

# Marking in the periodically weighted Aztec diamond

Nedialko Bradinoff

KTH Royal Institute of Technology  
Visiting LPSM Sorbonne university

Based on joint work with Tomas Berggren

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## The Aztec diamond with uniform weights

- There is an *arctic curve* (Jockusch-Propp-Schorr).
- Inside the arctic curve is a *liquid/rough* region
- Outside the arctic curve is a *solid/frozen* region
- Locally, near points on the arctic curve: *Airy process* (Johansson, 05)
- At *turning* points: **GUE-corners process** (Johansson- Nordenstam, 06).

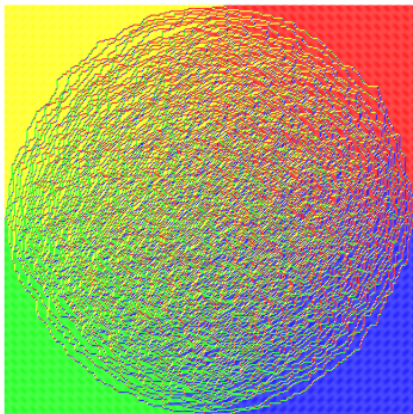


Figure: A sample of the Aztec diamond with uniform weights of size 300

## The Aztec diamond with periodic weights

- New *gaseous/smooth* disordered regions appear
- The critical phenomena are mostly not studied;
- What happens on the arctic curve, at the cusp points and **turning points**?

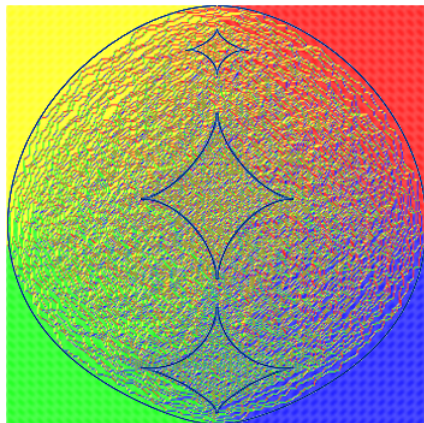


Figure:  $2 \times 4$ -periodic Aztec diamond

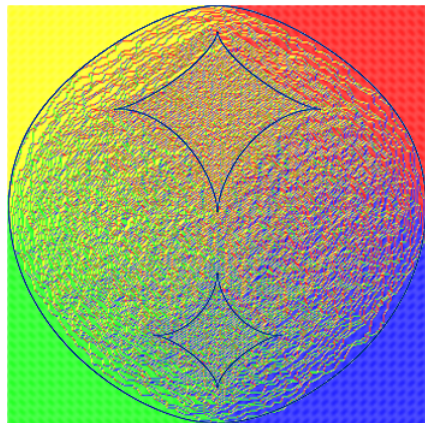
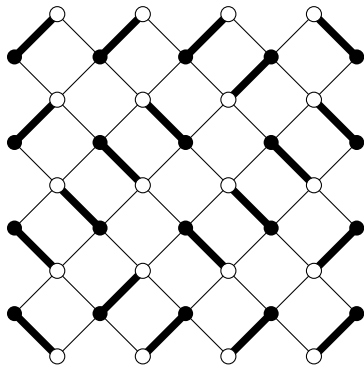
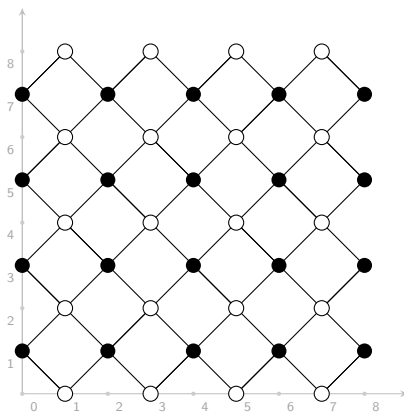


Figure:  $2 \times 3$ -periodic Aztec diamond

## The Aztec diamond as a dimer model

The *Aztec diamond graph* of size  $N$  is a bipartite graph defined on a subset of  $\{0, 1, 2, \dots, 2N\}^2 \subset \mathbb{Z}^2$  so that there are black vertices on points with even  $x$ -coordinate and odd  $y$ -coordinate and white vertices at points with odd  $x$ -coordinate and even  $y$ -coordinates.

