The convoy of the ASEP speed process

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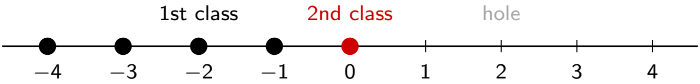
ASEP with a second class particle



Theorem (Mountford-Guiol '05; Ferrari-Pimentel '05; Aggarwal-Corwin-Ghosal '23)

Let $Y_0(t)$ be the position of the second class particle at time t, then

$$\lim_{t\to\infty} \frac{Y_0(t)}{(1-q)\,t} \overset{a.s.}{=} \mathcal{U}[-1,1] := U.$$

U is called the speed of the second class particle.

The Multi species ASEP

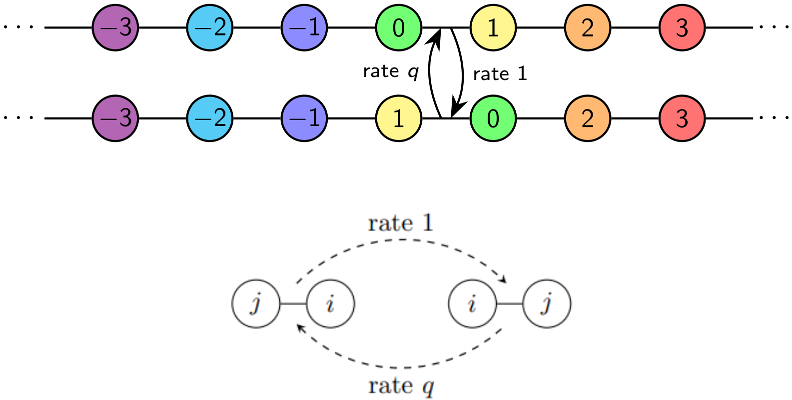


Figure 1: The multi-species ASEP with swap rate $(i > j)$

Color projection

Denote the position of particle 0 at time t as $X_t(0)$, consider the following color projection:

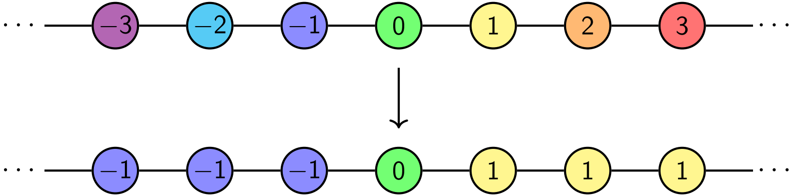


Figure 2: Projection regarding 0 as the second class particle

Color projection

Denote the position of particle 0 at time t as $X_t(0)$, consider the following color projection:

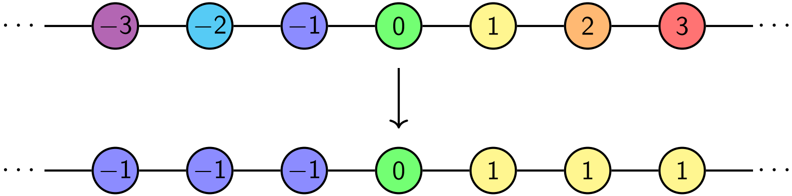


Figure 2: Projection regarding 0 as the second class particle

As a corollary of the speed of the second class particle,

$$\lim_{t\to\infty} \frac{X_0(t)}{(1-q)\,t} \overset{\text{a.s.}}{:=} \mathcal{U}_0.$$

Color projection

More generally, denote the position of particle i at time t as $X_t(i)$, consider the following color projection:

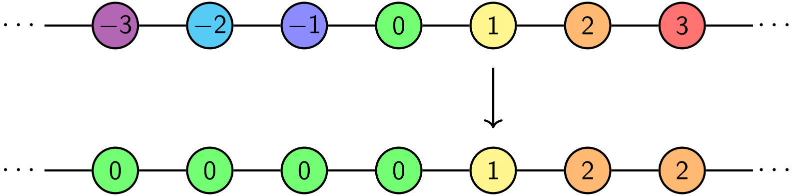


Figure 3: Projection regarding i ($i = 1$) as the second class particle

Similarly,

$$\lim_{t\to\infty} \frac{X_i(t) - i}{(1-q)\,t} \overset{\text{a.s.}}{:=} \mathcal{U}_i.$$

The ASEP speed process

Definition (Amir-Angel-Valkó '11)

Considering the color projection at each site, we get a family of uniform random variables $\{\mathcal{U}_n, n \in \mathbb{Z}\}$, which is called the **ASEP speed process** .

