FOCK'S DIMER MODEL ON THE AZTEC DIAMOND & BEYOND

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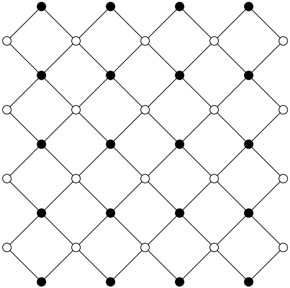
OUTLINE

- Dimer model on the Aztec diamond
- Fock's weights
- Inverse Formula
- Partition function formula
- Limit shapes
- Speyer graphs

DIMER MODEL ON THE AZTEC DIAMOND

- Aztec diamond $\mathsf{A}_n = (\mathsf{V}_n, \mathsf{E}_n)$ of size n



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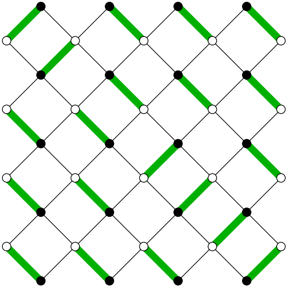


- Dimer configuration (perfect matching) M: subgraph with vertices of degree 1 $\rightsquigarrow \mathcal{M}(\mathsf{A}_n)$.
- Positive weight function: $\nu = (\nu_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}_n}$.
- Dimer Boltzmann measure:

$$\forall\, \mathsf{M} \in \mathcal{M}(\mathsf{A}_n), \quad \mathbb{P}_{\mathrm{dimer}}(\mathsf{M}) = \frac{\prod_{\mathsf{e}\in\mathsf{M}} \nu_{\mathsf{e}}}{Z(\nu)},$$

$Z(\nu) = \sum_{\mathsf{M}\in\mathcal{M}(\mathsf{A}_n)} \prod_{\mathsf{e}\in\mathsf{M}} \nu_{\mathsf{e}}$: partition function.

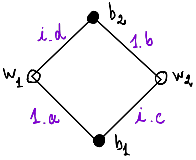
KASTELEYN MATRIX

- $\mathsf{V}_n = \mathsf{B}_n \sqcup \mathsf{W}_n$
- Edges are assigned modulus 1 complex numbers (ϕ_{wb}) such that for any face f (of degree 4) with vertices $\mathsf{w}_1, \mathsf{b}_1, \mathsf{w}_2, \mathsf{b}_2$,

$$\prod_{j=1}^{2} \frac{\phi_{\mathsf{w}_j \mathsf{b}_j}}{\phi_{\mathsf{w}_j \mathsf{b}_{j-1}}} = -1.$$

- Kasteleyn matrix K: weighted adjacency matrix of A_n

$$\mathsf{K}_{\mathsf{w},\mathsf{b}} = \begin{cases} \phi_{\mathsf{wb}} \nu_{\mathsf{wb}} & \text{if } \mathsf{w} \sim \mathsf{b} \\ 0 & \text{otherwise.} \end{cases}$$



$$K = \begin{array}{c c} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} w_1 \\ w_2 \end{array} & \begin{bmatrix} a & id \\ ic & 1b \end{bmatrix} \end{array}$$

$$|\det K| = ab + cd$$

PARTITION FUNCTION & BOLTZMANN MEASURE

THEOREM (KASTELEYN, TEMPERLEY-FISHER '61, KUPERBERG '98)

The partition function is equal to

$$Z(\nu) = |\det \mathsf{K}|.$$

THEOREM (KENYON'97)

Let $\{\mathsf{e}_1 = \mathsf{w}_1\mathsf{b}_1, \dots, \mathsf{e}_k = \mathsf{w}_k\mathsf{b}_k\}$ *be a subset of edges of* A_n. *The probability of dimer configurations containing this subset is equal to:*

$$\mathbb{P}_{\mathrm{dimer}}(\mathsf{e}_1, \dots, \mathsf{e}_k) = \left|\left(\prod_{j=1}^{k} \mathsf{K}_{\mathsf{w}_j,\mathsf{b}_j}\right) \det_{1\leq i,j\leq k} \mathsf{K}^{-1}_{\mathsf{b}_i,\mathsf{w}_j}\right|.$$

