Arctic curves of the Aztec diamond and related models

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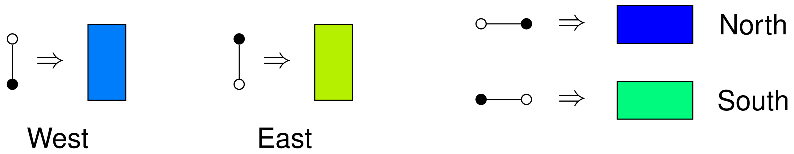
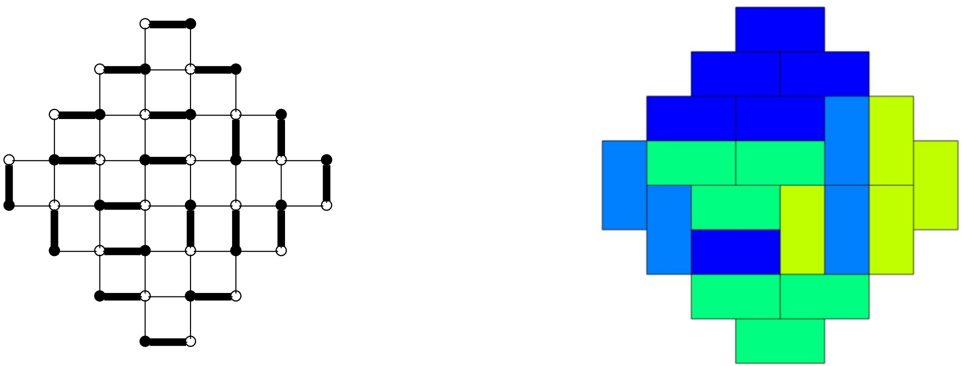
This presentation is based on

- M.P., *Arctic curves of periodic dimer models and generalized discriminants*, Selecta Mathematica, accepted.

The Aztec diamond as a dimer model





THE WEIGHTED AZTEC DIAMOND DIMER MODEL

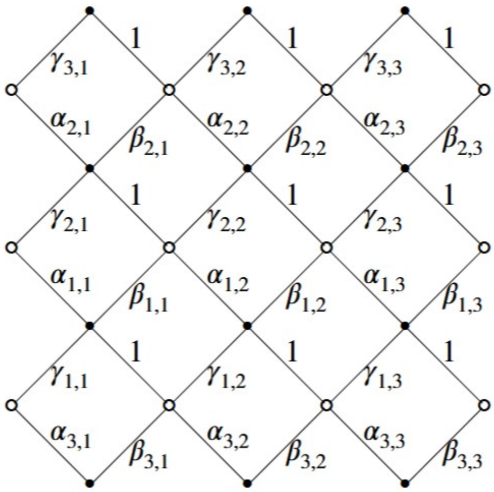
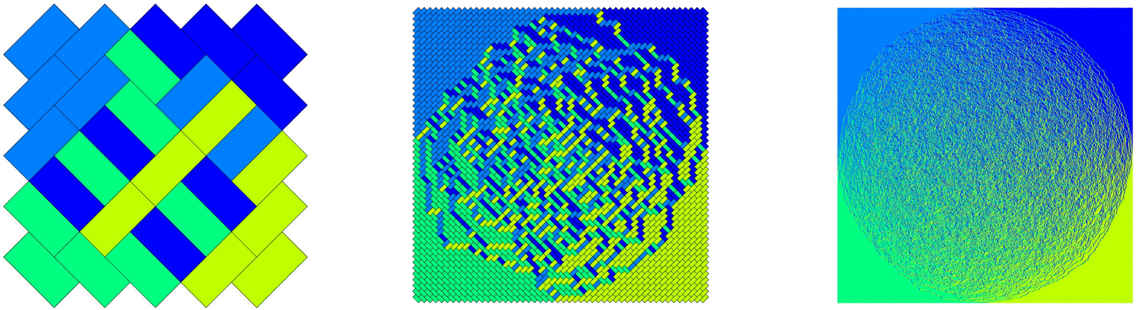


