Gaudin model, oscillating Young tableaux and cactus group

Anfisa Gurenkova

HSE, Skoltech

Paris, 2026

Notation

- $\mathfrak{g}$ – simple Lie algebra,
- $\Omega = \sum_a x_a \otimes x^a \in \mathfrak{g} \otimes \mathfrak{g}$ – Casimir element,
- $\Omega_{ij} = \sum_a 1 \otimes \cdots \otimes \underset{i}{x_a} \otimes \cdots \otimes \underset{j}{x^a} \otimes \cdots \otimes 1 \in U\mathfrak{g}^{\otimes n}$ – split Casimir element, living in i, j-th tensor factors,
- V_λ – finite-dimensional irreducible representation of highest weight λ.

Gaudin model

Gaudin Hamiltonians:

$$H_i(\underline{z}) = \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \in U\mathfrak{g}^{\otimes n},$$

where $\underline{z} \in \mathbb{C}^n, z_i \neq z_j$.

Gaudin model

Gaudin Hamiltonians:

$$H_i(\underline{z}) = \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \in U\mathfrak{g}^{\otimes n},$$

where $\underline{z} \in \mathbb{C}^n, z_i \neq z_j$.

- H_i pairwise commute and commute with diagonal $\mathfrak{g}$

Gaudin model

Gaudin Hamiltonians:

$$H_i(\underline{z}) = \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \in U\mathfrak{g}^{\otimes n},$$

where $\underline{z} \in \mathbb{C}^n, z_i \neq z_j$.

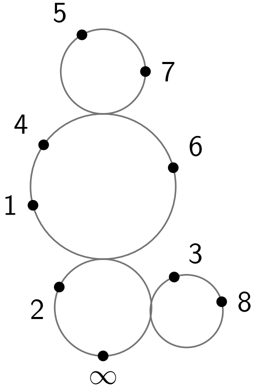
- H_i pairwise commute and commute with diagonal $\mathfrak{g}$
- Therefore $\{H_i(\underline{z})\}$ act in $\mathrm{Hom}(V_\mu, V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}) =: (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^\mu$

Deligne-Mumford space and its coverings

It is possible to extend the parameter space of Gaudin Hamiltonians to *Deligne-Mumford moduli space* $\overline{M}_{0,n+1}$.

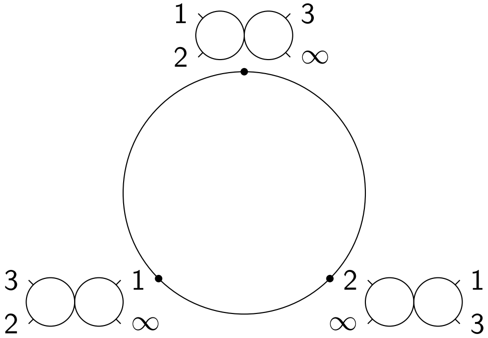
Deligne-Mumford space and its coverings

It is possible to extend the parameter space of Gaudin Hamiltonians to *Deligne-Mumford moduli space* $\overline{M}_{0,n+1}$.
Points in $\overline{M}_{0,n+1}$ look like a collection of $\mathbb{CP}^1$ which are attached to each other as cacti (cannot loop), with $n+1$ marked points.
Example (n=8):



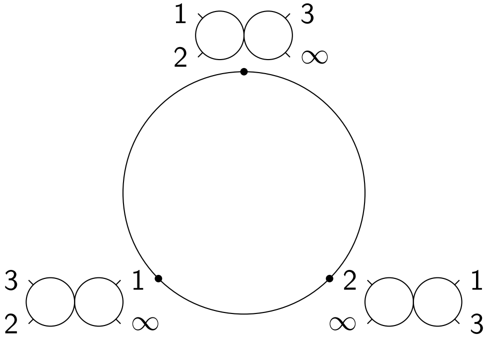
Deligne-Mumford space and its coverings

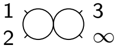
Example: $\overline{M}_{0,4} \cong \mathbb{CP}^1$:



Deligne-Mumford space and its coverings

Example: $\overline{M}_{0,4} \cong \mathbb{CP}^1$:

