

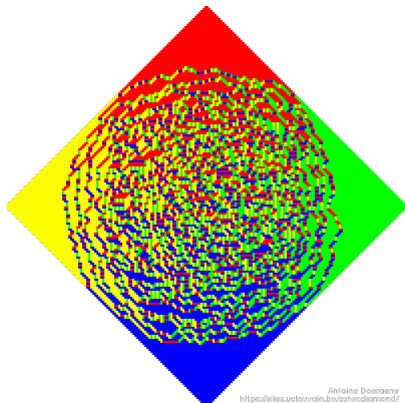
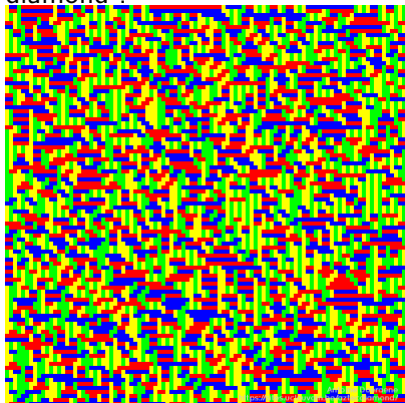
Aztec houses

Nys Charly

June 14, 2026

Introduction

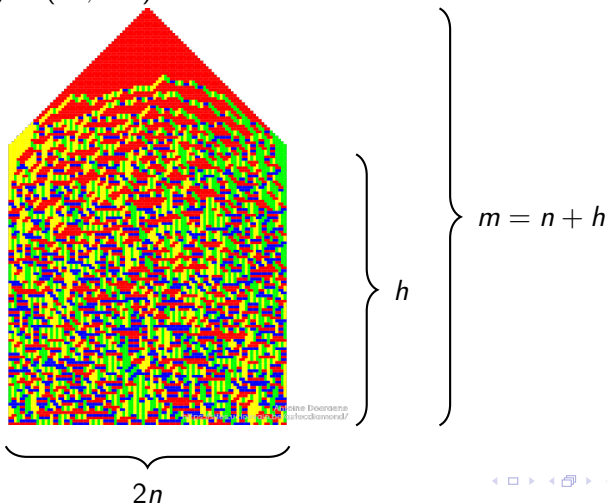
You probably all know about the rectangle tiling and the Aztec diamond :



Anoine Demanez
<https://ales.uct.ac.za/~ales/aztec/diamond/>

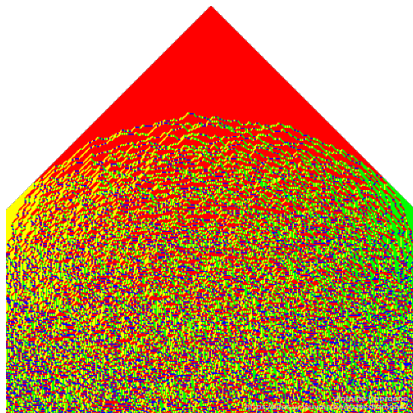
The Aztec House

What I am working on is an inbetween of those two : the Aztec House. The base is rectangular, with width $2n$ and height h , while the top resembles the upper half of a diamond of order n . Here, $(n, h) = (50, 100)$.



The Aztec house

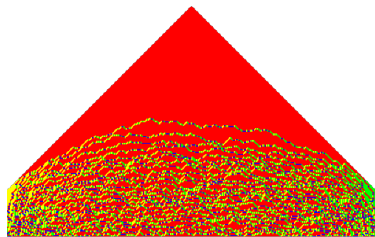
Here is a larger example, of order $(200,200)$.



Note that we observe three frozen regions: two small ones on the sides and a larger one at the top.

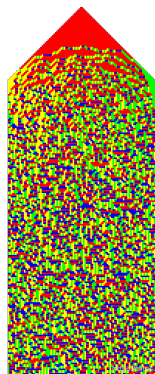
The Aztec house

Others obtained by playing with the parameters n and h .



Antoine Douaune
http://antoinedouaune.fr/

Figure: order : $(n, h) = (200, 50)$



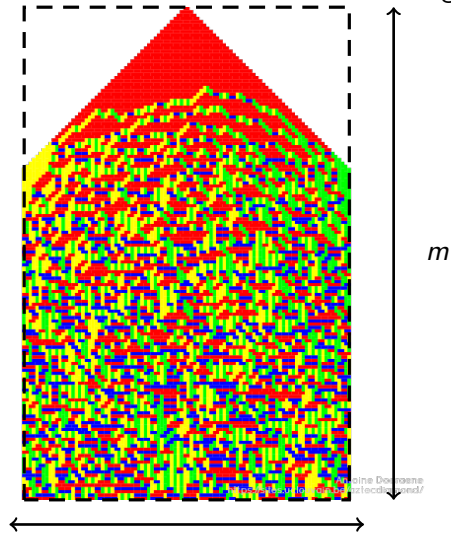
Antoine Douaune
http://antoinedouaune.fr/

Figure: order : $(n, h) = (50, 200)$

Again, we observe some arctic curves.

