On the Schubert support of Grothendieck polynomials

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Based on joint work with Anna Weigandt (Minnesota)

Grothendieck polynomials

- Lascoux–Schützenberger ('82) introduced **Schubert polynomials** $\mathfrak{S}_w$ as polynomial representatives of $H^*(Fl_n)$.
- They also introduced their K-theoretic analog: **Grothendieck polynomials** $\mathfrak{G}_w$.
- Both polynomials are defined recursively with divided difference formulas, and the lowest degree piece of $\mathfrak{G}_w$ is $\mathfrak{S}_w$.

Example. $\mathfrak{G}_w$ for $w \in S_3$.

$$\mathfrak{G}_{321} = x_1^2 x_2$$

$$\mathfrak{G}_{312} = x_1^2 \qquad \mathfrak{G}_{231} = x_1 x_2$$

$$\mathfrak{G}_{213} = x_1 \qquad \mathfrak{G}_{132} = x_1 + x_2 + \beta x_1 x_2$$

$$\mathfrak{G}_{123} = 1$$

Monomial support

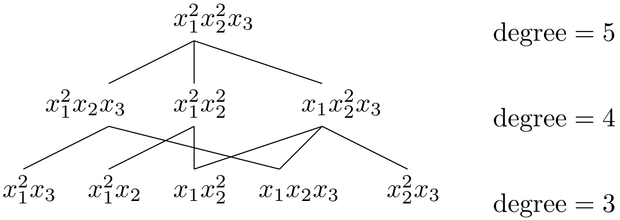
Definition

For $f = \sum_{\mathbf{a}\in\mathbb{Z}^n_{\geq 0}} c_\mathbf{a} x^\mathbf{a}$, the **monomial support** of f is

$$\mathrm{supp}(f) = \{\mathbf{a} \in \mathbb{Z}^n_{\geq 0} : c_\mathbf{a} \neq 0\}.$$

- Poset structure via component-wise comparison of exponent vectors.

Example. $\mathrm{supp}(\mathfrak{G}_{1432})$



Schubert support

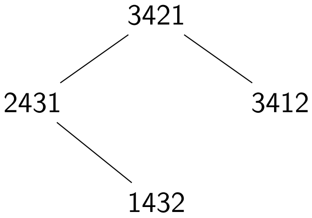
Definition

For $f = \sum_{u \in S_n} a_u \mathfrak{S}_u$, the **Schubert support** of f is

$$\mathrm{Supp}_{\mathfrak{S}}(f) := \{u \in S_n : a_u \neq 0\}.$$

- ◦ Poset structure via left weak order.

Example. $\mathrm{Supp}_{\mathfrak{S}}(\mathfrak{G}_{1432})$



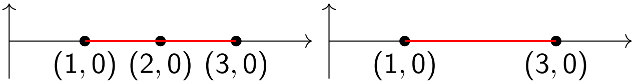
Newton polytopes

Definition

For $f = \sum_{\mathbf{a} \in \mathbb{Z}^n_{\geq 0}} c_{\mathbf{a}} x^{\mathbf{a}}$, Newton$(f)$ = convex hull of supp(f).

◦ f has **saturated Newton polytope** if $\mathbf{a} \in \text{Newton}(f) \implies c_{\mathbf{a}} \neq 0$.

Example. $f(x_1, x_2) = x_1 + x_1^2 + x_1^3$ and $g(x_1, x_2) = x_1 + x_1^3$



$\text{Newton}(f) = \text{Newton}(g) = \{(1,0), (2,0), (3,0)\}$.

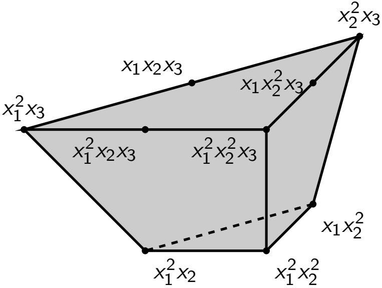
$\Rightarrow f$ has saturated Newton polytope but g does not.

Newton Polytopes for Grothendieck polynomials

Conjecture [Monical, Tokcan, and Yong, 2019]

For all $w \in S_n$, Grothendieck polynomials $\mathfrak{G}_w$ have saturated Newton polytope.

Example: $w = 1432$.



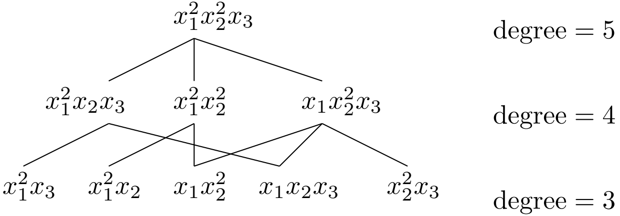
Known: Grassmannian, 0-1 Schubert, lattice-free, and others.