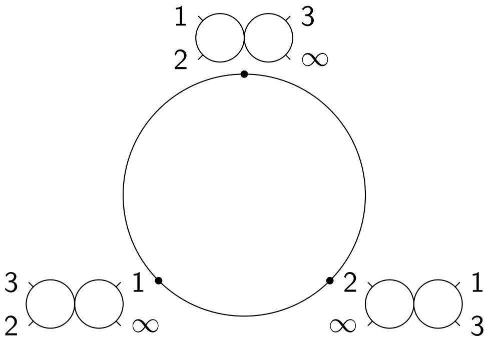


$H_3($ 

$) = \lim_{z_1 \to z_2}\left(\frac{\Omega_{13}}{z_3 - z_1} + \frac{\Omega_{23}}{z_3 - z_2}\right) = \frac{1}{z_3 - z_1}(\Omega_{13} + \Omega_{23})$

Deligne-Mumford space and its coverings

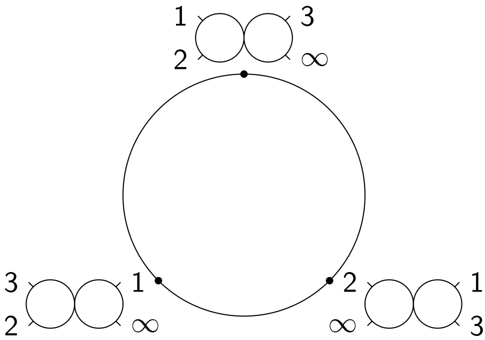
Example: $\overline{M}_{0,4} \cong \mathbb{CP}^1$:

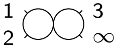


$$H_3\left({}^{1}_{2}\bigcirc\!\bigcirc{}^{3}_{\infty} \right) = \lim_{z_1 \to z_2}\left(\frac{\Omega_{13}}{z_3 - z_1} + \frac{\Omega_{23}}{z_3 - z_2}\right) = \frac{1}{z_3 - z_1}(\Omega_{13} + \Omega_{23})$$

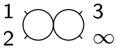
$$H_1\left({}^{1}_{2}\bigcirc\!\bigcirc{}^{3}_{\infty} \right) = \lim_{z_1 \to z_2}\left(\frac{\Omega_{12}}{z_1 - z_2} + \frac{\Omega_{13}}{z_1 - z_3}\right)$$

Example: $\overline{M}_{0,4} \cong \mathbb{CP}^1$:



$H_3($ 

$) = \lim_{z_1 \to z_2} \left(\frac{\Omega_{13}}{z_3 - z_1} + \frac{\Omega_{23}}{z_3 - z_2} \right) = \frac{1}{z_3 - z_1} (\Omega_{13} + \Omega_{23})$

$H_1($ 

$) = \lim_{z_1 \to z_2} \left(\frac{\Omega_{12}}{z_1 - z_2} + \frac{\Omega_{13}}{z_1 - z_3} \right) \equiv$

$\lim_{z_1 \to z_2} \left(\Omega_{12} + \frac{z_1 - z_2}{z_1 - z_3} \Omega_{13} \right) = \Omega_{12}$

Deligne-Mumford space and its coverings

In some cases $H_i(\underline{z})$ act in $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^\mu$ with simple joint spectrum for $\underline{z} \in \overline{M}_{0,n+1}(\mathbb{R})$.

Deligne-Mumford space and its coverings

In some cases $H_i(\underline{z})$ act in $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^\mu$ with simple joint spectrum for $\underline{z} \in \overline{M}_{0,n+1}(\mathbb{R})$.

$$\Downarrow$$

Unramified covering of eigenlines over $\overline{M}_{0,n+1}(\mathbb{R})$. Denoted by $\mathcal{E}^\mu_{\underline{\lambda}}$.

Deligne-Mumford space and its coverings

In some cases $H_i(\underline{z})$ act in $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^\mu$ with simple joint spectrum for $\underline{z} \in \overline{M}_{0,n+1}(\mathbb{R})$.

$$\Downarrow$$

Unramified covering of eigenlines over $\overline{M}_{0,n+1}(\mathbb{R})$. Denoted by $\mathcal{E}^\mu_{\underline{\lambda}}$.

Question

Describe the monodromy representation of this covering.

