^{-\frac{1}{2} \text{tr}(H^2)} dH.$$

- Eigenvalues form DPP with Christoffel-Darboux kernel;

$$K_n(x, y) = \sqrt{w(x)} \sqrt{w(y)} \sum_{j=0}^{n-1} p_j(x) p_j(y)$$

$$p(x) = a_n x^n + \dots + a_0$$

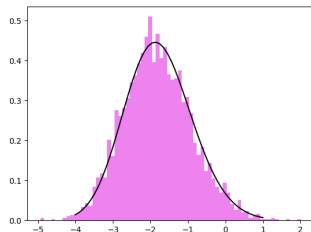
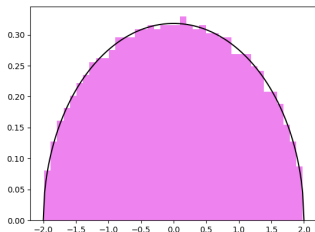
$$q(x) = b_m x^m + \dots + b_0$$


- Global statistics: **Wigner Semicircle law**.
- Local statistics (OP's characterized by RHP)

$$K^{Ai} = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x - y} \quad K^S = \frac{\sin[\pi(x - y)]}{\pi(x - y)}$$

- Gap probability

$$F(s) = \mathbb{E}\left[\prod (1 - \chi(\lambda_j))\right] = \det[I - K^{Ai}]$$



RHP for OPs

The 2×2 matrix-valued function

$$Y(z) = \begin{pmatrix} p_n(z)/\gamma_n & * \\ -2\pi i \gamma_{n-1} p_{n-1}(z) & * \end{pmatrix},$$

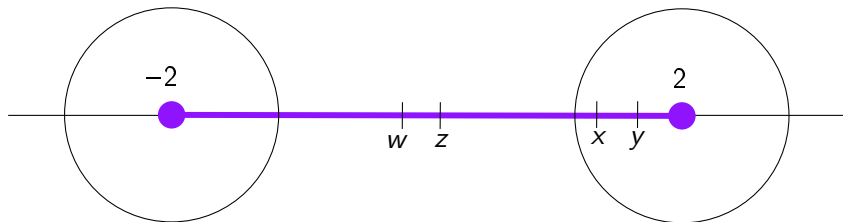
solves

- ① $Y(z)$ is analytic on $\mathbb{C}/\mathbb{R}$ with jump

$$Y_+(s) = Y_-(s) (I + w(s)E_{12}).$$

- ② As $z \rightarrow \infty$,

$$Y(z) = (I + O(z^{-1}))z^{n\sigma_3}.$$



Deformed Kernels

- Let $\sigma(x) : \mathbb{R} \rightarrow [0, 1]$ and consider a realization of a Point Process $\mathcal{X} = \{x_j\} \subset \mathbb{R}$;



- Thinned Process $\tilde{\mathcal{X}}$:** Get rid of x_j with probability $1 - \sigma(x_j)$;
- Multiplicative statistics

$$\mathbb{E} \left[\prod_{x_j \in \mathcal{X}} (1 - \sigma(x_j)) \right] = \det(I - K_\sigma),$$

where

$$K_\sigma(x, y) = \sqrt{\sigma(x)} K(x, y) \sqrt{\sigma(y)}.$$

Why people care about deformations?

It unlocks new universality limits.

- Its, Kuijlaars and Östensson (2008) define distribution on $\mathcal{H}_n$

$$p(H) dH = \frac{1}{Z_{n,N}} |\det H|^{2\alpha} e^{-N \operatorname{Tr} V(H)} dH.$$

$$K^{Ai}(x, y), \quad \sigma(x; \alpha) = |x|^\alpha.$$

- New universality law for K^α around the origin;

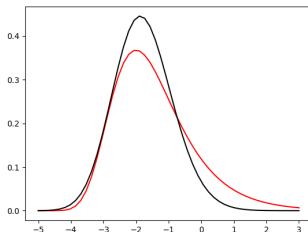
$$K^\alpha(x, y; s) = \frac{\varphi_1(x)\varphi_2(y) - \varphi_1(y)\varphi_2(x)}{2\pi i(x - y)}.$$

It allows us to interpolate between distributions.

