Domino tilings of the Aztec diamond in random environment

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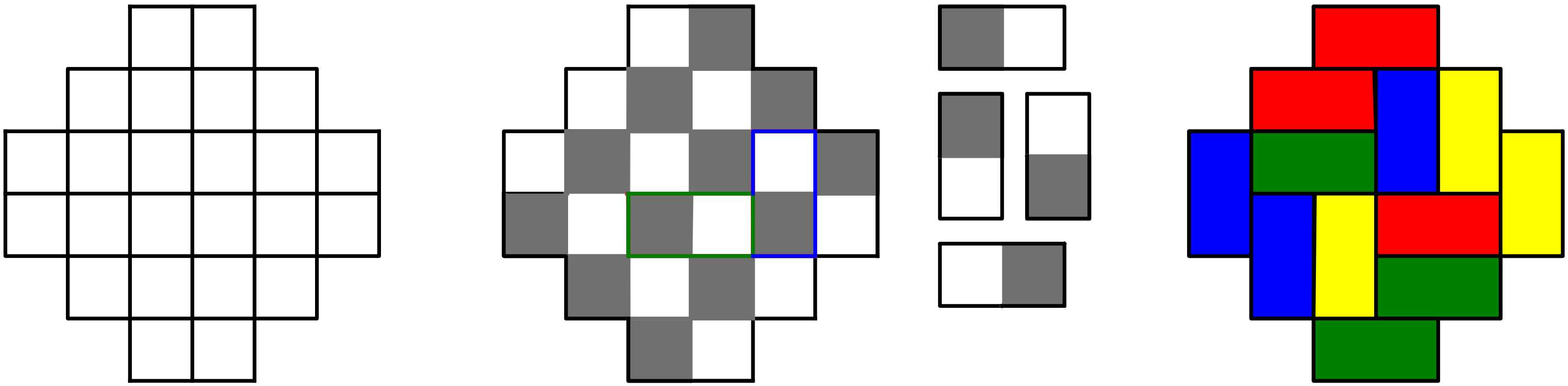
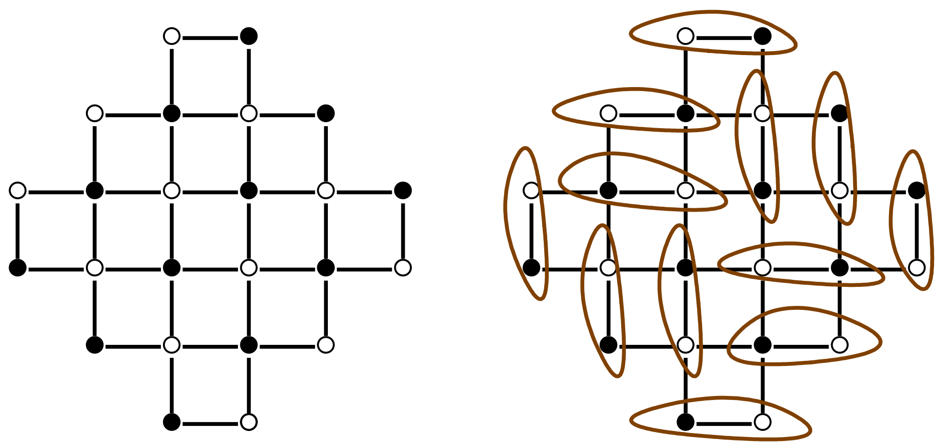
Domino tilings of the Aztec diamond were first introduced by Elkies-Kuperberg-Larsen-Propp'92.





This is a very well-studied object: Jockusch-Propp-Schor'98, Kenyon'00, Cohn-Kenyon-Propp'01, Johansson'03, Kenyon-Okounkov-Sheffield'06, Kenyon-Okounkov'07, Chhita-Johansson-Young'12, ...

The Disordered Aztec diamond

We are interested in the Aztec diamond with random edge weights.