Deligne-Mumford space and its coverings

In some cases $H_i(\underline{z})$ act in $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^{\mu}$ with simple joint spectrum for $\underline{z} \in \overline{M}_{0,n+1}(\mathbb{R})$.

$$\Downarrow$$

Unramified covering of eigenlines over $\overline{M}_{0,n+1}(\mathbb{R})$. Denoted by $\mathcal{E}^{\mu}_{\underline{\lambda}}$.

Question

Describe the monodromy representation of this covering.

Remarks

- Taking **real** $\underline{z} \in \overline{M}_{0,n+1}(\mathbb{R})$ ensures that $H_i(\underline{z})$ are Hermitian, therefore diagonalizable.

Deligne-Mumford space and its coverings

In some cases $H_i(\underline{z})$ act in $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^\mu$ with simple joint spectrum for $\underline{z} \in \overline{M}_{0,n+1}(\mathbb{R})$.

$$\Downarrow$$

Unramified covering of eigenlines over $\overline{M}_{0,n+1}(\mathbb{R})$. Denoted by $\mathcal{E}^{\mu}_{\underline{\lambda}}$.

Question

Describe the monodromy representation of this covering.

Remarks

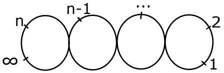
- Taking **real** $\underline{z} \in \overline{M}_{0,n+1}(\mathbb{R})$ ensures that $H_i(\underline{z})$ are Hermitian, therefore diagonalizable.
- In general one should consider higher Gaudin Hamiltonians to get simple joint spectrum.

Cactus group

S_n acts on $\overline{M}_{0,n+1}$ by permuting all points except ∞. The cactus group is $Cact_n := \pi_1^{S_n}(\overline{M}_{0,n+1}(\mathbb{R}))$.

Cactus group

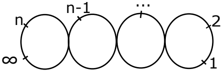
S_n acts on $\overline{M}_{0,n+1}$ by permuting all points except ∞. The cactus group is $Cact_n := \pi_1^{S_n}(\overline{M}_{0,n+1}(\mathbb{R}))$.

Take basepoints of the form 

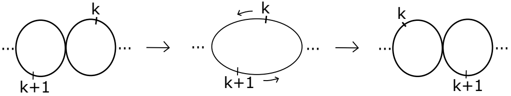
(caterpillar points).

Cactus group

S_n acts on $\overline{M}_{0,n+1}$ by permuting all points except ∞. The cactus group is $Cact_n := \pi_1^{S_n}(\overline{M}_{0,n+1}(\mathbb{R}))$.

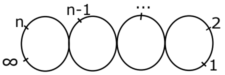
Take basepoints of the form 

(caterpillar points).

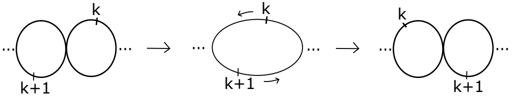
Paths t_k: 

Cactus group

S_n acts on $\overline{M}_{0,n+1}$ by permuting all points except ∞. The cactus group is $Cact_n := \pi_1^{S_n}(\overline{M}_{0,n+1}(\mathbb{R}))$.

Take basepoints of the form 

(caterpillar points).

Paths t_k: 

Proposition

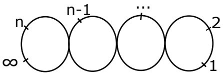
t_k generate $Cact_n$.

Gaudin algebra over caterpillar points

Suppose that $W \otimes V_{\lambda_i}$ is multiplicity-free for any irreducible W.

Gaudin algebra over caterpillar points

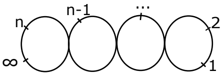
Suppose that $W \otimes V_{\lambda_i}$ is multiplicity-free for any irreducible W. Then the eigenbasis of Gaudin in $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^{\mu}$ over the point



has the following description.

Gaudin algebra over caterpillar points

Suppose that $W \otimes V_{\lambda_i}$ is multiplicity-free for any irreducible W. Then the eigenbasis of Gaudin in $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^{\mu}$ over the point

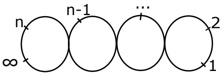


has the following description.

1. Decompose into irreducibles $V_{\lambda_1} \otimes V_{\lambda_2} = \bigoplus V_{\mu_2}$,

Gaudin algebra over caterpillar points

Suppose that $W \otimes V_{\lambda_i}$ is multiplicity-free for any irreducible W. Then the eigenbasis of Gaudin in $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^\mu$ over the point



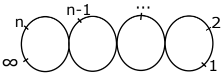
has the following description.