GAUGE EQUIVALENCE [KENYON, OKOUNKOV, SHEFFIELD '06]

- Two dimer models on A_n with weight functions ν, $\tilde{\nu}$ are gauge equivalent if, for any face f, with vertices $\mathsf{w}_1, \mathsf{b}_1, \mathsf{w}_2, \mathsf{b}_2$,

$$\prod_{j=1}^{2} \frac{\nu_{\mathsf{w}_j \mathsf{b}_j}}{\nu_{\mathsf{w}_j \mathsf{b}_{j-1}}} = \prod_{j=1}^{2} \frac{\tilde{\nu}_{\mathsf{w}_j \mathsf{b}_j}}{\tilde{\nu}_{\mathsf{w}_j \mathsf{b}_{j-1}}}.$$

- If this is the case, then
  - the partition functions are related by

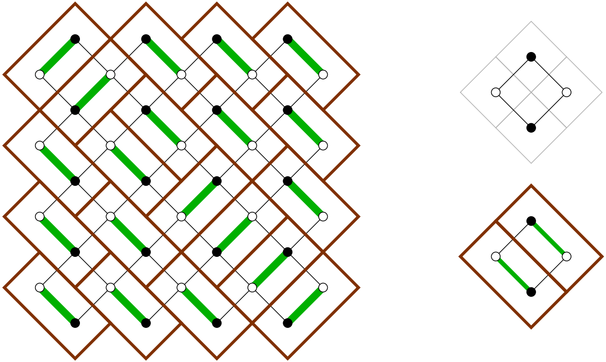
$$Z(\tilde{\nu}) = \frac{\tilde{\nu}(\mathsf{M}_0)}{\nu(\mathsf{M}_0)} Z(\nu),$$

    for any dimer configuration M_0.

  - ν and $\tilde{\nu}$ induce the same Boltzmann measure.

Dimer model on the Aztec diamond

Bijection with domino tilings



DIMER MODEL ON THE AZTEC DIAMOND

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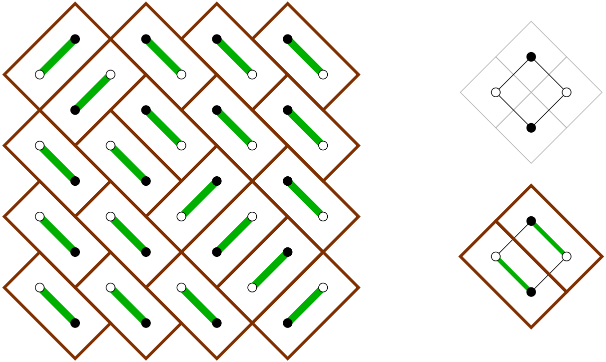

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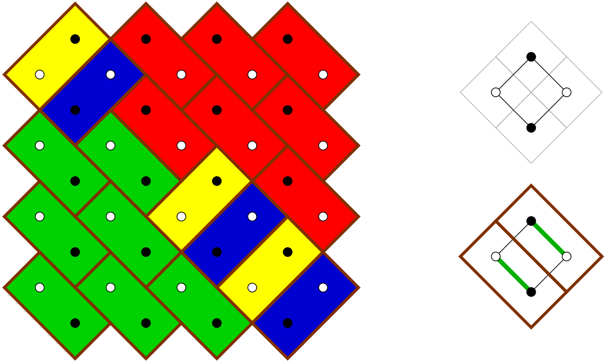


DIMER MODEL ON THE AZTEC DIAMOND

Long history . . .