Monomial support conjectures

Conjecture [Mészáros, Setiabrata, and St. Dizier, 2025]

1. If $\mathbf{a} \in \mathrm{supp}(\mathfrak{G}_w)$ and $|\mathbf{a}| < \deg(\mathfrak{G}_w)$,
   then there exists $\mathbf{b} \neq \mathbf{a}$, $\mathbf{b} \in \mathrm{supp}(\mathfrak{G}_w)$ such that $\mathbf{x}^{\mathbf{a}} \mid \mathbf{x}^{\mathbf{b}}$.
2. If $\mathbf{a} \in \mathrm{supp}(\mathfrak{G}_w)$ and $|\mathbf{a}| < \deg(\mathfrak{G}_w)$,
   then there exists i such that $\mathbf{x}^{\mathbf{a}} x_i = \mathbf{x}^{\mathbf{b}}$ and $\mathbf{b} \in \mathrm{supp}(\mathfrak{G}_w)$.

Example: $w = 1432$.

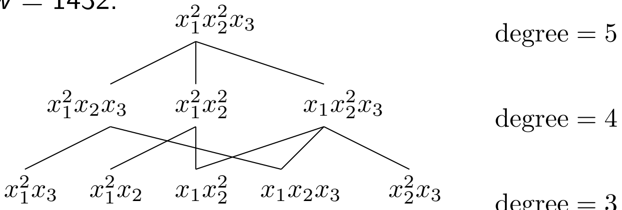


Known: Grassmannian, vexillary, fireworks, inverse fireworks, 0-1 Grothendieck, lattice-free, and others.

Conjecture [Mészáros, Setiabrata, and St. Dizier, 2025]

3. If $\mathbf{a}, \mathbf{b} \in \mathrm{supp}(\mathfrak{G}_w)$ and $\mathbf{x}^{\mathbf{a}} \mid \mathbf{x}^{\mathbf{b}}$, then $\{\mathbf{c} : \mathbf{a} \leq \mathbf{c} \leq \mathbf{b}\} \subseteq \mathrm{supp}(\mathfrak{G}_w)$.

Example: $w = 1432$.



Known: Grassmannian, vexillary, fireworks, lattice-free, and others.

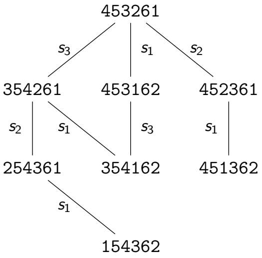
Schubert support conjecture

We conjecture Schubert support analogs:

Conjecture [C. and Weigandt, 2026+]

1. If $u \in \mathrm{Supp}_{\mathfrak{S}}(\mathfrak{G}_w)$ and $\ell(u) < \deg(\mathfrak{G}_w)$,
   then there exists $u \neq v$, $v \in \mathrm{Supp}_{\mathfrak{S}}(\mathfrak{G}_w)$ such that $u <_L v$.
2. If $u \in \mathrm{Supp}_{\mathfrak{S}}(\mathfrak{G}_w)$ and $\ell(u) < \deg(\mathfrak{G}_w)$,
   then there exists k such that $u < s_k u$ and $s_k u \in \mathrm{Supp}_{\mathfrak{S}}(\mathfrak{G}_w)$.

Example: $\mathfrak{G}_{154362}$



Remark: Conj. 1-2 imply the monomial analogs.

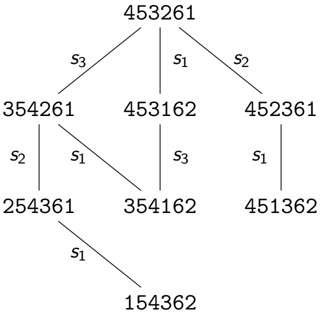
Schubert support conjectures

Conjecture [C. and Weigandt, 2026+]

3. The set $\mathrm{Supp}_{\mathfrak{S}}(\mathfrak{G}_w)$ is convex w.r.t left weak order. If $u, u' \in \mathrm{Supp}_{\mathfrak{S}}(\mathfrak{G}_w)$ and $u \leq_L u'$, then

$$[u, u']_L = \{v \in S_n : u \leq_L v \leq_L u'\} \subseteq \mathrm{Supp}_{\mathfrak{S}}(\mathfrak{G}_w).$$

Example: $\mathfrak{G}_{154362}$



Schubert support conjectures

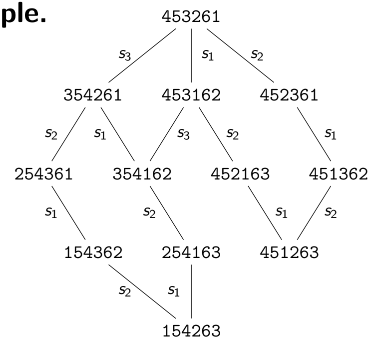
We give two more Schubert support conjectures:

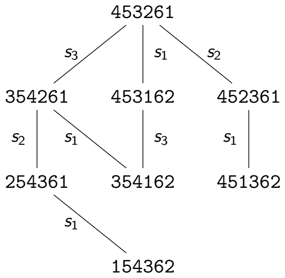
Conjecture [C. and Weigandt, 2026+]

Suppose $u \leq_L v$ and $\deg(\mathfrak{G}_u) = \deg(\mathfrak{G}_v)$.