- Bettea, Bouttier 2019,

$$K^{Ai}(x, y), \quad \sigma(x; \alpha) = \frac{e^{\alpha x}}{1 + e^{\alpha x}}.$$

- Interpolates between TW ($\alpha \rightarrow \infty$) and Gumbel ($\alpha \rightarrow 0$)

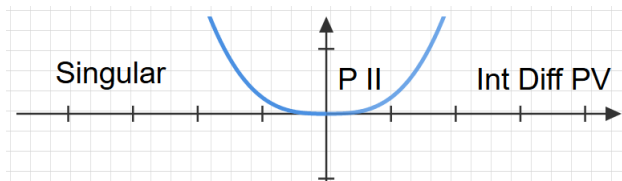


It allows us to “play” with the Point Process.

- Cafasso, Clayes, Ruzza 2021,

$$K^{Ai}(x, y), \quad \sigma\left(\frac{x}{t^{2/3}} + \frac{s}{t}\right).$$

- Recovers three regimes of asymptotic behavior;



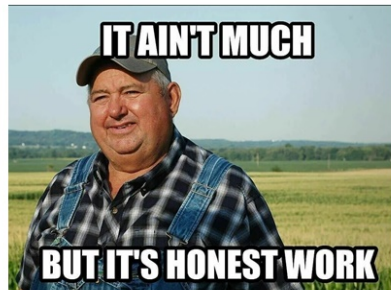
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The deformed Painlevé I kernel

- Painlevé I kernel:

$$K^{PI}(x, y; t_0, t_1) = \frac{\phi_1(x)\phi_2(y) - \phi_1(y)\phi_2(x)}{-2\pi i(x - y)},$$

where ϕ_1, ϕ_2 are solutions to the $P_I^{(2)}$.

- PI hierarchy:

$$1. \ddot{y} = y^2 + t$$

$$2. \dddot{y} = -40\dot{y}^2 - 80y\ddot{y} - 640y^3 - 256yt_1 + 256t_0$$

$$3. \overset{'''}{y} = \dots$$

- Deformed kernel with stretching and scale parameters in the deformation

$$K_{\sigma}(x, y) = \sqrt{\left(\frac{x}{s_0^{2/7}} - \frac{s_1}{s_0}\right)} K(x, y; t_0, t_1) \sqrt{\sigma\left(\frac{y}{s_0^{2/7}} - \frac{s_1}{s_0}\right)},$$

$s_1, t_0, t_1 \in \mathbb{R}$ and $s_0 > 0$, $\sigma : \mathbb{R} \rightarrow [0, 1]$ to be non-decreasing, possibly discontinuous at $r_1, \dots, r_j$.

- Goals:
 - 1 Study the underlying **Integrable Systems**;
 - 2 Connect these systems to multiplicative statistics

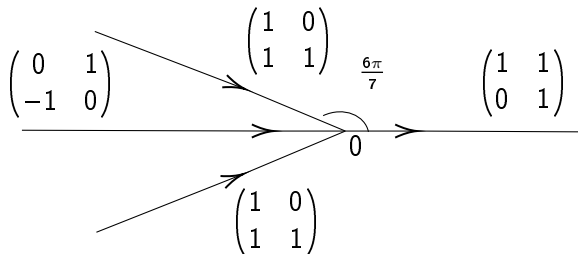
$$Q_{\sigma} := \mathbb{E}_{PI} \left[\prod_{j=1}^{\infty} \left(1 - \sigma \left(\frac{\lambda_j}{s_0^{2/7}} - \frac{s_1}{s_0} \right) \right) \right] = \det(I - \mathbb{K}_{\sigma}^{PI});$$

- 3 Find asymptotics for such systems.