1. Decompose into irreducibles $V_{\lambda_1} \otimes V_{\lambda_2} = \bigoplus V_{\mu_2}$,
2. Decompose $V_{\mu_2} \otimes V_{\lambda_3} = \bigoplus V_{\mu_3}$ for each μ_2,

   . . .

Gaudin algebra over caterpillar points

Suppose that $W \otimes V_{\lambda_i}$ is multiplicity-free for any irreducible W.
Then the eigenbasis of Gaudin in $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^{\mu}$ over the point



has the following description.

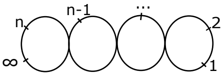
1. Decompose into irreducibles $V_{\lambda_1} \otimes V_{\lambda_2} = \bigoplus V_{\mu_2}$,
2. Decompose $V_{\mu_2} \otimes V_{\lambda_3} = \bigoplus V_{\mu_3}$ for each μ_2,

   ...

The eigenbasis is encoded by the sequences $(\lambda_1, \mu_2, \ldots, \mu_{n-1}, \mu)$.

Gaudin algebra over caterpillar points

Suppose that $W \otimes V_{\lambda_i}$ is multiplicity-free for any irreducible W. Then the eigenbasis of Gaudin in $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^{\mu}$ over the point



has the following description.

1. Decompose into irreducibles $V_{\lambda_1} \otimes V_{\lambda_2} = \bigoplus V_{\mu_2}$,
2. Decompose $V_{\mu_2} \otimes V_{\lambda_3} = \bigoplus V_{\mu_3}$ for each μ_2,

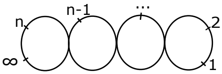
   ...

The eigenbasis is encoded by the sequences $(\lambda_1, \mu_2, \ldots, \mu_{n-1}, \mu)$.

Example: $\mathfrak{g} = \mathfrak{sp}_{2n}$. Highest weights are Young diagrams of length $\leq n$.

Gaudin algebra over caterpillar points

Suppose that $W \otimes V_{\lambda_i}$ is multiplicity-free for any irreducible W. Then the eigenbasis of Gaudin in $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^{\mu}$ over the point



has the following description.

1. Decompose into irreducibles $V_{\lambda_1} \otimes V_{\lambda_2} = \bigoplus V_{\mu_2}$,
2. Decompose $V_{\mu_2} \otimes V_{\lambda_3} = \bigoplus V_{\mu_3}$ for each μ_2,

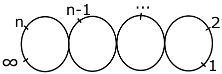
   ...

The eigenbasis is encoded by the sequences $(\lambda_1, \mu_2, \ldots, \mu_{n-1}, \mu)$.

Example: $\mathfrak{g} = \mathfrak{sp}_{2n}$. Highest weights are Young diagrams of length $\leq n$. Take $V_{\lambda_i} = V = \mathbb{C}^{2n}$, i.e. $\lambda_i = \square$.

Gaudin algebra over caterpillar points

Suppose that $W \otimes V_{\lambda_i}$ is multiplicity-free for any irreducible W. Then the eigenbasis of Gaudin in $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^{\mu}$ over the point



has the following description.

1. Decompose into irreducibles $V_{\lambda_1} \otimes V_{\lambda_2} = \bigoplus V_{\mu_2}$,
2. Decompose $V_{\mu_2} \otimes V_{\lambda_3} = \bigoplus V_{\mu_3}$ for each μ_2,

   ...

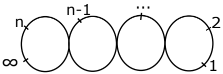
The eigenbasis is encoded by the sequences $(\lambda_1, \mu_2, \ldots, \mu_{n-1}, \mu)$.

Example: $\mathfrak{g} = \mathfrak{sp}_{2n}$. Highest weights are Young diagrams of length $\leq n$. Take $V_{\lambda_i} = V = \mathbb{C}^{2n}$, i.e. $\lambda_i = \square$.

Tensor product rule: $V_\lambda \otimes V = \bigoplus\limits_{\mu = \lambda \pm \square} V_\mu$.

Gaudin algebra over caterpillar points

Suppose that $W \otimes V_{\lambda_i}$ is multiplicity-free for any irreducible W.
Then the eigenbasis of Gaudin in $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^{\mu}$ over the point



has the following description.