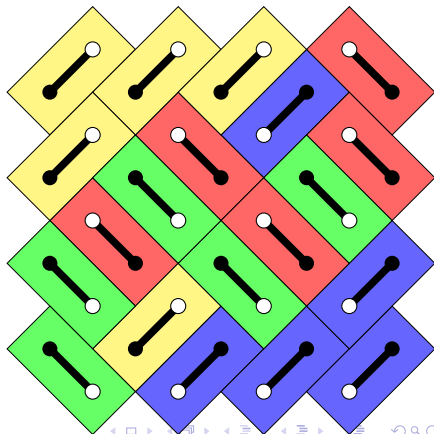
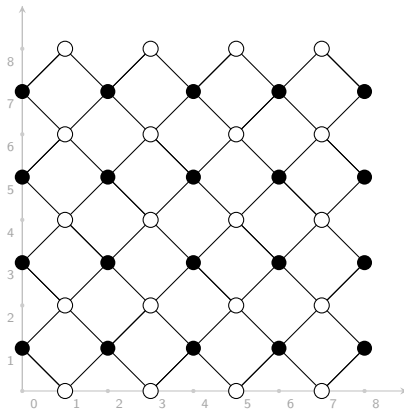
$$\mathbb{P}_{\text{dimer}}(\mathcal{D}) = \frac{\prod_{e \in \mathcal{D}} w(e)}{\sum_{\mathcal{D}} \prod_{e \in \mathcal{D}} w(e)}$$



## The Aztec diamond

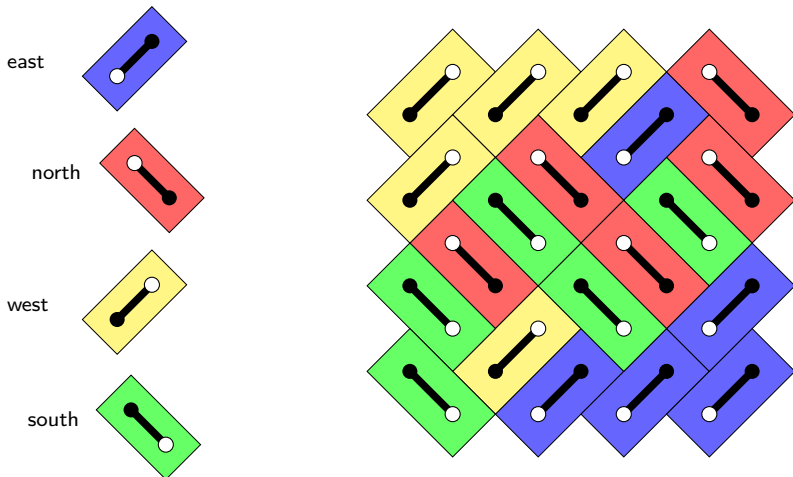
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# The Aztec diamond as a dimer model

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# An interlacing particle system at the turning point

How do we study the process at the turning point?

