Stationary inverse-Wishart polymers

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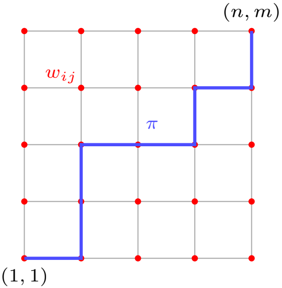
Integrable combinatorics

Directed polymers

Consider a model defined on $\mathbb{Z}^2_{\geqslant 1}$.

- $w = (w_{ij})_{i,j \geqslant 1}$: i.i.d. random variables in $\mathbb{R}_+$.
- Partition function

$$Z^w(n,m) := \sum_{\text{paths}} \prod_{(i,j)\in\text{path}} w_{ij}$$
$$= w_{nm}\Big(Z^w(n-1,m) + Z^w(n,m-1)\Big).$$



Conjecture (Kardar–Parisi–Zhang universality)

Under mild assumptions on the distribution of w, there exist constants f and g such that

$$\frac{\log Z^w(n,n) - fn}{(gn)^{1/3}} \xrightarrow[n\to\infty]{} \mathrm{TW}_{\mathrm{GUE}}\,.$$

The Tracy-Widom distribution governs the fluctuations of the top eigenvalue of large Hermitian matrices (Gaussian Unitary Ensemble).

An integrable model

- A random variable $w \sim \mathrm{Gamma}^{-1}(\theta)$ if $w^{-1} \sim \mathrm{Gamma}(\theta)$. It is supported on $\mathbb{R}_{>0}$ and has density $\frac{1}{\Gamma(\theta)} x^{-\theta} e^{-x^{-1}}$ with respect to $\mathrm{d}\mu(x) = \frac{\mathrm{d}x}{x}$.
- When $w_{ij} \overset{i.i.d.}{\sim} \mathrm{Gamma}^{-1}(2\theta)$ for some $\theta > 0$, the partition functions $(Z^\theta(n,m))_{n,m \geqslant 1}$ is called the log-gamma polymer [Seppäläinen 2012].

Theorem (Seppäläinen 2012)

$$\frac{\log Z^\theta(n,n)}{n} \xrightarrow[n\to\infty]{a.s.} -2\psi(\theta),$$

where $\psi(\theta) := \big(\log \Gamma(\theta)\big)'$ *is the digamma function.*

- Using the geometric Robinson–Schensted–Knuth correspondence, the Laplace transform $\mathbb{E}\left[e^{-uZ^\theta(n,m)}\right]$ is explicit in terms of the Whittaker functions [Corwin–O'Connell-Seppäläinen–Zygouras 2014].
- $n^{1/3}$ fluctuation scale of $\log Z^\theta(n,n)$ and the Tracy-Widom limit [Borodin–Corwin–Remenik 2013].

Directed polymer with matrix-valued disorder

Consider a model defined on $\mathbb{Z}^2_{\geqslant 1}$.

- Let $\mathcal{P}_d$ denote the space of $d \times d$ symmetric positive definite matrices.
- Define a product on $\mathcal{P}_d$ by $A \star B := B^{1/2} A B^{1/2}$.
- $(W_{ij})_{i,j\geqslant 1} \overset{i.i.d.}{\sim}$ inverse Wishart matrices with parameter $2\theta > \frac{d-1}{2}$: supported on $\mathcal{P}_d$ and have density

$$\frac{1}{\Gamma_d(2\theta)} |x^{-1}|^{2\theta} e^{-\operatorname{tr}[x^{-1}]} \mathsf{d}\mu(x),$$

with respect to $\mathsf{d}\mu(x) = |x|^{-\frac{d+1}{2}} \prod_{1\leqslant i\leqslant j\leqslant d} dx_{ij}$.

Definition (Arista–Bisi–O'Connell 2023)

The partition function of the inverse-Wishart polymer is defined by

$$Z^\theta(n,m) = W(n,m) \star \Big(Z^\theta(n-1,m) + Z^\theta(n,m-1) \Big).$$

- $d = 1$ recovers the log-gamma polymer.
- The Laplace transform of $Z^\theta(n,m)$ is explicit in terms of the Whittaker functions of matrix arguments [Arista–Bisi–O'Connell 2023].