4. Then $\mathrm{Supp}_{\mathfrak{S}}(\mathfrak{G}_v) \subseteq \mathrm{Supp}_{\mathfrak{S}}(\mathfrak{G}_u)$.
5. Then $\mathrm{Supp}_{\mathfrak{S}}(\mathfrak{G}_v) = \{w \in \mathrm{Supp}_{\mathfrak{S}}(\mathfrak{G}_u) : w \geq v \text{ in strong order}\}$.

Example.





Schubert support theorems

Theorem [C. and Weigandt, 2026+]

- Conj. 1-2 hold for inverse fireworks permutations
- Conj. 1-5 hold for Grassmannian permutations
- Conj. 1-3 hold for permutations of the form $1 \times u$, with u dominant.

Proof technique:

- Inverse fireworks: linear relations via differential operators [Pechenik, Speyer, and Weigandt, 2024]
- Grassmannian: Changing bases formula via tableaux combinatorics [Lenart, 2000]
- $1\times$ dominant: Changing bases formula via pipe dreams combinatorics [Weigandt, 2025]

Grassmannian permutations to partitions

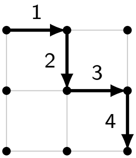
- Permutations w are **Grassmannian** if it has at most one descent.
- They correspond to partitions $\lambda(w)$, and

$$\mathfrak{G}_w(\mathbf{x}) = \mathfrak{G}_{\lambda(w)}(x_1, \ldots, x_h).$$

Idea: Reverse the first increasing sequence and subtract by $(h, h-1 \ldots, 1)$.

Example.

- Let $w = 24|13$. $\lambda(w) = (4,2) - (2,1) = (2,1)$, $h = 2$.



Changing bases formulas

Theorem [Lenart, 2000]

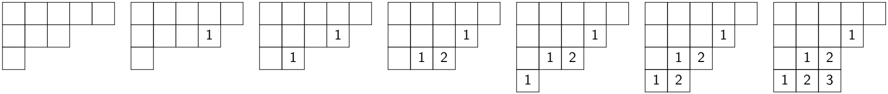
Let $\lambda = (\lambda_1, \dots, \lambda_h)$ be a partition. Then

$$\mathfrak{G}_\lambda(x_1, \dots, x_h) = \sum_{\mu \supseteq \lambda} |\mathrm{inc}(\mu/\lambda, h)| \mathfrak{S}_\mu(x_1, \dots, x_h),$$

where $\mathrm{inc}(\mu/\lambda, h) = \{$strictly increasing tableaux of shape μ/λ with at most h rows and entries in row i at most $i-1\}$

- The Schubert support $\mathrm{Supp}_{\mathfrak{S}}(\mathfrak{G}_\lambda) = \{\mu : |\mathrm{inc}(\mu/\lambda, h)| \neq 0\}$.

Example: $\lambda = (5, 3, 1, 0)$, $h = 4$.



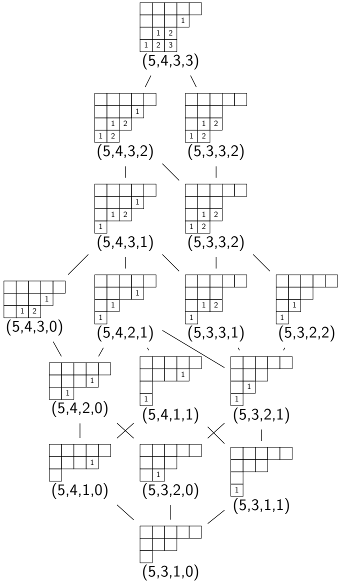
(5, 3, 1, 0) (5, 4, 1, 0) (5, 4, 2, 0) (5, 4, 3, 0) (5, 4, 3, 1) (5, 4, 3, 2) (5, 4, 3, 3)

- λ corresponds to $136924578 \in S_9$.

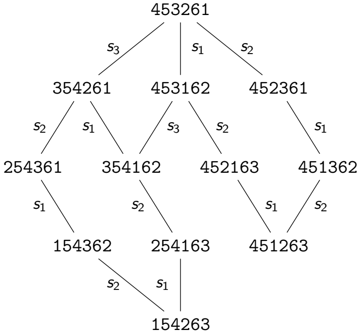
Schubert support of symmetric Grothendieck polynomials

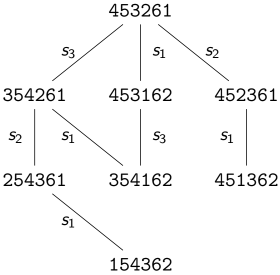
Proof idea:

- Identify the unique maximal element
- Left weak order corresponds to containment order on partitions
- Characterize the entire support



Thank you for your attention!





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