

Tensor decomposition of Demazure crystals.

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Outline

- Crystals.
- Demazure crystals.

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- Problem.
- Main results.

Crystals

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Crystals

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Definition

A crystal B is a set together with maps

$$wt : B \rightarrow P, \varphi_i, \varepsilon_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}, e_i, f_i : B \rightarrow B \cup \{0\}, \forall i \in I$$

satisfying the following properties.

- $\varphi_i(b) = \varepsilon_i(b) + \langle wt(b), \alpha_i^\vee \rangle,$
- $e_i(b) = b' \iff b = f_i b',$
- $wt(e_i(b)) = wt(b) + \alpha_i, \quad \varepsilon_i(e_i(b)) = \varepsilon_i(b) - 1,$
- $\varphi_i(b) = -\infty, e_i(b) = f_i(b) = 0.$

Tensor product of crystals

Normal Crystals:

$$\varepsilon_i(b) = \max\{k \mid e_i^k(b) \neq 0\}$$

$$\varphi_i(b) = \max\{k \mid f_i^k(b) \neq 0\}$$

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Definition:

For $b_1 \otimes b_2 \in B_1 \otimes B_2$, define the maps wt, e_i, f_i as follows

$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2);$$

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes e_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes f_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

Demazure Crystals

For $i \in I$ and $S \subseteq B$, define

$$T_i S = \{f_i^k b \mid b \in S, k \geq 0\} \setminus \{0\} \subseteq B.$$

For $w = s_{i_1} \cdots s_{i_k}$ in W , define $T_w(S) = T_{i_1} \cdots T_{i_k} S$.

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- $B(\infty)$ - generated by b_∞ with weight 0.

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- $B(\lambda)$ - generated by b_λ with weight λ .
- $B(\infty)$ - generated by b_∞ with weight 0.
- $B_w(\lambda) = T_w\{b_\lambda\}$ - **Demazure crystal**.
- $B_w(\infty) = T_w\{b_\infty\}$ - Limiting Demazure crystal.

Extremal subsets

For $i \in I$ and $b \in B(\lambda)$ with $e_i(b) = 0$, the i -string for b is

$$S = \{f_i^k(b); k \geq 0\} \setminus \{0\}.$$

Extremal subsets

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$$S = \{f_i^k(b); k \geq 0\} \setminus \{0\}.$$

Definition:

For $\lambda \in P^+ \cup \{\infty\}$ a non-empty subset X of $B(\lambda)$ is said to be **extremal** if for any i -string S with highest weight vector b , we have

$$S \cap X = \emptyset, S, \{b\}.$$

Remark: $[2, K]$ - Demazure crystals are extremal subsets but not all extremal subsets are Demazure crystals.

Problem

For $\lambda, \mu \in P^+$ and $v, w \in W$, consider the tensor product of Demazure crystals

$$B_v(\lambda) \otimes B_w(\lambda)$$

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Question: [1, A]

When the tensor product of Demazure crystals is a disjoint union of Demazure crystals?

Notations

For $\lambda, \mu \in P^+$ and $b \in B$

- $W_\lambda = \{w \in W \mid w\lambda = \lambda\}$ - Stabilizer of λ in W .
- w_{min}^λ - minimal length representative of the coset wW_λ .

Notations

For $\lambda, \mu \in P^+$ and $b \in B$

- $W_\lambda = \{w \in W \mid w\lambda = \lambda\}$ - Stabilizer of λ in W .
- w_{min}^λ - minimal length representative of the coset wW_λ .
- b is primitive if $e_i(b) = 0, \forall i$.
- $B(\mu)^\lambda = \{b \in B(\mu) \mid b_\lambda \otimes b \text{ is primitive in } B(\lambda) \otimes B(\mu)\}$.
- $B_w(\mu)^\lambda = B(\mu)^\lambda \cap B_w(\mu)$
- If $w = e$ then $B_e(\lambda) = \{b_\lambda\}$

Decomposition of $B(\lambda) \otimes B(\mu)$

Theorem: [3, PL]

For $\lambda, \mu \in P^+$,

$$B(\lambda) \otimes B(\mu) = \coprod_{b \in B(\mu)^\lambda} \mathcal{F}(b_\lambda \otimes b)$$

where $\mathcal{F}(b_\lambda \otimes b) \cong B(\lambda + wt(b))$.