1. Decompose into irreducibles $V_{\lambda_1} \otimes V_{\lambda_2} = \bigoplus V_{\mu_2}$,
2. Decompose $V_{\mu_2} \otimes V_{\lambda_3} = \bigoplus V_{\mu_3}$ for each μ_2,

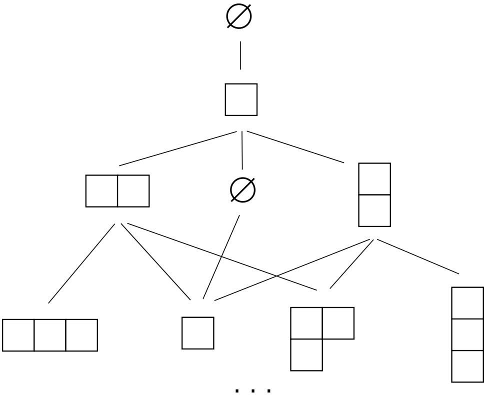
   ...

The eigenbasis is encoded by the sequences $(\lambda_1, \mu_2, \ldots, \mu_{n-1}, \mu)$.

Example: $\mathfrak{g} = \mathfrak{sp}_{2n}$. Highest weights are Young diagrams of length $\leq n$. Take $V_{\lambda_i} = V = \mathbb{C}^{2n}$, i.e. $\lambda_i = \square$.
Tensor product rule: $V_\lambda \otimes V = \bigoplus_{\mu = \lambda \pm \square} V_\mu$.

Basis $\leftrightarrow$ oscillating tableaux.

Oscillating tableaux



Results

Let $\mathfrak{g} = \mathfrak{sp}_{2n}$, $V_{\lambda_i} = V = \mathbb{C}^{2n}$.

Results

Let $\mathfrak{g} = \mathfrak{sp}_{2n}$, $V_{\lambda_i} = V = \mathbb{C}^{2n}$.

- The monodromy along t_k acts locally: it does not change μ_j for $j \neq k$, and depends only on μ_{k-1} and μ_{k+1}.

Results

Let $\mathfrak{g} = \mathfrak{sp}_{2n}$, $V_{\lambda_i} = V = \mathbb{C}^{2n}$.

- The monodromy along t_k acts locally: it does not change μ_j for $j \neq k$, and depends only on μ_{k-1} and μ_{k+1}.
- If $\mu_{k-1} \neq \mu_{k+1}$, there are at most 2 possibilities for μ_k. If there are 2, t_k exchanges them.

Results

Let $\mathfrak{g} = \mathfrak{sp}_{2n}$, $V_{\lambda_i} = V = \mathbb{C}^{2n}$.

- The monodromy along t_k acts locally: it does not change μ_j for $j \neq k$, and depends only on μ_{k-1} and μ_{k+1}.
- If $\mu_{k-1} \neq \mu_{k+1}$, there are at most 2 possibilities for μ_k. If there are 2, t_k exchanges them.
- The action preserves the standard tableaux. The restriction of t_k on SYT is a special case of the **Bender-Knuth involution**.

Bender-Knuth involutions on oscillating tableaux

If $\mu_{k-1} = \mu_{k+1}$, we order the tableaux as follows.

Bender-Knuth involutions on oscillating tableaux

If $\mu_{k-1} = \mu_{k+1}$, we order the tableaux as follows.

- If $\mu_k = \mu_{k-1} + \square$ and $\mu'_k = \mu_{k-1} - \square$, then $\mu_k > \mu'_k$,

Bender-Knuth involutions on oscillating tableaux

If $\mu_{k-1} = \mu_{k+1}$, we order the tableaux as follows.

- If $\mu_k = \mu_{k-1} + \Box$ and $\mu'_k = \mu_{k-1} - \Box$, then $\mu_k > \mu'_k$,
- The set of $\mu_k = \mu_{k-1} + \Box$ is ordered by the inverse number of row of the added box (upper row is the greatest),

Bender-Knuth involutions on oscillating tableaux

If $\mu_{k-1} = \mu_{k+1}$, we order the tableaux as follows.