PI RHP

Let $\Phi(z)$ the 2×2 matrix solves the RHP (CIK2009):

- ① Φ is analytic on $\mathbb{C} \setminus \Sigma$ with jumps



- ② As $z \rightarrow \infty$,

$$\Phi(z) = z^{-\frac{\sigma_3}{4}} N \left(I + \frac{1}{z^{1/2}} \begin{pmatrix} -h & 0 \\ 0 & h \end{pmatrix} + \frac{1}{2z} \begin{pmatrix} h^2 & iy \\ -iy & h^2 \end{pmatrix} + \dots \right) e^{-\theta(z)\sigma_3},$$

$$\text{where } \theta(z) = \frac{2}{7} z^{7/2} + \frac{2t_1}{3} z^{3/2} - 2t_0 z^{1/2}.$$

- $\phi_1(z) := \Phi_{11}(z; t_0, t_1)$ and $\phi_2(z) := \Phi_{21}(z; t_0, t_1)$;
- Connection to P_I^2 :

$$\frac{\partial_{t_0}^4 y}{64} + \frac{5}{8} \left((\partial_{t_0} y)^2 + 2y \partial_{t_0}^2 y \right) + 10y^3 + 4yt_1 - 4t_0 = 0.$$

- The deformed kernel can be expressed as

$$K_{\sigma}^{PI}(u, v) = \frac{f^T(u)g(v)}{u - v}$$

$$f(u) = \frac{\sqrt{\sigma(\cdot)}}{\sqrt{\pi i}} \begin{pmatrix} \phi_1(u) \\ \phi_2(u) \end{pmatrix}, \quad g(u) = \frac{\sqrt{\sigma(\cdot)}}{2\sqrt{\pi i}} \begin{pmatrix} -\phi_2(u) \\ \phi_1(u) \end{pmatrix}.$$

IIS RHP

- IIS Theory: let $Y(z)$ be the 2×2 matrix-valued function

$$Y(z) = I - \int_{\mathbb{R}} \frac{(1 - K_{\sigma}^{PI})^{-1} f(u) g^T(u)}{u - z} du,$$

solves the following RH problem

- 1 Y is analytic on $\mathbb{C} \setminus \mathbb{R} \cup \{r_j\}$,

$$Y_+(z) = Y_-(z)[I - 2\pi i f(z) g^T(z)].$$

- 2 As $z \rightarrow \infty$,

$$Y(z) = I + O(z^{-1}).$$

- Consider $\zeta = z/s_0^{2/7} - s_1/s_0$ and set

$$X(\zeta) = A s_0^{\frac{\sigma_3}{14}} Y(z(\zeta)) \Phi(z(\zeta)).$$

Then $X(\zeta)$ solves the following RH problem

- $X(\zeta)$ is analytic on $\mathbb{C} \setminus \mathbb{R} \cup \{r_j\}$,

$$X_+(\zeta) = X_-(\zeta) \begin{pmatrix} 1 & 1 - \sigma(\zeta) \\ 0 & 1 \end{pmatrix}.$$

- As $\zeta \rightarrow \infty$,

$$X(\zeta) = A \zeta^{-\frac{\sigma_3}{4}} N \left[I + \sum_{j \geq 1} \frac{X^{(j)}}{\zeta^{j/2}} \right] e^{-\left(\tau_7 \zeta^{\frac{7}{2}} + \tau_5 \zeta^{\frac{5}{2}} + \tau_3 \zeta^{\frac{3}{2}} + \tau_1 \zeta^{\frac{1}{2}} \right) \sigma_3} \\ \times \begin{cases} I & |\arg \zeta| < \pi - \epsilon \\ \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} & \pi - \epsilon < \pm \arg \zeta < \pi \end{cases}.$$

- The τ_j can be explicitly expressed by means of $\{t_1, t_0, s_1, s_0\}$

$$\tau_1 = \frac{s_1 t_1}{s_0^{4/7}} - 2s_0^{1/7} t_0 + \frac{5s_1^3}{8s_0^2} \quad \tau_5 = s_1$$

$$\tau_3 = \frac{2s_0^{3/7} t_1}{3} + \frac{5s_1^2}{4s_0} \quad \tau_7 = \frac{2s_0}{7}.$$