Decomposition of $B_e(\lambda) \otimes B_w(\mu)$

Theorem: [1, A]

For $\lambda \in P^+$, $\mu \in P^+ \cup \{\infty\}$, and $w \in W$, then

- 1 $B_e(\lambda) \otimes B_w(\mu)$ is isomorphic to a disjoint union of Demazure crystals.
- 2 For all $b \in B_w(\mu)^\lambda$ there exist $y_{w,b}^\lambda \in W$ such that

$$B_e(\lambda) \otimes B_w(\mu) = \coprod_{b \in B_w(\mu)^\lambda} T_{y_{w,b}^\lambda}(b_\lambda \otimes b)$$

$$\text{where } T_{y_{w,b}^\lambda}(b_\lambda \otimes b) \cong \begin{cases} B_{y_{w,b}^\lambda}(\lambda + wt(b)), & \text{if } \mu < \infty, \\ B_{y_{w,b}^\lambda}(\infty; \lambda + wt(b)), & \text{if } \mu = \infty. \end{cases}$$

Note: $B_w(\mu)^\lambda$ is a finite set.

Main results

Theorem: [4, S]

For $\lambda, \mu \in P^+$ and $v, w \in W$, the following statements are equivalent.

- 1 $B_v(\lambda) \otimes B_w(\mu)$ is isomorphic to a disjoint union of Demazure crystals.
- 2 $v_{\min}^\lambda \in \langle \{s_i \mid \langle w\mu, \alpha_i^\vee \rangle \leq 0\} \rangle$.
- 3 $B_v(\lambda) \otimes B_w(\mu)$ is an extremal subset of $B(\lambda) \otimes B(\mu)$.

Decomposition of $B_v(\lambda) \otimes B_w(\mu)$

Theorem: [4, S]

Let $\lambda, \mu \in P^+$, and $v, w \in W$. Assume that $v_{\min}^\lambda \in \langle \{s_i \mid \langle w\mu, \alpha_i^\vee \rangle \leq 0\} \rangle$, then for all $b \in B_w(\mu)^\lambda$ there exist $u(b, v) \in W$ such that

$$B_v(\lambda) \otimes B_w(\mu) \cong \coprod_{b \in B_w(\mu)^\lambda} B_{u(b, v)}(\lambda + wt(b)).$$

Main results

Theorem: [4, S]

For $\lambda \in P^+$ and $v, w \in W$, the following statements are equivalent.

- 1 $B_v(\lambda) \otimes B_w(\infty)$ is isomorphic to a disjoint union of Demazure crystals.
- 2 $v_{\min}^\lambda \in \langle \{s_i \mid l(s_i w) < l(w)\} \rangle$.
- 3 $B_v(\lambda) \otimes B_w(\infty)$ is an extremal subset of $B(\lambda) \otimes B(\infty)$.

Decomposition of $B_v(\lambda) \otimes B_w(\infty)$

Theorem: [4, S]

Let $\lambda \in P^+$, and $v, w \in W$. Assume that $v_{\min}^\lambda \in \langle \{s_i \mid l(s_i w) < l(w)\} \rangle$ then for all $b \in B_w(\infty)^\lambda$ there exist $u(b, v) \in W$ such that

$$B_v(\lambda) \otimes B_w(\infty) \cong \coprod_{b \in B_w(\infty)^\lambda} B_{u(b, v)}(\infty; \lambda + wt(b)).$$

- **Key Polynomials.**
- Specialized non-symmetric **Macdonald polynomials** at $t = 0$.
- Product of Demazure characters.

Thank You

References I

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References II

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