- If $\mu_k = \mu_{k-1} + \square$ and $\mu'_k = \mu_{k-1} - \square$, then $\mu_k > \mu'_k$,
- The set of $\mu_k = \mu_{k-1} + \square$ is ordered by the inverse number of row of the added box (upper row is the greatest),
- The set of $\mu_k = \mu_{k-1} - \square$ is ordered by the number of row of the deleted box.

Bender-Knuth involutions on oscillating tableaux

If $\mu_{k-1} = \mu_{k+1}$, we order the tableaux as follows.

- If $\mu_k = \mu_{k-1} + \square$ and $\mu'_k = \mu_{k-1} - \square$, then $\mu_k > \mu'_k$,
- The set of $\mu_k = \mu_{k-1} + \square$ is ordered by the inverse number of row of the added box (upper row is the greatest),
- The set of $\mu_k = \mu_{k-1} - \square$ is ordered by the number of row of the deleted box.

The monodromy along t_k reverses this order.

Bender-Knuth involutions on oscillating tableaux

If $\mu_{k-1} = \mu_{k+1}$, we order the tableaux as follows.

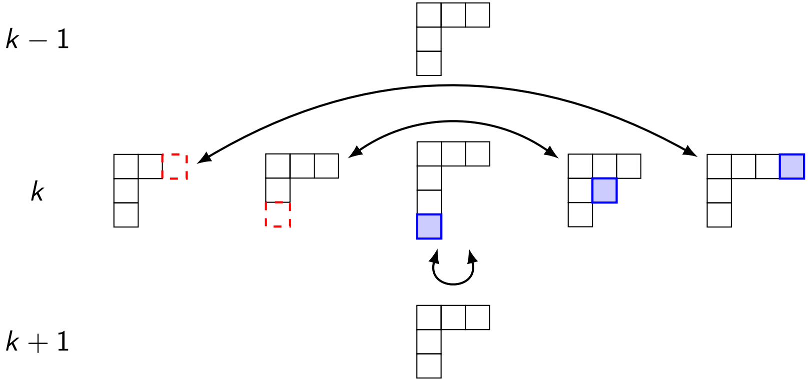
- If $\mu_k = \mu_{k-1} + \square$ and $\mu'_k = \mu_{k-1} - \square$, then $\mu_k > \mu'_k$,
- The set of $\mu_k = \mu_{k-1} + \square$ is ordered by the inverse number of row of the added box (upper row is the greatest),
- The set of $\mu_k = \mu_{k-1} - \square$ is ordered by the number of row of the deleted box.

The monodromy along t_k reverses this order.

Remark

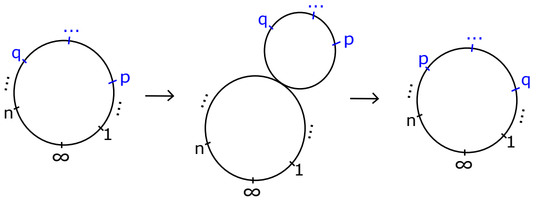
This order is given by the eigenvalues of H_k.

Bender-Knuth involutions on oscillating tableaux



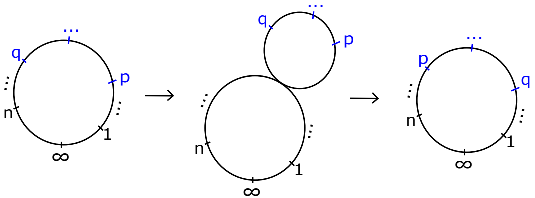
Cactus group and crystals

Another generating set of $Cact_n$ is given by s_{pq}:



Cactus group and crystals

Another generating set of $Cact_n$ is given by s_{pq}:



Relations come from codimension 2 cells:

$$s_{p,q}^2 = 1$$

$$s_{p_1,q_1}s_{p_2,q_2} = s_{p_2,q_2}s_{p_1,q_1} \quad \text{if } [p_1,q_1] \cap [p_2,q_2] = \emptyset$$

$$s_{p_1,q_1}s_{p_2,q_2}s_{p_1,q_1} = s_{p_1+q_1-q_2,p_1+q_1-p_2} \quad \text{if } [p_1,q_1] \supset [p_2,q_2].$$

Cactus group and crystals

Cactus group and crystals

- Crystal commutors do not satisfy the braid relation in general.