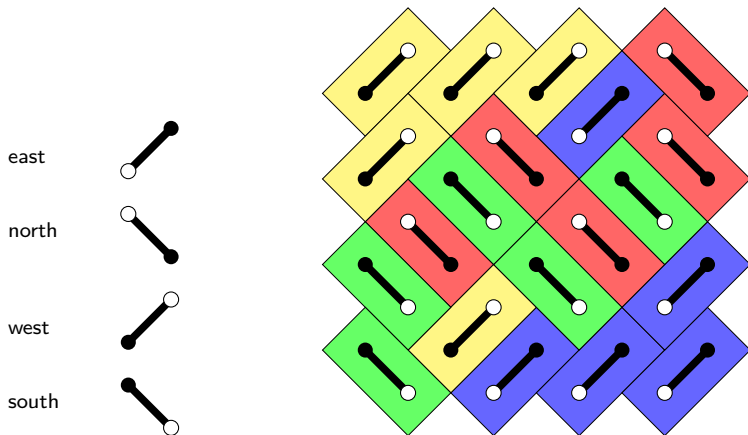


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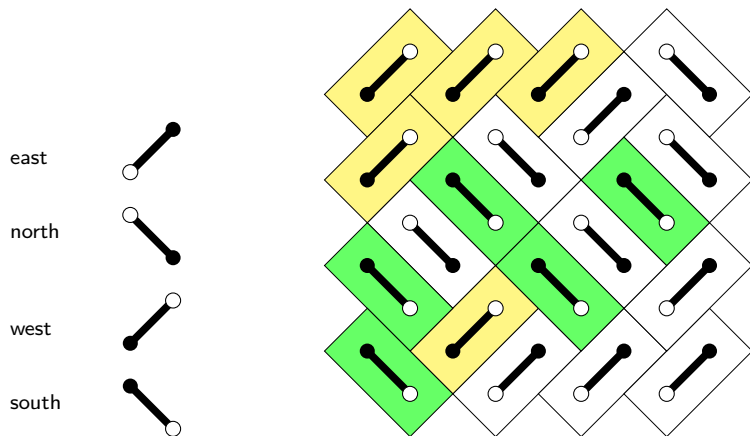


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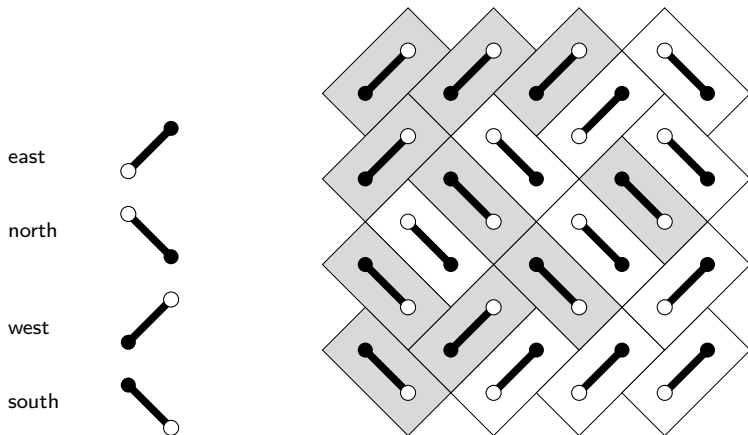


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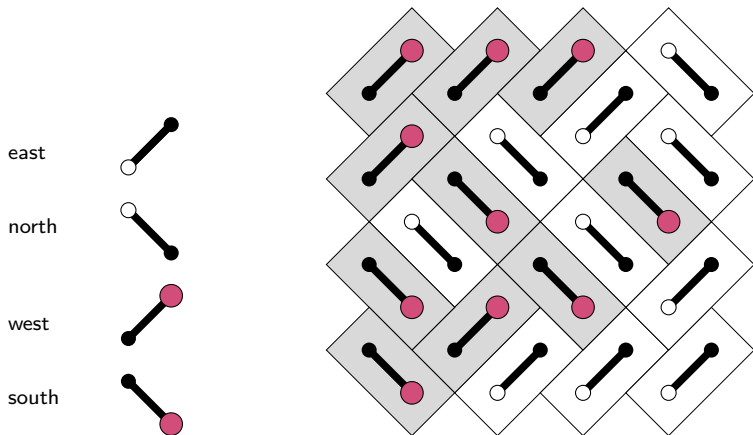
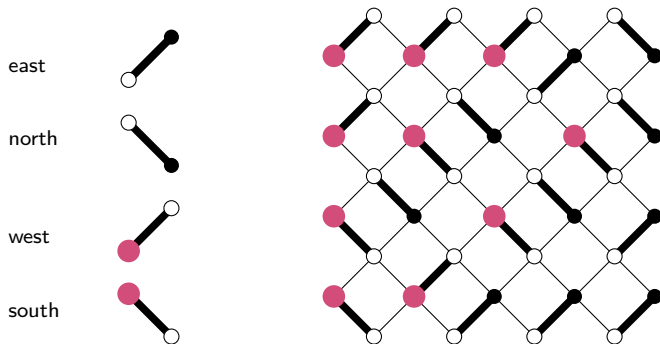


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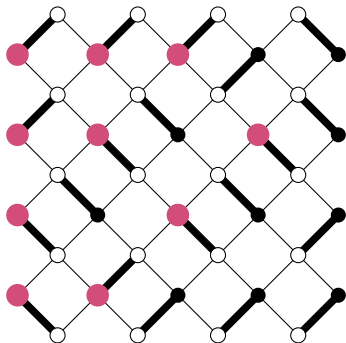
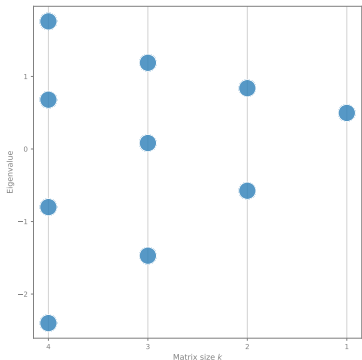
How do we study the process at the turning point?