Figure. Weighted Aztec diamond graph of size 3

Example: Uniform weighting, $\alpha_{i,j} = \beta_{i,j} = \gamma_{i,j} = 1$



Random tiling with uniform weights of size 5, 50, 500 (generated with code kindly provided by Christophe Charlier)

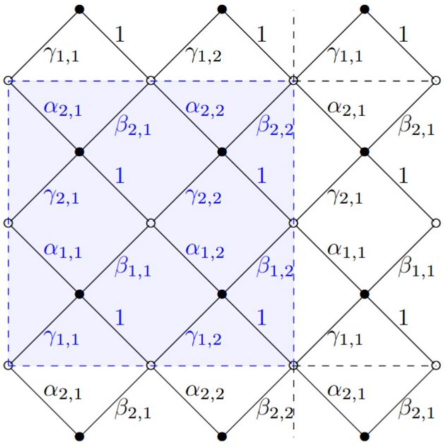
[Jockusch, Propp, Shor '98]: **Arctic Circle Theorem**

$\Rightarrow$ **What about <u>nonuniform</u> weights?**

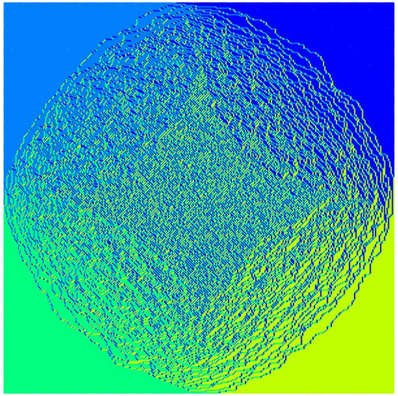
- $k \times \ell$-periodic Aztec diamonds were studied by [Berggren, Borodin '25]
- **Idea**: Distribute the weights in a *doubly periodic fashion*:

$$\alpha_{i+km,j+\ell n} = \alpha_{i,j}$$
$$\beta_{i+km,j+\ell n} = \beta_{i,j}, \qquad m, n \in \mathbb{Z}$$
$$\gamma_{i+km,j+\ell n} = \gamma_{i,j}$$

- Example: a 2×2 weighting of a 3×3-graph

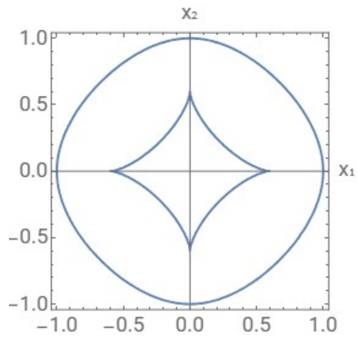


Tilings of $k \times \ell$-periodic models of the Aztec diamond exhibit three types of regions: **frozen, rough, smooth**.



- **frozen**: dominos are perfectly correlated (no randomness),
- **rough**: domino correlations decay quadratically with distance,
- **smooth**: domino correlations decay exponentially with distance.

Classification goes back to the work of [Kenyon, Okounkov, Sheffield '06].
$\Rightarrow$ **deep connection to algebraic geometry**



We define $\mathcal{AC}$ to be

$$\mathcal{AC} = \{(x_1, x_2) \in [-1,1]^2 \colon (x_1, x_2) \text{ is on the boundary of the rough region}\}.$$

Corollary ([Berggren, Borodin '25])

There exist (explicit) meromorphic differentials η_j, $j = 0, 1, 2$ on some compact Riemann surface $\mathcal{R}$ of genus g (the spectral curve) such that

$$\mathcal{AC} = \{(x_1, x_2) \in [-1,1]^2 \colon \eta(x_1, x_2) = \eta_0 + x_1\eta_1 + x_2\eta_2 \textit{ has a higher order zero}\}.$$

$\Rightarrow$ we need to define a 'discriminant-like object' Δ on the Riemann surface $\mathcal{R}$.

Genus 0-case: $p(z) = az^2 + bz + c \quad \Rightarrow \quad z_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$, with $\Delta = \Delta(a, b, c) = b^2 - 4ac$.