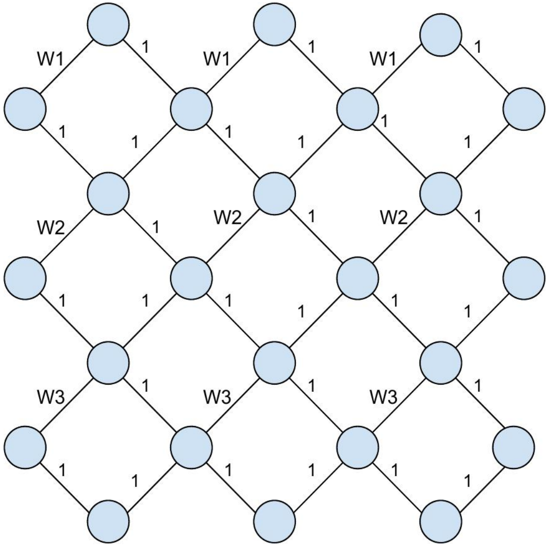
Physics literature: There are studies for the Aztec diamond with all edge weights being chosen i.i.d. Perret-Ristivojevic-Le Doussal-Schehr-Wiese'12. This model is conjectured to belong in a broader class of random interfaces that have "super-rough" fluctuation.

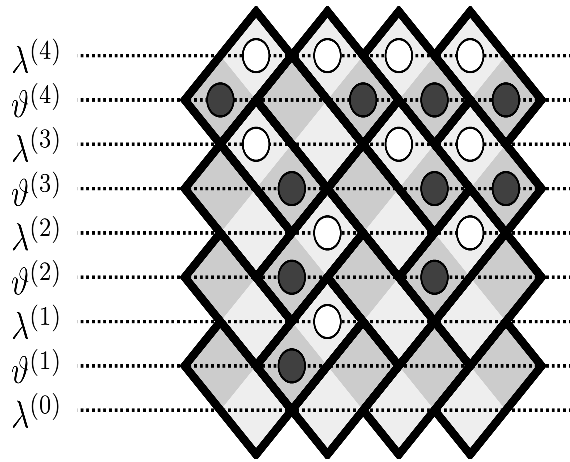
Mathematics literature: Disordered dimer models on the square lattice — the random weights either repeat in one direction Bufetov-Petrov-Z.'25, Moulard-Toninelli'25, Z.'25, or, they are all i.i.d. and Gamma-distributed Duits-Van Peski'25. In both cases despite the disorder integrable structure is preserved.

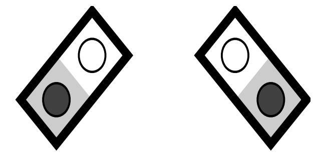
In the context of random environment it is common to distinguish between two types of expectations:

- Quenched: We first fix the random environment and then compute expectations.
- Annealed: We average in both over the randomness in the tilings and the randomness in the environment.

$$\sum_{i=1}^{N} f\left(\frac{\lambda_i^{(N)}+N-i}{N}\right) - \mathbb{E}\left[f\left(\frac{\lambda_i^{(N)}+N-i}{N}\right)\right] \Rightarrow ?, \quad \frac{N}{M} \to \eta$$