**Figure:** Left: The particles defined from the south and west dimers visualized in purple. Right: A dimer cover of the Aztec diamond of size 4. The particles form an interlacing particle system.

We denote this interlacing particle system:

$$(u_s^t)_{1 \leq s \leq t \leq 2\ell N}, \quad u_s^{t+1} \leq u_s^t \leq u_{s+1}^{t+1}.$$



## Theorem (Johansson-Nordenstamm 2006)

For the Aztec diamond with uniform weights,

$$\left( \frac{u_s - \frac{N}{2}}{\sqrt{N/2}} \right)_{1 \leq s \leq t} \rightarrow (\xi_s^t)_{1 \leq s \leq t},$$

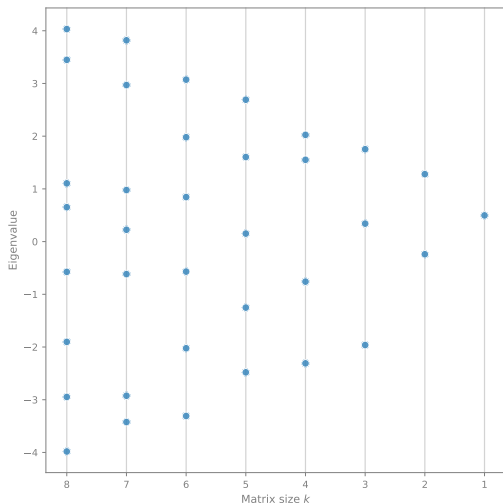
in the sense of weak convergence, as  $N \rightarrow \infty$ .

$(\xi_s^t)_{1 \leq s \leq t, t \in \mathbb{N}}$  is the *GUE-corners process*.

## GUE-corners process

We sample a sequence of Hermitian matrices by choosing independent random variables,

- $X_{ij} \sim \frac{1}{\sqrt{2}}(N(0, 1) + iN(0, 1))$  for  $i > j$ ,
- $X_{ii} \sim N(0, 1)$ .



We plot the eigenvalues of the  $t \times t$  matrix given by  $X^{(t)} = (X_{i,j})_{1 \leq i, j \leq t}$ .

$$\begin{pmatrix} \boxed{X_{11}} & X_{12} & X_{13} & X_{14} \\ X_{21} & \boxed{X_{22}} & X_{23} & X_{24} \\ X_{31} & X_{32} & \boxed{X_{33}} & X_{34} \\ X_{41} & X_{42} & X_{43} & \boxed{X_{44}} \end{pmatrix}$$

The ordered eigenvalues of  $X^{(t)}$ ,

$$\xi_1^t \leq \xi_2^t \leq \dots \leq \xi_t^t,$$

interlace:

$$\xi_s^{t+1} \leq \xi_s^t \leq \xi_{s+1}^{t+1}.$$

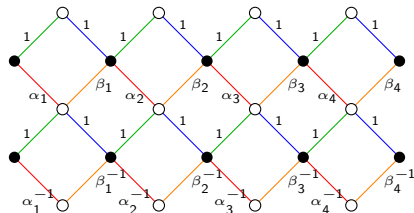
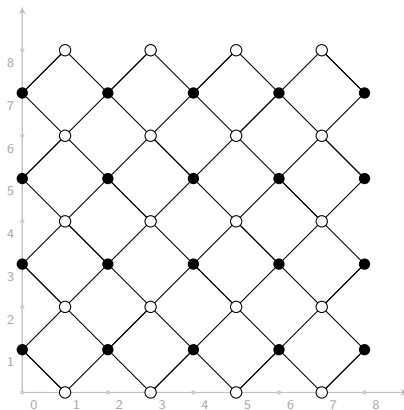
### Definition

The random set  $(\xi_s^t)_{1 \leq s \leq t, t \in \mathbb{N}}$  defines a point process on  $\Lambda = \mathbb{N} \times \mathbb{R}$ , the **GUE-corners process**.