- The problem depends analytically on $\{\tau_j\}$, so that it satisfies Lax equations

$$\frac{\partial}{\partial \tau_j} X = B_j X,$$

KdV connection

Theorem 1

Let $u = X_{11}^{(1)} + iX_{12}^{(1)}$. The function $v(\tau_1, \tau_3, \tau_5, \tau_7) := \partial_{\tau_1} u$, satisfies the first three terms in the KdV hierarchy,

$$\frac{\partial}{\partial \tau_3} v = \frac{1}{4} v''' + 3vv'$$

$$\frac{\partial}{\partial \tau_5} v = \frac{1}{16} v'''' + \frac{5}{2} v' v'' + \frac{5}{4} vv''' + \frac{15}{2} v^2 v'.$$

$$\begin{aligned} \frac{\partial}{\partial \tau_7} v = & \frac{35}{2} v^3 v' + \frac{35}{8} v^2 v''' + \frac{35}{2} vv' v'' + \frac{7}{16} vv'''' \\ & + \frac{35}{8} (v')^3 + \frac{21}{16} v' v'''' + \frac{35}{16} v'' v''' + \frac{1}{64} v'''''' \end{aligned}$$

Moreover, $u(\tau_1, \tau_3, \tau_5, \tau_7)$, solves the first three homogeneous potential KdV equations.

Step 1 : For $j \in \{1, 3, 5, 7\}$, use the asymptotic condition for X to obtain the matrices B_j

$$B_j := \frac{\partial}{\partial \tau_j} X X^{-1}.$$

Step 2 Apply the compatibility condition

$$\partial_{\tau_j} B_k - \partial_{\tau_k} B_j = [B_j, B_k].$$

Step 3 : Combine the equations in order to simplify the expressions for the entries of the B_j and obtain KdVs;

Theorem 2

For any $\tau_1, \tau_3, \tau_5 \in \mathbb{R}$ and $\tau_7 > 0$ it holds that

$$\frac{\partial}{\partial \tau_1} \log Q_\sigma = u + \left(\frac{2}{7}\right)^{1/7} \frac{h}{\tau_7^{1/7}} - \frac{3}{98} \frac{\tau_3 \tau_5^2}{\tau_7^2} + \frac{15}{2744} \frac{\tau_5^4}{\tau_7^3} + \frac{1}{7} \frac{\tau_1 \tau_5}{\tau_7}. \quad (1)$$

$$\tau_5 / s_0^{5/7} \rightarrow \infty$$

By the Small Norm Theorem, it follows that

$$Y(z) = I + O(e^{-k\tau_5/s_0^{5/7}}).$$

Lemma 3

Let $\tau_5 \geq M\tau_7^{5/7}$ and $\tau_3 \leq \frac{15\tau_5}{28s_0} + \tau_1 \frac{s_0}{\tau_5}$. Then, there exist a real constant $k > 0$ such that

$$u = - \left(\frac{2}{7\tau_7} \right)^{1/7} h - \left(\frac{\tau_1\tau_5}{7\tau_7} - \frac{3\tau_3\tau_5^2}{98\tau_7^2} + \frac{15\tau_5^4}{2744\tau_7^3} \right) + O(e^{-k\tau_5/\tau_7^{5/7}}),$$

$$v = \left(\frac{2}{7\tau_7} \right)^{2/7} y - \frac{\tau_5}{7\tau_7} + O(e^{-k\tau_5/\tau_7^{5/7}}),$$

as $\tau_7 \rightarrow 0$, where y solves the $P_I^{(2)}$ equation and h is such that $h_{t_0} = 2y$.