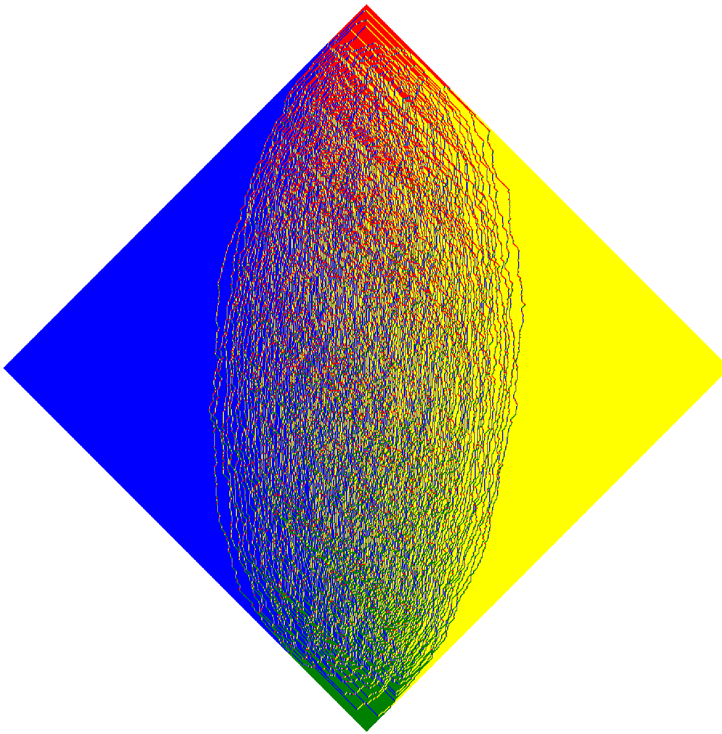


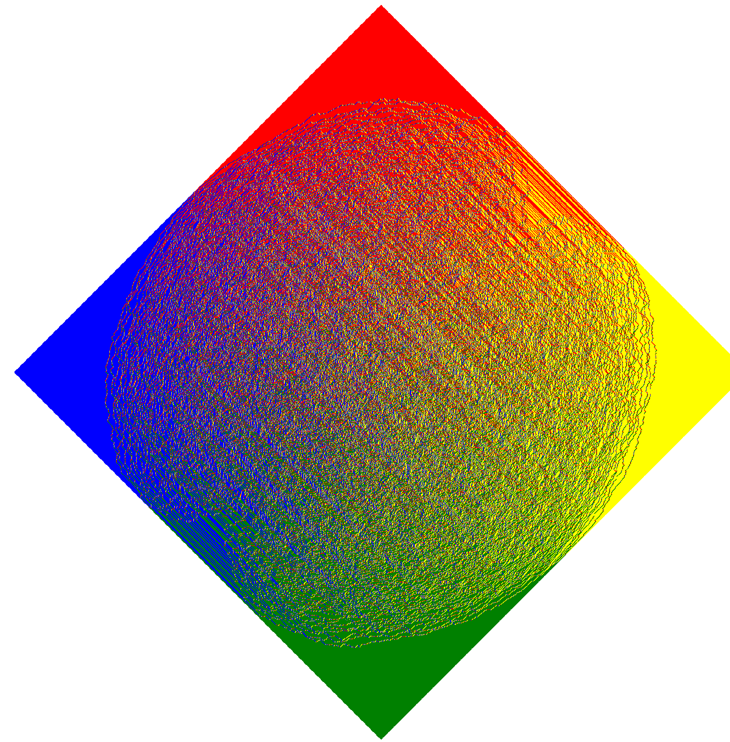




Let us consider $w_i = \frac{\beta_i}{1-\beta_i}$, where $\beta_1, \ldots, \beta_M$ are random.

- Domino tilings of the Aztec diamond of size M are in bijection with sequences of partitions $\{\lambda^{(M)}, \theta^{(M)}, \ldots, \lambda^{(1)}, \theta^{(1)}\}$ that satisfy certain interlacing properties.





Theorem Bufetov-Petrov-Z.'25 (independent edge weights)

In the case of decreasing variance ($\lim_{M\to\infty}\mathbb{E}[\beta_1]=b$ and $\lim_{M\to\infty}M\mathrm{Var}(\beta_1)=\sigma^2$), the collection

$$(p_k^{(\lfloor \eta M\rfloor)}-\mathbb{E}[p_k^{(\lfloor \eta M\rfloor)}])_{k\in\mathbb{Z}_{>0},\eta\in(0,1)}, \quad \text{where} \quad p_k^{(N)}=\sum_{i=1}^{N}\left(\frac{\lambda_i^{(N)}+N-i}{N}\right)^k,$$