Different kinds of weights

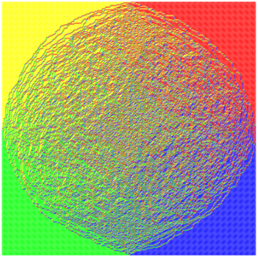
Uniform, 1-periodic

2-periodic, $2 \times k$-periodic, $\ell \times k$-periodic (generic)

- Enumeration
- Explicit expression for K^{-1}
- Limit shapes + more

- [Elkies, Kuperberg, Larsen, Propp '92] [Propp '94] [Jokusch, Propp, Shor '98]
- [Helfgott'00] [Johansson'02] [Chhita, Young'14] [Chhita, Johannsson, Young'14] [Chhita, Johannsson '16] [Boutillier, Bouttier, Chapuy, Corteel, Ramassamy '17]
- [Duits, Kuijlaars '21] [Berggren, Duits '19] [Berggren '21] [Borodin, Duits '23] [Berggren, Borodin '23] [Kuijlaars, Piorkowski '24]
- [Cohn, Kenyon, Propp '01] [Okounkov, Reshetikhin '03] [Kenyon, Okounkov '07] [Di Francesco, Soto-Garrido'14] [Astala, Duse, Prause, Zhong '23] [Kenyon, Prause '23] [Bobenko, Bobenko '24]

Dimer model on the Aztec diamond



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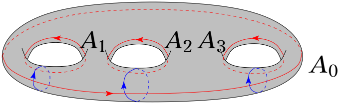
A large array of techniques

- Shuffling algorithms
- Connection to lattice paths and particle systems
- Generating functions
- Connection to Schur processes
- Matrix valued orthogonal polynomials
- Block Toepliz matrices and Wiener Hopf factorization

DIMER MODEL WITH FOCK'S WEIGHTS

- Tool 1. Geometric data and theta functions.
  - Maximal curve Σ of genus $g \geq 0$. Compact Riemann surface with σ, anti-holomorphic involution; Real locus: $g+1$ top. circles $A_0, A_1, \dots, A_g$, fixed by σ.



  - Jacobian variety: $\mathrm{Jac}(\Sigma) = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$
    Ω is pure imaginary period matrix constructed from Σ.
  - Theta function on $\mathbb{C}^g$

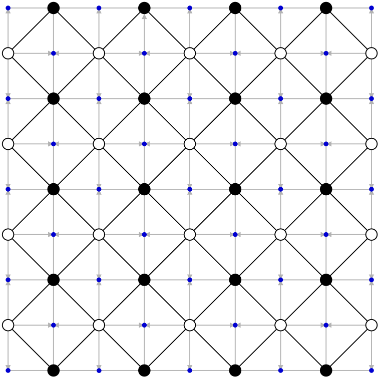
$$\theta(z) = \sum_{n \in \mathbb{Z}^g} \exp(-i\pi\langle n, \Omega n\rangle + 2i\pi\langle z, n\rangle),$$

  - Abel map: $\Sigma \to \mathrm{Jac}(\Sigma) \rightsquigarrow$ theta function on Σ.
  - Prime form E on $\Sigma \times \Sigma$
    Building block of meromorphic functions on Σ.

Dimer model with Fock's weights

- Tool 2. Another type of geometric data.
  - Aztec diamond $\mathsf{A}_n \rightsquigarrow$ quad-graph $\mathsf{A}_n^\diamond$
  - Oriented train-tracks $\vec{\mathcal{T}}$ – Naturally split into 4
  - Angle map : $\vec{\mathcal{T}} \to A_0$

$$\boldsymbol{\alpha} = (\alpha_j)_{j=1}^n,\ \boldsymbol{\beta} = (\beta_j)_{j=1}^n,\ \boldsymbol{\gamma} = (\gamma_j)_{j=1}^n,\ \boldsymbol{\delta} = (\delta_j)_{j=1}^n.$$



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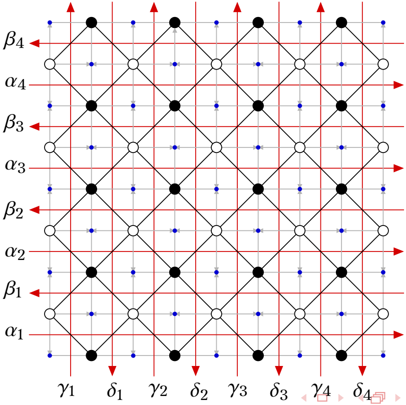
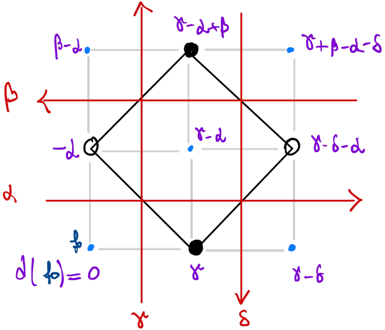