$$\tau_5 / s_0^{5/7} = O(1)$$

We start with $X(\zeta)$ and change variables

$$\xi = s_0^{2/7} \left(\frac{\zeta}{s_0^{2/7}} - \frac{\tau_5}{s_0} \right).$$

- ❶ $Z(\xi)$ is analytic on $\mathbb{C} \setminus \mathbb{R}$.
- ❷ $Z(\xi)$ satisfies the following jump relations

$$Z_+(\xi) = Z_-(\xi) \begin{pmatrix} 1 & 1 - \sigma(s_0^{-2/7} \xi) \\ 0 & 1 \end{pmatrix}.$$

- ❸ As $\xi \rightarrow \infty$,

$$Z(\xi) = \xi^{-\frac{\sigma_3}{4}} N \left[I + \sum_{j \geq 1} \frac{Z^{(j)}}{\xi^{j/2}} \right] e^{-\left(\frac{2}{7} \xi^{\frac{7}{2}} + \frac{\tau_5}{s_0^{5/7}} \xi^{\frac{5}{2}} + \frac{\tau_3}{s_0^{3/7}} \xi^{\frac{3}{2}} + \frac{\tau_1}{s_0^{1/7}} \xi^{\frac{1}{2}} \right) \sigma_3}$$

- Suppose $\sigma(z) \rightarrow \iota$ as $z \rightarrow \infty$. Then $Z(\xi) \sim \tilde{Z}(\xi)$, where

$$\tilde{Z}_+(\xi) = \tilde{Z}_-(\xi) \begin{pmatrix} 1 & 1 - \iota \chi_{(0,\infty)} \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R}.$$

Lemma 4

Let $|\tau_3| \leq C_1 s_0^{3/7}$, $|\tau_1| \leq C_2 s_0^{1/7}$ and $|\tau_5| \leq C_3 \tau_7^{5/7}$ for real positive constants C_1, C_2, C_3 . Then, as $\tau_7 \rightarrow 0$, it follows that

$$u = \left(\frac{2}{7\tau_7} \right)^{1/7} \frac{1}{c^{1/2}} \left(\frac{q}{2} + p \right) + O(\tau_7^{1/7})$$

$$v = \left(\frac{2}{7\tau_7} \right)^{1/7} \frac{1}{c^{1/2}} \left(\frac{\partial}{\partial \tau_1} \frac{q}{2} - \left(\frac{2}{7\tau_7} \right)^{1/7} \frac{1}{c^{1/2}} \frac{q^2}{2} \right) + O(\tau_7^{1/7}),$$

where $c = 2^{-10/7}$ and q is the solution to the $P_{II}^{(3)}$.

$$\tau_5/s_0^{5/7} \rightarrow -\infty$$

- Suppose $\tau_5/s_0^{5/7} \rightarrow -\infty$ no faster than $s_0^{-1/7}$ as $s_0 \rightarrow 0$ and set

$$\eta = -\frac{2}{7}\xi^3 s_0^{1/7} - \frac{\tau_5}{s_0^{4/7}}\xi^2 - \frac{\tau_3}{s_0^{2/7}}\xi - \tau_1.$$

- Recall that as $\xi \rightarrow \infty$

$$Z(\xi) = \xi^{-\frac{\sigma_3}{4}} N \left[I + \sum_{j \geq 1} \frac{Z^{(j)}}{\xi^{j/2}} \right] e^{-\left(\frac{2}{7}\xi^{\frac{7}{2}} + \frac{\tau_5}{s_0^{5/7}}\xi^{\frac{5}{2}} + \frac{\tau_3}{s_0^{3/7}}\xi^{\frac{3}{2}} + \frac{\tau_1}{s_0^{1/7}}\xi^{\frac{1}{2}} \right) \sigma_3}.$$

- Then

$$-\left(\frac{2}{7}\xi^{\frac{7}{2}} + \frac{\tau_5}{s_0^{5/7}}\xi^{\frac{5}{2}} + \frac{\tau_3}{s_0^{3/7}}\xi^{\frac{3}{2}} + \frac{\tau_1}{s_0^{1/7}}\xi^{\frac{1}{2}} \right) = (s_0^{-2/7}\xi)^{1/2}\eta.$$