Cactus group and crystals

- Crystal commutors do not satisfy the braid relation in general. Instead:

$$\begin{array}{ccc} A \otimes B \otimes C & \xrightarrow{1 \otimes \sigma_{B,C}} & A \otimes C \otimes B \\ \big\downarrow{\scriptstyle \sigma_{A,B} \otimes 1} & & \big\downarrow{\scriptstyle \sigma_{A,C \otimes B}} \\ B \otimes A \otimes C & \xrightarrow{\sigma_{B \otimes A,C}} & C \otimes B \otimes A \end{array}$$

Cactus group and crystals

- Crystal commutors do not satisfy the braid relation in general. Instead:

$$\begin{array}{ccc} A \otimes B \otimes C & \xrightarrow{1 \otimes \sigma_{B,C}} & A \otimes C \otimes B \\ \downarrow{\scriptstyle \sigma_{A,B} \otimes 1} & & \downarrow{\scriptstyle \sigma_{A,C \otimes B}} \\ B \otimes A \otimes C & \xrightarrow{\sigma_{B \otimes A, C}} & C \otimes B \otimes A \end{array}$$

- $\sigma_{pq} : B_1 \otimes \cdots B_p \cdots B_q \cdots \otimes B_n \to B_1 \otimes \cdots B_q \cdots B_p \cdots \otimes B_n$ are well-defined and satisfy the cactus group relations

Cactus group and crystals

- Crystal commutors do not satisfy the braid relation in general. Instead:

$$\begin{array}{ccc} A\otimes B\otimes C & \xrightarrow{1\otimes\sigma_{B,C}} & A\otimes C\otimes B \\ \downarrow{\scriptstyle \sigma_{A,B}\otimes 1} & & \downarrow{\scriptstyle \sigma_{A,C\otimes B}} \\ B\otimes A\otimes C & \xrightarrow{\sigma_{B\otimes A,C}} & C\otimes B\otimes A \end{array}$$

- $\sigma_{pq} : B_1 \otimes \cdots B_p \cdots B_q \cdots \otimes B_n \to B_1 \otimes \cdots B_q \cdots B_p \cdots \otimes B_n$ are well-defined and satisfy the cactus group relations

Theorem [HKRW17]

There exists a natural structure of crystal on the fiber of $\mathcal{E}_{\underline{\lambda}}$ such that $\mathcal{E}_{\underline{\lambda}} \cong B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}$, and the monodromy action of s_{pq} coincides with the commutor action σ_{pq}.

Cactus group and crystals

- Crystal commutors do not satisfy the braid relation in general. Instead:

$$\begin{array}{ccc} A\otimes B\otimes C & \xrightarrow{1\otimes\sigma_{B,C}} & A\otimes C\otimes B \\ \downarrow{\scriptstyle \sigma_{A,B}\otimes 1} & & \downarrow{\scriptstyle \sigma_{A,C\otimes B}} \\ B\otimes A\otimes C & \xrightarrow{\sigma_{B\otimes A,C}} & C\otimes B\otimes A \end{array}$$

- $\sigma_{pq} : B_1 \otimes \cdots B_p \cdots B_q \cdots \otimes B_n \to B_1 \otimes \cdots B_q \cdots B_p \cdots \otimes B_n$ are well-defined and satisfy the cactus group relations

Theorem [HKRW17]

There exists a natural structure of crystal on the fiber of $\mathcal{E}_{\underline{\lambda}}$ such that $\mathcal{E}_{\underline{\lambda}} \cong B_{\lambda_1} \otimes \cdots \otimes B_{\lambda_n}$, and the monodromy action of s_{pq} coincides with the commutor action σ_{pq}.

The relation between t_k and the cactus group were studied in [BK96].

References

Arkady Berenstein and Anatol N. Kirillov.
Groups Generated by Involutions, Gelfand–Tsetlin Patterns, and Combinatorics of Young Tableaux.
St. Petersburg Mathematical Journal, 7(1):77–127, 1996.
https://pages.uoregon.edu/arkadiy/bk1.pdf.

Iva Halacheva, Joel Kamnitzer, Leonid Rybnikov, and Alex Weekes.
Crystals and Monodromy of Bethe Vectors, 2017.
https://arxiv.org/abs/1708.05105.