DIMER MODEL WITH FOCK'S WEIGHTS

- Tool 3. **Discrete Abel map d**
  - Function $\mathbf{d}$ on vertices of $\mathsf{A}_n^\diamond$: $\mathbf{d}(\mathsf{f}_0) = 0$ for given face f_0, then local rule

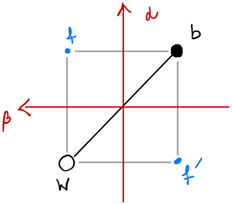


- Well chosen point $t \in \mathrm{Jac}(\Sigma)$: $t \in (\mathbb{R}/\mathbb{Z})^g$.

DIMER MODEL WITH FOCK'S WEIGHTS

- Fock's adjacency matrix

$$K_{w,b} = \begin{cases} \dfrac{E(\alpha,\beta)}{\theta(t+\mathbf{d}(f))\theta(t+\mathbf{d}(f'))} & \text{if } w \sim b \\ 0 & \text{otherwise.} \end{cases}$$



THEOREM (BOUTILLIER-CIMASONI-DT '23)

If the following conditions hold:

- *Σ is a maximal-curve,*
- *angle map $\vec{\mathfrak{T}} \to A_0$ is such that $\boldsymbol{\alpha < \gamma < \beta < \delta}$*
- $t \in (\mathbb{R}/\mathbb{Z})^g$

then, Fock's adjacency matrix is a Kasteleyn matrix for a dimer model on A_n (positive weights).

$\rightsquigarrow$ We are doing probability...

EXAMPLES

- Genus 0
  - $\Sigma = \hat{\mathbb{C}}$ (Riemann sphere), $A_0 = S^1 = \{z \in \hat{\mathbb{C}} : |z| = 1\}$
  - Riemann theta function: const. function 1
  - Prime form: $E(u, v) = v - u$
  - Bijection $S^1 \leftrightarrow \overline{A}_0 = \mathbb{R}/\pi\mathbb{Z}$: $\alpha = e^{2i\bar{\alpha}}$

$$\mathsf{K}_{\mathsf{w,b}} = e^{2i\bar{\beta}} - e^{2i\bar{\alpha}} = 2ie^{i(\bar{\alpha}+\bar{\beta})}\sin(\bar{\beta} - \bar{\alpha})$$

Kenyon's "critical" weights

EXAMPLES

- Genus 1
  - $\Sigma = \mathbb{T}(\tau) = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ (torus), $\tau \in i\mathbb{R}_+$
  - $A_0 = \mathbb{R}/\mathbb{Z}$
  - Riemann theta function: Jacobi's theta function $\theta_3(\pi u|\tau)$
  - Prime form: $E(u,v) = \frac{\theta_1(\pi(v-u)|\tau)}{\pi\theta_1'(0)}$

$$\mathsf{K}_{\mathsf{w},\mathsf{b}} = \frac{\theta_1(\pi(\beta - \alpha))}{\pi\theta_1'(0)\theta_3(\pi(t + \mathbf{d}(\mathsf{f})))\theta_3(\pi(t + \mathbf{d}(\mathsf{f}')))}$$

DIMER MODEL WITH FOCK'S WEIGHTS

PROPOSITION (BOUTILLIER-DT)

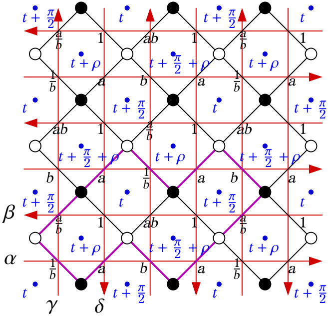
For any $n \in \mathbb{N}^*$, and any choice of positive edge weights ν on A_n, there exists

- *an M-curve Σ of genus g,*
- *a parameter $t \in (\mathbb{R}/\mathbb{Z})^g$,*
- *angles $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta} \in A_0$ assigned to oriented train-tracks of A_n satisfying $\boldsymbol{\alpha} < \boldsymbol{\gamma} < \boldsymbol{\beta} < \boldsymbol{\delta}$ on A_0,*

such that the dimer models with Fock's weights and weight function ν are gauge equivalent.