- ① $H_\sigma(\zeta)$ is analytic on $\mathbb{C} \setminus \mathbb{R}$, and

$$H_{\sigma,+}(\zeta) = H_{\sigma,-}(\zeta) \begin{pmatrix} 1 & 1 - \sigma(\zeta) \\ 0 & 1 \end{pmatrix}, \quad \zeta \in \mathbb{R}$$

- ② As $\zeta \rightarrow \infty$,

$$H_\sigma(\zeta) = \left[I + \frac{1}{\zeta} \begin{pmatrix} q_\sigma & p_\sigma \\ r_\sigma & -q_\sigma \end{pmatrix} + O(\zeta^{-2}) \right] \zeta^{-\frac{\sigma_3}{4}} N e^{\zeta^{1/2} \eta \sigma_3}.$$

Same RHP as the one studied in Clayes, Cafasso, Ruzza 2021!

Lemma 5

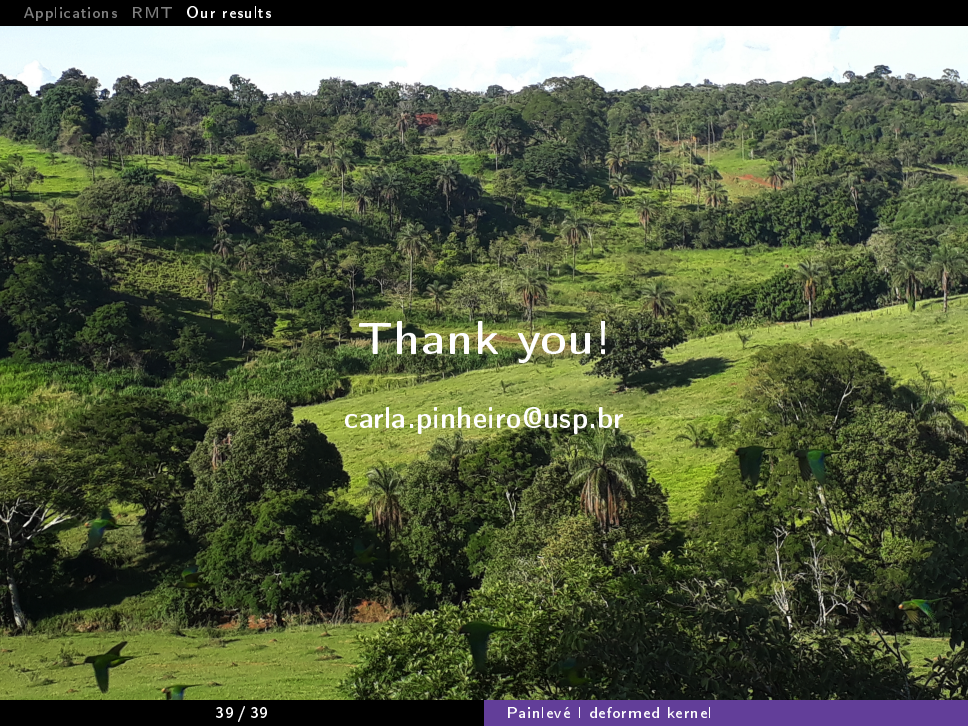
Let $-M_1\tau_7^{4/7} \leq \tau_5 \leq -M_2\tau_7^{5/7}$, $|\tau_3| \leq M_3\tau_7^{2/7}$, $-M_4 \leq \tau_1 \leq x$ for $x = 2(-2^{6/7}\tau_7^{1/7}/7^{6/7} + \tau_5(7\tau_7/2)^{-4/7} - |\tau_3|(7\tau_7/2)^{-2/7})$ and M_1, M_2, M_3, M_4 real positive constants. As $\tau_7 \rightarrow 0$ it follows that

$$u = p_\sigma(-\tau_1) + O(x),$$

$$v = p_\sigma^2(-\tau_1) - 2q_\sigma(\tau_1) + O(x).$$

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Thank you!

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