# The Aztec diamond with periodic weights

"We assign to each edge  $e$  a weight  $w(e) > 0$ "

$$\mathbb{P}_{dimer}(\mathcal{D}) = \frac{\prod_{e \in \mathcal{D}} w(e)}{\sum_{\mathcal{D}} \prod_{e \in \mathcal{D}} w(e)}$$



**Figure:** The Aztec diamond graph of size 4 on the left and the fundamental domain of a  $2 \times 4$  periodic Aztec diamond dimer model on the right. North, East, South, and West edges are colored blue, orange, red, and green respectively.

## A family of $2 \times \ell$ periodic weights

For the weights we study there are exactly 4 **turning points**. We focus on the *turning point* on the right side.

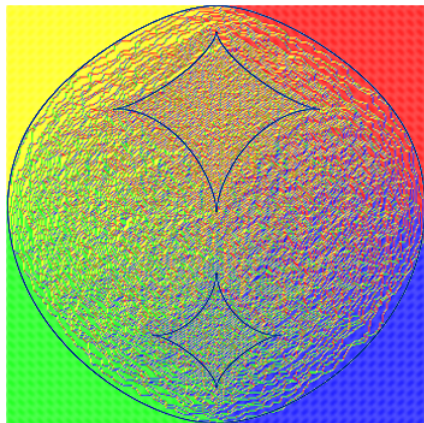


Figure:  $2 \times 3$ -periodic Aztec diamond

$$\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_\ell > 0$$

$$\prod_{k=1}^{\ell} \alpha_k = \prod_{k=1}^{\ell} \beta_k.$$

Assume that the Aztec diamond is of size  $2\ell N$ .

### Lemma

The right-most turning point is unique and has coordinates  $(2\ell N, 2\tau N)$ , where setting  $a_k = \alpha_k \beta_{k-1}^{-1}$  and  $b_k = \alpha_k^{-1} \beta_k$

$$\tau = \sum_{k=1}^{\ell} \frac{1 + a_k + a_k b_k + a_k b_k a_{k+1}}{(1 + a_k)(1 + b_k)(1 + a_{k+1})}.$$

## Convergence of correlation functions

The point processes we study are **determinantal**, meaning that the correlation/density functions of having particles at positions,  $\zeta_1, \zeta_2, \dots, \zeta_m \in X$  are given by

$$\rho(\zeta_1, \zeta_2, \dots, \zeta_m) = \det K(\zeta_j, \zeta_k)_{j,k=1}^m$$

for a function  $K : X \times X \rightarrow \mathbb{C}$ , known as the **correlation kernel** of the determinantal point process.

### Theorem (Berggren-B, 26)

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbf{Ga} \times N^{\frac{1}{2}} K_{Int}(\ell x_1 + i_1, 2y_1 + j_1; \ell x_2 + i_2, 2y_2 + j_2) \\ = \nu(t, j) \sigma^{-1} K_{GUE}(t_1, \sigma^{-1} \mu_1; t_2, \sigma^{-1} \mu_2), \end{aligned}$$

where

$$\nu(t, j) = \begin{cases} \frac{1}{1 + \alpha_{\ell+1-t} \beta_{\ell-t}^{-1}}, & j = 0, \\ \frac{\alpha_{\ell+1-t} \beta_{\ell-t}^{-1}}{1 + \alpha_{\ell+1-t} \beta_{\ell-t}^{-1}}, & j = 1. \end{cases}$$

A

## interlacing particle system at the turning point

How do we interpret the convergence of correlation functions?

