EXERCISES: CRYSTALS AND SYMMETRIC FUNCTIONS

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1. Background

We briefly summarize some definitions associated to crystals that will be helpful for the exercises. Further details can be found in [1].

1.1. **Kashiwara crystals.** Let $\mathfrak{sl}_n$ be the Lie algebra of type A_{n-1}, and let $I = \{1, 2, \dots, n-1\}$. We denote the simple roots by $\{\alpha_i \mid i \in I\}$, the simple coroots by $\{\alpha_i^\vee \mid i \in I\}$, and the fundamental weights by $\{\Lambda_i \mid i \in I\}$. Let Λ denote the integral span of the fundamental weights.

Definition 1.1. A *Kashiwara $U_q(\mathfrak{sl}_n)$-crystal* consists of a nonempty set B together with maps

$$e_i, f_i \colon B \to B \sqcup \{0\},$$
$$\mathsf{wt} \colon B \to \Lambda,$$

for each $i \in I$. For $x \in B$, define the string lengths $\varepsilon_i, \varphi_i \colon B \to \mathbb{Z}_{\geqslant 0}$ by

$$\varepsilon_i(x) = \max\{k \mid e_i^k(x) \neq 0\},$$
$$\varphi_i(x) = \max\{k \mid f_i^k(x) \neq 0\}.$$

We require that the following conditions hold for all $i \in I$:

(1) $f_i(x) = y$ if and only if $e_i(y) = x$ for all $x, y \in B$;
(2) $\mathsf{wt}(f_i(x)) = \mathsf{wt}(x) - \alpha_i$ for all $x \in B$ such that $f_i(x) \neq 0$;
(3) $\varphi_i(x) = \varepsilon_i(x) + \langle \alpha_i^\vee, \mathsf{wt}(x) \rangle$ for all $x \in B$.

Crystals associated to a $U_q(\mathfrak{sl}_n)$-representation need to satisfy further conditions, for example the Stembridge axioms [2].

1.2. **Tensor products of crystals.** If B and C are two crystals associated to the same root system, then we define the *tensor product* $B \otimes C$. As a set, it is the Cartesian product, but we denote the ordered pair (x, y) with $x \in B$ and $y \in C$ by $x \otimes y$. We define $\mathsf{wt}(x \otimes y) = \mathsf{wt}(x) + \mathsf{wt}(y)$,

$$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y & \text{if } \varphi_i(y) \leqslant \varepsilon_i(x), \\ x \otimes f_i(y) & \text{if } \varphi_i(y) > \varepsilon_i(x), \end{cases}$$

and

$$e_i(x \otimes y) = \begin{cases} e_i(x) \otimes y & \text{if } \varphi_i(y) < \varepsilon_i(x), \\ x \otimes e_i(y) & \text{if } \varphi_i(y) \geqslant \varepsilon_i(x). \end{cases}$$

It is understood that $x \otimes 0 = 0 \otimes x = 0$.

Exercise 1. Let

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$$

be the standard crystal $B_\square$ of type A_2 and let $\mathsf{wt}(\boxed{i}) = \mathbf{e}_i$.

(1) Compute the connected components of $B_\square^{\otimes 3}$.
(2) Characterize the element in the component of $B_\square^{\otimes 2}$ containing the element $\boxed{2} \otimes \boxed{1}$.
(3) How can you characterize the elements in $\left(B_\square^{\otimes 2}\right) \otimes \left(B_\square^{\otimes 2}\right)$ containing the element

$$\left(\boxed{2} \otimes \boxed{1}\right) \otimes \left(\boxed{2} \otimes \boxed{1}\right).$$

Exercise 2. Let $B_\square$ be the crystal

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{\bar{r}} \xrightarrow{r-1} \cdots \xrightarrow{1} \boxed{\bar{1}}$$

of type C_r with $\mathsf{wt}(\boxed{i}) = \mathbf{e}_i$ and $\mathsf{wt}(\boxed{\bar{i}}) = -\mathbf{e}_i$.

(1) Write down the simple roots for type C_r and check that $B_\square$ is a Kashiwara crystal.
(2) Characterize the elements in $B_\square^{\otimes 2}$ in type C_2 containing $\boxed{2} \otimes \boxed{1}$.
(3) In general, can you character the elements in $B_\square^{\otimes k}$ of type C_r containing

$$\boxed{k} \otimes \boxed{k-1} \otimes \cdots \otimes \boxed{1}\,?$$

Exercise 3. Let B be a Kashiwara crystal (corresponding to a representation) with index set I. For $b \in B$ and $i \in I$, define $k = \langle \alpha_i^\vee, \mathsf{wt}(b) \rangle$. Define

$$s_i(b) := \begin{cases} f_i^k(b) & \text{if } k \geqslant 0, \\ e_i^{-k}(b) & \text{if } k < 0. \end{cases}$$

(1) Show that
$$\mathsf{wt}(s_i(b)) = r_{\alpha_i}(\mathsf{wt}(b)),$$
where $r_{\alpha_i}(\mu) = \mu - \langle \alpha_i^\vee, \mu \rangle \, \alpha_i$.
(2) Show that the character of the crystal
$$\chi_B(x) = \sum_{b \in B} x^{\mathsf{wt}(b)}.$$
is invariant under the Weyl group generated by r_{α_i} for $i \in I$.
(3) For type A_{r-1}, the crystal B_λ consists of all semistandard Young tableaux of shape λ in the alphabet $\{1, 2, \ldots, r\}$. Show that in this case the character is a symmetric polynomial, that is, is invariant under interchanging x_i and x_{i+1} for all $1 \leqslant i < r$.

References

[1] Daniel Bump and Anne Schilling. *Crystal bases*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. Representations and combinatorics.

[2] John R. Stembridge. A local characterization of simply-laced crystals. *Trans. Amer. Math. Soc.*, 355(12):4807–4823, 2003.

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