Theorem ([P. '26])

Let $\eta_0, \dots, \eta_n$ be linearly independent meromorphic sections of some holomorphic line bundle on a compact Riemann surface $\mathcal{R}$ of genus $\mathbf{g}$. Then the corresponding discriminant Δ of $x_0\eta_0 + \cdots + x_n\eta_n$ exists and can be constructed using Riemann theta functions.

Moreover, if $x_0\eta_0 + \cdots + x_n\eta_n$ has $\mathbf{z}$ zeros, then (as a polynomial in $x_0, \dots, x_n$)

$$\deg \Delta = 2\mathbf{z} + 2\mathbf{g} - 2.$$

As shown in [Berggren, Borodin '25], we have

- The genus $\mathbf{g}$ is equal to the **number of smooth regions**.
- The number of poles $\mathbf{p}$ of $\eta(x_1, x_2)$ equals the **number frozen regions**.

$\Rightarrow \mathbf{z} = \mathbf{p} + 2\mathbf{g} - 2$ as η is a meromorphic differential.

$\Rightarrow \deg \Delta = 2\mathbf{z} + 2\mathbf{g} - 2 = \underline{6\mathbf{g} + 2\mathbf{p} - 6}.$

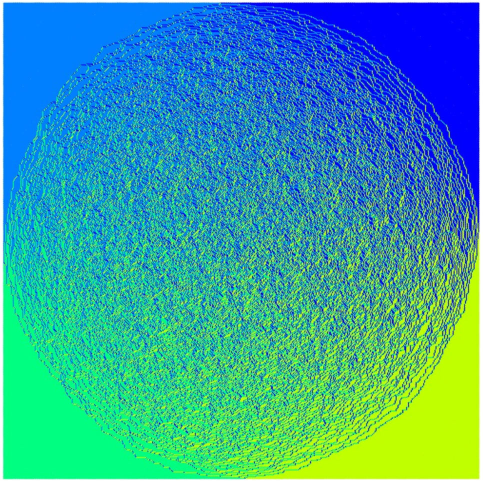
Ùsing [Berggren, Borodin '25] we obtain the following **generalization of the arctic circle theorem**.

Theorem ([P. '26])

For a $k \times \ell$-periodic model of the Aztec diamond we have

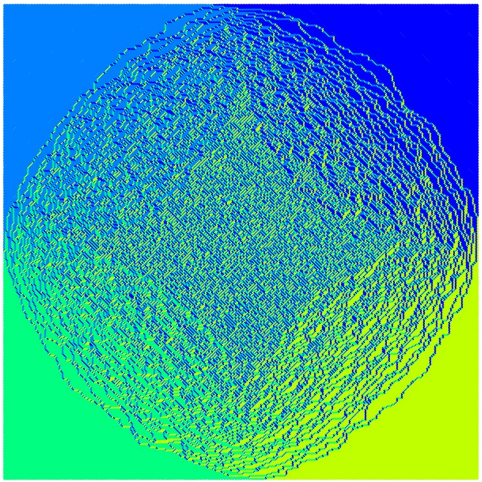
$$\textbf{\textit{degree of arctic curve}} = 6 \cdot \#\textbf{\textit{smooth regions}} + 2 \cdot \#\textbf{\textit{frozen regions}} - 6$$

degree of arctic curve $= 6 \cdot \#$**smooth regions** $+ 2 \cdot \#$**frozen regions** $- 6$



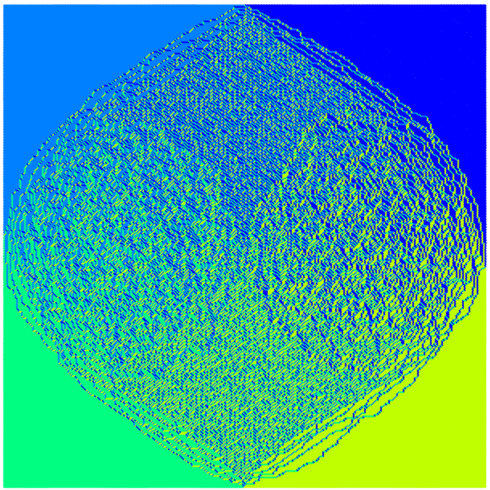
- number of **smooth** regions $= 0$,
- number of **frozen** regions $= 4$,
- degree: $6 \cdot 0 + 2 \cdot 4 - 6 = 2$
  $\Rightarrow$ arctic circle.