The Aztec house

Every Aztec house can be embedded in a minimal rectangle of



order $(2n, m)$:

$2n$

Formalism

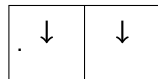
My bigger tool to reach that goal is Lieb's transfer matrix (1967)³, and some formalism from quantum physics.

Instead of using colors, I will code my domino configuration with arrows. Each domino has two cells. I will draw an arrow in those

two, as follows :

↓
↑

and



To one arrow, I associate an vector :

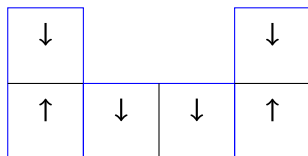
$$\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2 \text{ and } \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2.$$

Formalism

In summary, we have 3 ways to describe our Aztec house :

dominos \Leftrightarrow arrows $\Leftrightarrow (C^2)^{\otimes n}$ for each row

As an example, the drawing's first line can be coded as



$$\begin{aligned} &= |\uparrow\downarrow\downarrow\uparrow\rangle \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

Lieb's transfer matrix

It would be interesting to find a matrix that, starting from a vector, i.e., a configuration, would generate all possible configurations of the upper row. This matrix does indeed exist and is called Lieb's transfer matrix (1967). If $|x\rangle$ is a configuration, then

$$T|x\rangle = |y_1\rangle + |y_2\rangle + |y_3\rangle + \dots \quad (1)$$

where the states $|y_i\rangle$ correspond to all possible configurations for the next line.

Example

If we take the previous picture, then

$$T|\uparrow\downarrow\downarrow\uparrow\rangle = |\downarrow\downarrow\downarrow\downarrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle. \quad (2)$$

This is what we were expecting, for after $|\uparrow\downarrow\downarrow\uparrow\rangle$, the only 2 compatible configurations are :

↓	↓	↓	↓
↑	↓	↓	↑

One lying in the middle, rest being forced.

↓	↑	↑	↓
↑	↓	↓	↑

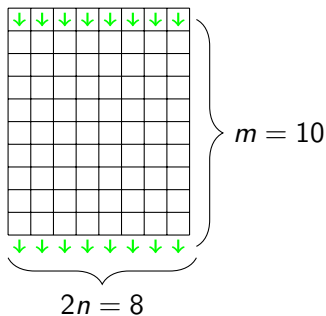
Two standing in the middle, rest being forced.

Some information

Before going further, we need some information/tools. First, we introduce the following quantity, derived directly from the eigenvalues of the Transfer matrix, for $1 \leq a \leq n$:

$$\mu_a = \sqrt{1 + \cos^2 \frac{\pi a}{2n+1}} + \cos \frac{\pi a}{2n+1}. \quad (3)$$

Second is how to describe a tiling. We have a clear example here of how to describe a rectangle with arrows.



Partition functions

Why is T so important? T can help us compute some partition functions. For example, the partition function of a rectangle of width $2n$ and height m is given by

$$Z_{2n,m} = \langle \downarrow \dots \downarrow | T^m | \downarrow \dots \downarrow \rangle \quad (4)$$

$$= \prod_{a=1}^n \frac{\mu_a^{m+1} - (-\mu_a)^{-(m+1)}}{\mu_a + \mu_a^{-1}} \quad (5)$$

$$= \prod_{a=1}^n G_a(m+1). \quad (6)$$

This product coincides with Kasteleyn's formula (1961)⁴

$$Z_{2n,m} = \prod_{a=1}^n \prod_{b=1}^m \sqrt{4 \cos^2\left(\frac{\pi a}{2n+1}\right) + 4 \cos^2\left(\frac{\pi b}{m+1}\right)}$$

Partition functions

The case of the Aztec houses of order (n, h) is slightly different. I have to define $|s_J\rangle$ as some initial condition (that I will not detail here), of size $2n$, such that the number of all Aztec houses of order (n, h) is given by

$$AH_{n,h} = \langle \downarrow^{\otimes 2n} | T^{n+h} | s_J \rangle \quad (7)$$

$$= \langle \downarrow^{\otimes 2n} | T^m | s_J \rangle \quad (8)$$

$$= \left(\frac{4}{2n+1} \right)^{r/2} \prod_{a=1}^n G_a(m+1) \det(SDS^t) \quad (9)$$

$$= Z_{2n,m} \left(\frac{4}{2n+1} \right)^{r/2} \det(SDS^t), \quad (10)$$

where $r = 2\lfloor n/2 \rfloor$.