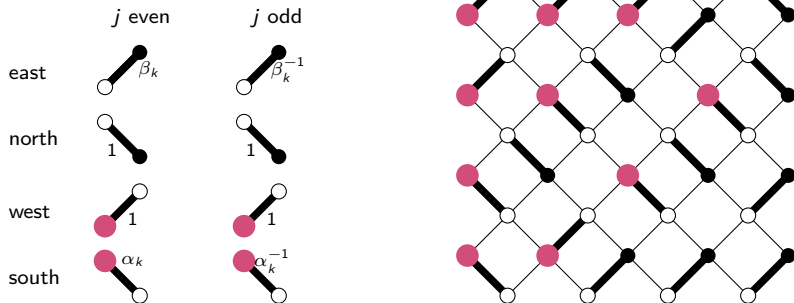
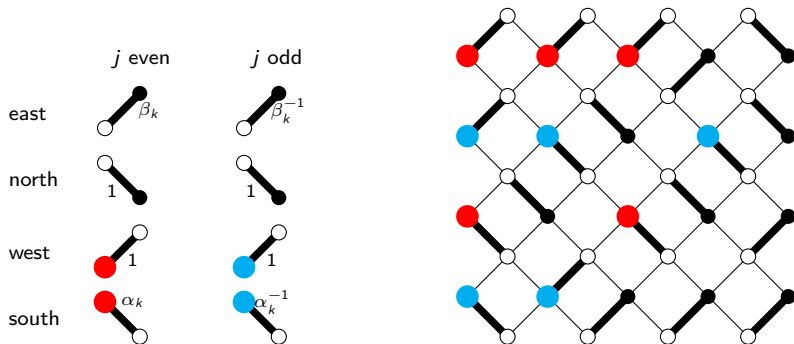


Figure:

## A colored interlacing particle system at the turning point

How do we interpret the convergence of correlation functions?



**Figure:** Left: The particles defined from the dimers visualized in red and cyan. The color depends on the parity of the coordinates of the particle. Right: A dimer cover of the Aztec diamond of size 4. The particles form an interlacing particle system.

We denote this **colored** interlacing particle system:

$$(u_{s,j}^t)_{1 \leq s \leq t \leq 2\ell N}, \quad u_{s,j}^{t+1} \leq u_{s,j}^t \leq u_{s+1,j}^{t+1}$$

## The marked GUE-corners process

Let  $\theta : \Lambda \rightarrow [0, 1]$  be a measurable function. Then:

### Definition

The *marked GUE-corners process*,  $(\xi_{s,j}^t)_{1 \leq s \leq t, t \in \mathbb{N}}$  is the point process on  $\Lambda_{\{0,1\}} = \Lambda \times \{0, 1\}$  constructed out of the GUE-corners process on  $\Lambda$  by independently assigning to each point a binary value  $m \sim \text{Ber}(\theta(t, \mu))$ , where  $(t, \mu) \in \Lambda$  is the location of the point.

Suppose the location of the turning point is  $(2\ell N, 2\tau N)$  and let

$$\theta(t, \mu) = \theta(t) = \frac{\alpha_{\ell+1-t} \beta_{\ell-t}^{-1}}{1 + \alpha_{\ell+1-t} \beta_{\ell-t}^{-1}}.$$

### Theorem (Berggren-B., 2026)

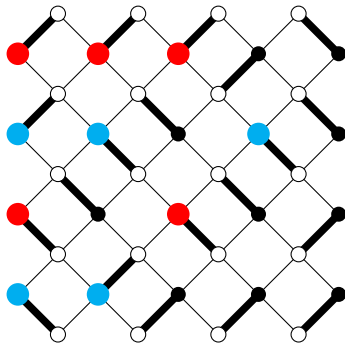
There is a  $\sigma > 0$  such that:

$$\left( \frac{u_{s,j}^t - 2N\tau}{2\sigma\sqrt{N}} \right)_{1 \leq s \leq t} \rightarrow (\xi_{s,j}^t)_{1 \leq s \leq t},$$

in the sense of weak convergence as  $N \rightarrow \infty$ , and  $(\xi_{s,j}^t)_{1 \leq s \leq t}$  is the marked GUE-corners process with an underlying marking function  $\theta$ .

## The thinned GUE-corners process

For a measurable function  $\theta : \Lambda \rightarrow [0, 1]$ , the *thinned GUE-corners process* is obtained from the GUE-corners process by deleting each particle independently in with probability  $\theta$ .











### Corollary (Berggren-B., 26)

For the same  $\sigma > 0$  if we restrict our attention to particles with  $j = 0$  (or  $j = 1$ )

$$\left( \frac{u_{s,j}^t - 2N\tau}{2\sigma\sqrt{N}} \right)_{1 \leq s \leq t} \rightarrow \left( \xi_{s,j}^t \right)_{1 \leq s \leq t},$$

in the sense of weak convergence as  $N \rightarrow \infty$ , and  $(\xi_{s,j}^t)_{1 \leq s \leq t}$  is the thinned GUE-corners process with a thinning function  $\theta$  (or  $1 - \theta$ ).

## References

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