degree of arctic curve $= 6 \cdot \#$**smooth regions** $+ 2 \cdot \#$**frozen regions** $- 6$



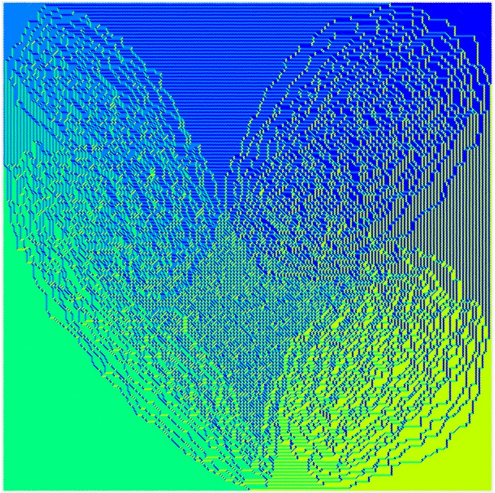
- number of **smooth** regions $= 1$,
- number of **frozen** regions $= 4$,
- degree: $6 \cdot 1 + 2 \cdot 4 - 6 = 8$
  $\Rightarrow$ octic circle.

degree of arctic curve $= 6 \cdot \#$**smooth regions** $+ 2 \cdot \#$**frozen regions** $- 6$



- number of **smooth** regions $= 2$,
- number of **frozen** regions $= 4$,
- degree: $6 \cdot 2 + 2 \cdot 4 - 6 = 14$

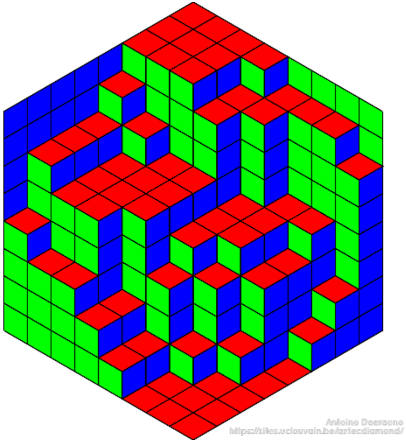
degree of arctic curve $= 6 \cdot \#$**smooth regions** $+ 2 \cdot \#$**frozen regions** $- 6$



- number of **smooth** regions $= 1$,
- number of **frozen** regions $= 8$,
- degree: $6 \cdot 1 + 2 \cdot 8 - 6 = 16$

Using the recent work of [Bobenko, Bobenko '25], these degrees formulas can be extended to

- tiling models of the **hexagon** (and Aztec diamond) with **quasi-periodic weights** (or Fock weights),



Corollary (P '26)

For quasi-periodic models of the hexagon studied in [Bobenko, Bobenko '25] having a smooth region with 6 cusps, the degree of the arctic curve is given by:

$$\textbf{\textit{degree of arctic curve}} = 6 \cdot \#\textbf{\textit{smooth regions}} + 2 \cdot \#\textbf{\textit{frozen regions}} - 6.$$

For models without such a 6-cusp smooth region we have

$$\textbf{\textit{degree of arctic curve}} = 6 \cdot \#\textbf{\textit{smooth regions}} + 2 \cdot \#\textbf{\textit{frozen regions}} - 10.$$

Note the change in the constant term!