Stationary inverse-Wishart polymer

Theorem (Barraquand–O. 2025)

The model $Z^{\theta,u}(n,m)$ with boundary condition

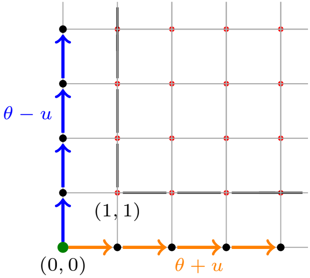
$$\Big(Z^{\theta,u}(0,0), Z^{\theta,u}(k,0)\star(Z^{\theta,u}(k-1,0))^{-1}, Z^{\theta,u}(0,k)\star(Z^{\theta,u}(0,k-1))^{-1}\Big)_{k\geqslant 1}$$
$$\sim \mathsf{d}\mu \times \mathrm{Wishart}^{-1}(\theta+u)^{\mathbb{N}} \times \mathrm{Wishart}^{-1}(\theta-u)^{\mathbb{N}}.$$

is stationary.

- We conjecture that for the model without b.c.

$$\frac{\log|Z^{\theta}(n,n)|}{n} \xrightarrow[n\to\infty]{a.s.} f_d^{\theta} \neq -2\psi_d(\theta),$$

where $\psi_d(\theta) := \big(\log\Gamma_d(\theta)\big)'$ is the multivariate digamma function.



Stationary inverse-Wishart polymer

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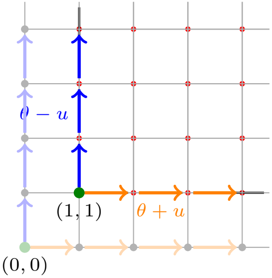
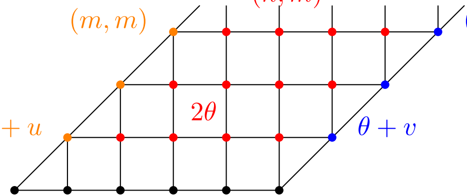


Model on a strip



- When $d=1$, the stationary measure of log-gamma polymer on a strip is given in terms of the skew Whittaker functions [Barraquand–Corwin–Yang 2024].

Definition

For parameter $\alpha > \frac{d-1}{2}$ and arguments $\lambda, \mu \in \mathcal{P}_d^2$, the skew Whittaker function of matrix arguments is defined by

$$\Psi_\alpha(\lambda/\mu) := |\mu_1(\lambda_1)^{-1}|^\alpha |\mu_2(\lambda_2)^{-1}|^\alpha e^{-\operatorname{tr}[\mu_1(\lambda_1)^{-1}+\mu_2(\lambda_2)^{-1}+\lambda_2(\mu_1)^{-1}]}.$$

- $d=1$ recovers the skew-Whittaker functions.

Cauchy summation identity

- For $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_m)$,

$$\sum_{\pi \in \mathbb{Y}} s_\pi(\alpha) s_\pi(\beta) = \prod_{i,j} \frac{1}{1 - \alpha_i \beta_j} =: \Pi(\alpha; \beta).$$

- For $\lambda, \nu \in \mathbb{Y}$,

$$\sum_{\pi \in \mathbb{Y}} s_{\pi/\lambda}(\alpha) s_{\pi/\nu}(\beta) = \Pi(\alpha; \beta) \sum_{\pi \in \mathbb{Y}} s_{\lambda/\pi}(\beta) s_{\nu/\pi}(\alpha).$$

- Let $\mathrm{Sign}_n = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n, \lambda_1 \geqslant \ldots \geqslant \lambda_n\}$,

$$\sum_{\pi \in \mathrm{Sign}_2} s_{\pi/\lambda}(\alpha) s_{\pi/\nu}(\beta) = \sum_{\pi \in \mathrm{Sign}_2} s_{\lambda/\pi}(\beta) s_{\nu/\pi}(\alpha).$$

Lemma (Cauchy-type identity)

For $\alpha, \beta \in \mathbb{R}$ *and* $\lambda, \mu \in \mathcal{P}_d^2$,

$$\int_{\mathcal{P}_d^2} \Psi_\alpha(\pi/\lambda) \Psi_\beta(\pi/\mu) \mathrm{d}\mu(\pi) = \int_{\mathcal{P}_d^2} \Psi_\beta(\lambda/\pi) \Psi_\alpha(\mu/\pi) \mathrm{d}\mu(\pi).$$

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Thank you !