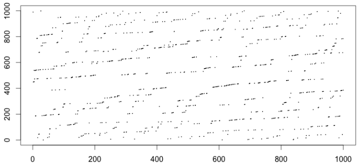


Figure 10: TASEP

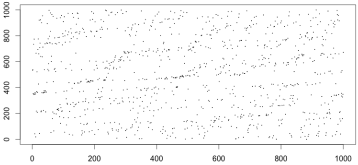


Figure 11: ASEP ($q = 0.8$)

The (T)ASEP speed process, images from [Martin '20].

The convoy of the ASEP speed process

Definition (Amir-Angel-Valkó '11)

The convoy of particle 0 is the random set of indices corresponding to particles that have the same speed as particle 0. Formally,

$$\mathcal{C} = \{j \in \mathbb{Z} : \mathcal{U}_j = \mathcal{U}_0\}.$$

- The system is translation invariant, so we only study the convoy of particle 0.
- To study the asymptotics, we also define

$$\mathcal{C}^n = \{j \in [1, n] : \mathcal{U}_j = \mathcal{U}_0\}.$$

- Combinatorial representations are constructed in [Amir-Angel-Valkó '11] and in [Martin '20].

Question: How large is the set $\mathcal{C}^n$ for large n ?

Main results: The asymptotic size of the convoy

Notation: u is the speed of particle 0, $x = \frac{1+u}{2}$, $c = x(1-x)$
$\mathbb{E}_x[\cdot] = \mathbb{E}[\cdot | x = \frac{1+U_0}{2}]$.

Theorem (T. '25)

The asymptotic expected size of the ASEP convoy of particle 0 *is of order* $\sqrt{n}$. *More precisely, for any fixed* $q \in [0,1)$ *and fixed* $x \in [0,1]$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \mathbb{E}_x[\#\mathcal{C}^n] = \sqrt{\frac{4x(1-x)}{\pi}}.$$

Conjecture (T. '25)

Under the same assumption,

$$\lim_{n\to\infty} \frac{\#\mathcal{C}^n}{\sqrt{n}} \overset{(d)}{=} |\mathcal{N}(0, 2x(1-x))|,$$

The conjecture holds when $q = 0$.

Main results: An exact formula

Definition (Han-Zeng '99)

The q-Gandhi polynomials $B_n(x,q)$ are defined by

$$B_n(x,q) := \Delta_q\left(x^2 B_{n-1}(x,q)\right),$$

with $B_1(x,q) = 1$, where

$$\Delta_q f(x) := \frac{f(1+qx) - f(x)}{(1+qx) - x}.$$

The q-Genocchi number is the q-Gandhi polynomials at $x = 1$.

Theorem (T. '25)

$$\mathbb{E}_x[\#\mathcal{C}^n] = \sum_{k=0}^{n} (1-q)^{2k+1}(-x(1-x))^{k+1}\binom{n+1}{k+1}B_k(1,q).$$

Main results: The $q \to 1^-$ limit

Theorem (T. '25)

Consider $q_n = e^{-\frac{\gamma}{\sqrt{n}}}$, we have

$$\lim_{n\to\infty} \mathbb{E}_x \left[\frac{\#\mathcal{C}^n}{\sqrt{n}} \right] = C^\gamma.$$

In addition, we have

$$\lim_{\gamma\to 0} C^\gamma = 0, \qquad \lim_{\gamma\to\infty} C^\gamma = \sqrt{\frac{4x(1-x)}{\pi}}.$$

Ingredients of our proof

- Step 1: We developed a coupling argument based on Martin's construction.

$$\mathbb{E}\left[\#\mathcal{C}^n\right] = \mathbb{E}\left[Q_{n+1} - P_{n+1}\right].$$

- Step 2: We found an appropriate family of orthogonal polynomials to describe the transition probability

$$\mathbb{P}\left(Q_{n+1} = j | Q_1 = i\right) = \frac{1}{(q;q)_j} \int_{1-4c}^{1} x^n P_i(x) P_j(x) \psi(x) \, dx.$$

- Step 3: We derived the scaling limit of Q_{n+1} by analyzing the endpoint of the orthogonality interval.

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Thanks for your attention!