Model without 6-cusp smooth region

degree of arctic curve $= 6 \cdot \#$**smooth regions** $+ 2 \cdot \#$**frozen regions** $- 10$

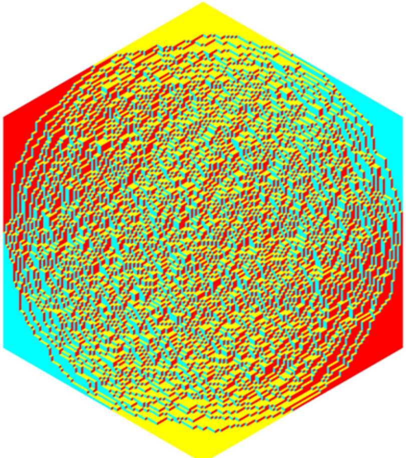


Figure. Image produced by Christophe Charlier

- number of **smooth** regions $= 0$,
- number of **frozen** regions $= 6$,
- degree: $6 \cdot 0 + 2 \cdot 6 - \mathbf{10} = 2$
  $\Rightarrow$ arctic circle.

MODEL WITH SMOOTH REGION HAVING 6 CUSPS (3×3-PERIODIC)

$$\textbf{degree of arctic curve} = 6 \cdot \#\textbf{smooth regions} + 2 \cdot \#\textbf{frozen regions} - 6$$

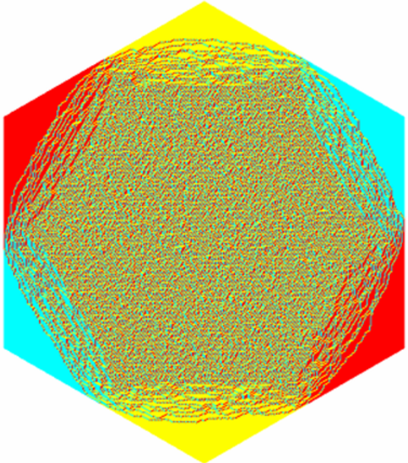


Figure. Taken from [Kuijlaars '25]

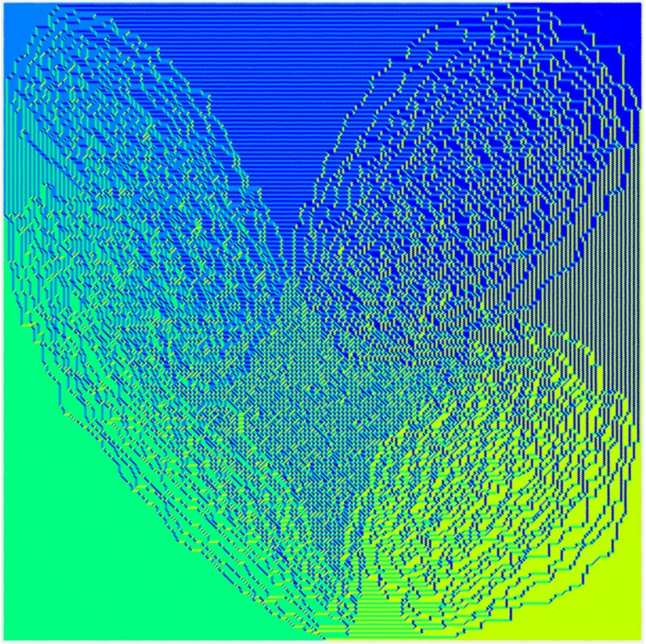
- number of **smooth** regions $= 1$,
- number of **frozen** regions $= 6$,
- degree: $6 \cdot 1 + 2 \cdot 6 - 6 = 12$

T. Berggren and A. Borodin, *Geometry of the doubly periodic Aztec dimer model*, Comm. Amer. Math. Soc. 5, 475–570 (2025).

A.I. Bobenko and N. Bobenko, *Dimers and M-curves: Limit Shapes from Riemann Surfaces*, to appear in Duke Math. Journal.

A.B.J Kuijlaars and M. P., *Wiener-Hopf factorizations and matrix-valued orthogonal polynomials*, Probab. Math. Phys. **6**(2), 547–580 (2025).

M. P., *Arctic curves of periodic dimer models and generalized discriminants*, Selecta Mathematica, accepted.



Thank you for your attention!