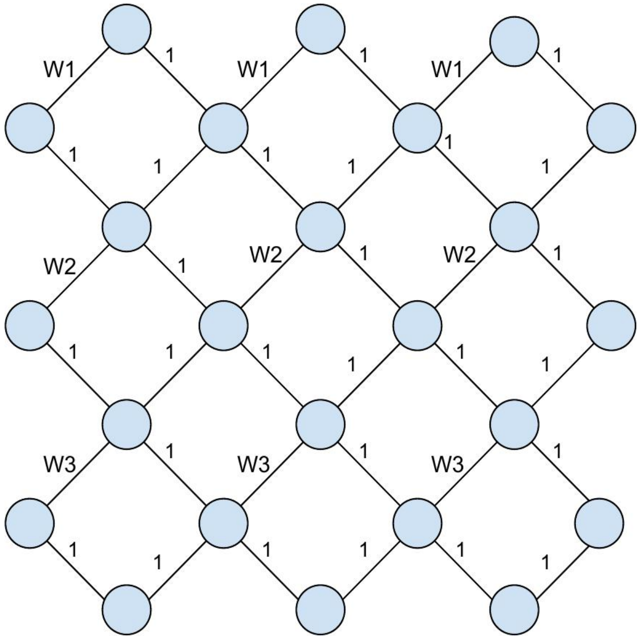
converges to a jointly Gaussian distribution with covariance given by

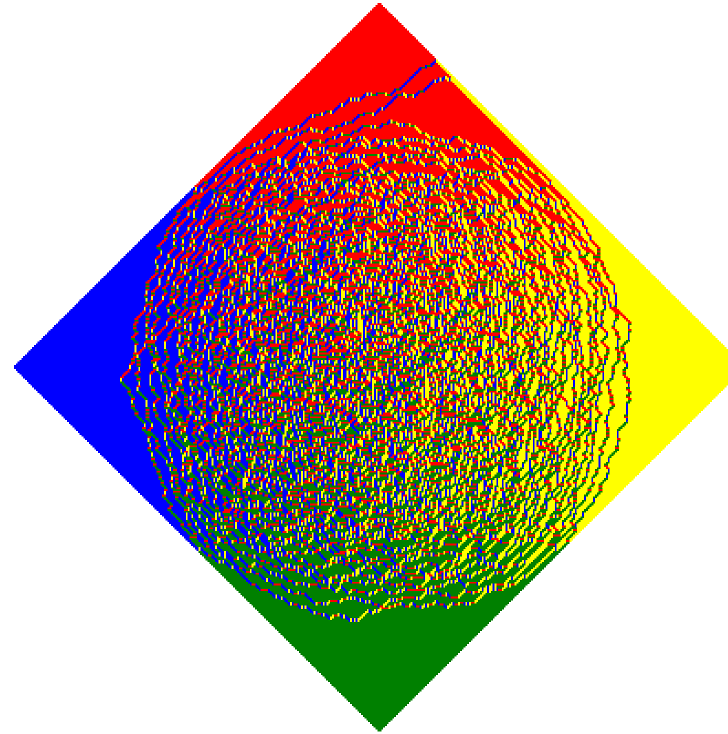
$$\lim_{M\to\infty}\mathrm{Cov}(p_{k_1}^{(\lfloor \eta_1 M\rfloor)},p_{k_2}^{(\lfloor \eta_2 M\rfloor)})=\frac{1}{(2\pi \mathrm{i})^2}\oint_{|w|=\varepsilon}\oint_{|z|=2\varepsilon}\left(\frac{\eta_1}{z}+\eta_1+\frac{(1+z)(1-\eta_1)b}{1+bz}\right)^{k_1}$$

$$\times\left(\frac{\eta_2}{w}+\eta_2+\frac{(1+w)(1-\eta_2)b}{1+bw}\right)^{k_2}\left(\frac{\min\{1-\eta_1,1-\eta_2\}\sigma^2}{(1+bz)^2(1+bw)^2}+\frac{1}{(z-w)^2}\right)dzdw,$$

Theorem Bufetov-Petrov-Z.'25 (independent edge weights)

In the case where the probability distribution of β_i does not depend on M, the collection $(\sqrt{M}(p_k^{(\lfloor \eta M\rfloor)}-\mathbb{E}[p_k^{(\lfloor \eta M\rfloor)}]))_{k\in\mathbb{Z}_{>0},\eta\in(0,1)}$ converges to a standard Brownian motion.