This can be interpreted as "partition function of the rectangle" time a deformation term.

Probability of an \uparrow

Beyond the artic curve, all the dominoes are lying horizontally (red region) or standing (sides regions). In other words, the probability of having a standing domino is either 0 or 1. The key to compute this probability is to know how many configurations has an \uparrow at position (x, y) . This number is given by the operator :

$$T^{m-y} \sigma_x^+ \sigma_x^- T^y \quad (11)$$

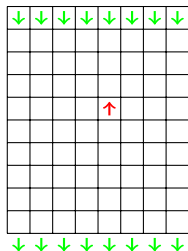
The operator $\sigma_x^+ \sigma_x^-$ annihilates all the configurations without an \uparrow at (x, y) .

Probability of an \uparrow in the rectangle

For the rectangle, we have,

$$P_{Z_{2n,m}}[(x, y) = \uparrow] = \frac{\langle \downarrow^{\otimes 2n} | T^{m-y} \sigma_x^+ \sigma_x^- T^y | \downarrow^{\otimes 2n} \rangle}{\langle \downarrow^{\otimes 2n} | T^m | \downarrow^{\otimes 2n} \rangle} \quad (12)$$

$$= \left(\frac{4}{2n+1} \right) \sum_{a=1}^n \sin^2 \left(\frac{\pi a x}{2n+1} \right) \frac{G_a(y) G_a(m-y)}{G_a(m+1)} \quad (13)$$



$(2n, m) = (8, 10)$ and $(x, y) = (5, 6)$

It is possible to show that, in the middle of the rectangle, this quantity tends to $1/4$.

Probability of an \uparrow in the AH

We can now compute the probability to have an \uparrow at position (x, y) in the rectangle and in the Aztec House :

$$P_{AH_{n,h}}[(x, y) = \uparrow] = \frac{\langle \downarrow^{\otimes 2n} | T^{m-y} \sigma_x^+ \sigma_x^- T^y | S_J \rangle}{\langle \downarrow^{\otimes 2n} | T^m | S_J \rangle} \quad (14)$$

$$= \left(\frac{4}{2n+1} \right) \left[\text{tr}((SDS^t)^{-1} SES^t) \right] \quad (15)$$

$$+ \sum_{a=1}^n \sin^2 \left(\frac{\pi ax}{2n+1} \right) \frac{G_a(y) G_a(m-y)}{G_a(m+1)} \quad (16)$$

$$= \left(\frac{4}{2n+1} \right) \text{tr}[(SDS^t)^{-1} SES^t] + P_{Z_{2n,m}}[(x, y) = \uparrow] \quad (17)$$

So, the probability of an \uparrow in an $AH_{n,h}$ is the probability of an \uparrow in a $Z_{2n,m}$ plus a deformation term.

Conclusion

The conclusion of this talk is

- ▶ Lieb's transfer matrix allows us to predict all the configurations.
- ▶ The behaviour of the Aztec house is deeply related to the one of the embedding minimal rectangle.

Thank you !

Thank you !

Questions

Any questions ?

Additional information

The different matrices used are defined here as :

$$S_{jk} := \sin\left(\frac{\pi jk}{2n+1}\right), j \in J \text{ for some set } J \text{ of size } r, \text{ and } k = 1, \dots, n. \quad (18)$$

$$D_{a,b} := (-1)^a \frac{G_a(m)}{G_a(m+1)} \delta_{a,b}, \text{ is a } n\text{-diagonal matrix} \quad (19)$$

$$E_{a,b} := (-1)^b \sin\left(\frac{\pi xa}{2n+1}\right) \sin\left(\frac{\pi xb}{2n+1}\right) \quad (20)$$

$$\times \begin{cases} \frac{M_a(m-y)P_b(m+1-y)}{P_a(m+1)P_b(m+1)} & \text{if } (-1)^{n+x} = 1 \\ \frac{M_b(m-y)P_b(m+1-y)}{P_b(m+1)P_a(m+1)} & \text{if } (-1)^{n+x} = -1 \end{cases} \quad (21)$$

Bibliography

- [1] Philippe Ruelle (2025).
Counting the Domino Tilings of Aztec Diamond-Like Domains (Working Notes).
- [2] Arthur Doeraene.
*[https://sites.uclouvain.be/aztecdiamond/
domino-shuffling-implementation.html](https://sites.uclouvain.be/aztecdiamond/domino-shuffling-implementation.html)*
- [3] Elliott H. Lieb (1967).
Residual Entropy of Square Ice.
Physical Review 162(1), 162–172.
- [4] P. W. Kasteleyn (1961).
The Statistics of Dimers on a Lattice I. The Number of Dimer Arrangements on a Quadratic Lattice.
Physica 27, 1209–1225.