Let $X \in \{0,\dots,M\}$ and $a,b \in (0,1)$. Consider the edge weights

$$w_M = w_{M-1} = \cdots = w_{M-X+1} = \frac{a}{1-a} \qquad \text{and} \qquad w_{M-X} = \cdots = w_1 = \frac{b}{1-b}.$$

Bufetov-Z.'24: Discrete analog of the Baik-Ben Arous-Péché phase transition from random matrices.

Theorem Z.'25: Assume that X has binomial distribution with parameter p_M, such that $Mp_M \to l$. Then the collection $(p_k^{(\lfloor \eta M \rfloor)} - \mathbb{E}[p_k^{(\lfloor \eta M \rfloor)}])_{k \in \mathbb{Z}_{>0}, \eta \in (0,1)}$ converges in the sense of moments:

$$\lim_{M\to\infty} \mathbb{E}[(p_{k_1}^{(\lfloor \eta_1 M \rfloor)} - \mathbb{E}[p_{k_1}^{(\lfloor \eta_1 M \rfloor)}]) \ldots (p_{k_\nu}^{(\lfloor \eta_\nu M \rfloor)} - \mathbb{E}[p_{k_\nu}^{(\lfloor \eta_\nu M \rfloor)}])]$$

$$= \mathbb{E}[(\boldsymbol{G}_1 + c_1(\boldsymbol{P} - l)) \ldots (\boldsymbol{G}_\nu + c_\nu(\boldsymbol{P} - l))],$$

where $(\boldsymbol{G}_1, \ldots, \boldsymbol{G}_\nu)$ is a Gaussian vector and $\boldsymbol{P}$ is Poisson(l). Moreover, $(\boldsymbol{G}_1, \ldots, \boldsymbol{G}_\nu)$, $\boldsymbol{P}$ are independent and $c_1, \ldots, c_\nu$ are constants given by

$$c_i = \frac{1}{2\pi i} \oint_0 \left(\frac{\eta_i}{z} + \eta_i + \frac{(1+z)(1-\eta_i)b}{1+bz} \right)^{k_i} \left(\frac{a}{1+az} - \frac{b}{1+bz} \right) dz.$$

Theorem Z.'25 (quenched CLT for independent edge weights)

In the case where the distribution of β_i does not depend on M, for almost every realization of the disorder, the vector $(p_k^{(\lfloor \eta M \rfloor)} - \mathbb{E}_\lambda[p_k^{(\lfloor \eta M \rfloor)}])_{k \in \mathbb{Z}_{>0}, \eta \in (0,1)}$ converges to a jointly Gaussian distribution, and

$$\lim_{M\to\infty} \mathrm{Cov}_\lambda(p_{k_1}^{\lfloor \eta_1 M \rfloor}, p_{k_2}^{\lfloor \eta_2 M \rfloor}) = \frac{1}{(2\pi i)^2} \oint_{|w|=\varepsilon} \oint_{|z|=2\varepsilon} \left((1-\eta_1)\mathbb{E}\left[\frac{\beta_1+\beta_1 z}{1+\beta_1 z}\right] + \eta_1 + \frac{\eta_1}{z} \right)^{k_1}$$

$$\left((1-\eta_2)\mathbb{E}\left[\frac{\beta_1+\beta_1 w}{1+\beta_1 w}\right] + \eta_2 + \frac{\eta_2}{w} \right)^{k_2} \frac{dzdw}{(z-w)^2}, \quad \text{almost surely.}$$