EXAMPLE. BIASED 2×2 PERIODIC WEIGHTS [BORODIN-DUITS '23]

- Let $a > 0, b \in (0,1]$. Consider biased 2×2 periodic weights.



- On Fock's side, impose that $\Sigma = \mathbb{T}(q)$ (genus 1), and:
  - for all $j \in \{1, \ldots, n\}$,

$$\alpha_j = \alpha,\ \beta_j = \beta,\ \gamma_j = \gamma,\ \delta_j = \delta, \quad \alpha < \gamma < \beta < \delta,$$

  - $\beta - \alpha = \delta - \gamma = \frac{1}{2} \quad \Rightarrow \quad \gamma - \alpha = \delta - \beta := \rho \in (0, \frac{1}{2})$

Example. Biased 2×2 periodic weights [Borodin-Duits '23]

Proposition

Consider biased 2×2 periodic weights, and $\alpha, \beta, \gamma, \delta \in A_0$ as above. Then, there exists $\rho \in (0, \frac{1}{2})$ and $\tau \in i\mathbb{R}^+$ such that, for $t = \frac{1}{4}$, both dimer models are gauge equivalent.

Proof.

Compute alternate products in each case

$$\frac{1}{a^2} = \frac{\theta_3(\pi(t+\frac{1}{2}))^2}{\theta_3(\pi t)^2} \frac{\theta_1(\pi\rho)^2}{\theta_1(\pi(\frac{1}{2}-\rho))^2}, \qquad \frac{1}{a^2} = \frac{\theta_3(\pi t)^2}{\theta_3(\pi(t+\frac{1}{2}))^2} \frac{\theta_1(\pi\rho)^2}{\theta_1(\pi(\frac{1}{2}-\rho))^2},$$

$$(ab)^2 = \frac{\theta_3(\pi(t+\rho))^2}{\theta_3(\pi(t+\frac{1}{2}+\rho))^2} \frac{\theta_1(\pi(\frac{1}{2}-\rho))^2}{\theta_1(\pi\rho)^2}, \qquad \frac{a^2}{b^2} = \frac{\theta_3(\pi(t+\frac{1}{2}+\rho))^2}{\theta_3(\pi(t+\rho))^2} \frac{\theta_1(\pi(\frac{1}{2}-\rho))^2}{\theta_1(\pi\rho)^2}.$$

Relations imply $t = 1/4$, and (using Jacobi's elliptic functions)

$$a = (k')^{-\frac{1}{2}} \operatorname{cs}(\tilde{\rho})$$

$$b = \frac{\operatorname{cs}(\tilde{\rho}) - \operatorname{dn}(\tilde{\rho})}{\operatorname{cs}(\tilde{\rho}) \operatorname{dn}(\tilde{\rho}) - k'} = \frac{\sqrt{k'^{-1} + a^2} - \sqrt{k'^{-1} + a^{-2}}}{\sqrt{k' + a^2} - \sqrt{k' + a^{-2}}}$$

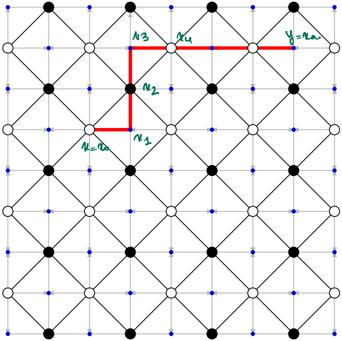
EXPLICIT EXPRESSION FOR K^{-1}

- Forms g in the kernel of K (bulk)

$$g : \mathsf{A}_n^{\diamond} \times \mathsf{A}_n^{\diamond} \times \Sigma \to \{\text{forms on } \Sigma\}$$

$$(\mathsf{x}, \mathsf{y}, u) \mapsto g_{\mathsf{x},\mathsf{y}}(u)$$

Path $\mathsf{x} = \mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_n = \mathsf{y}$ in $\mathsf{A}_n^{\diamond}$: $g_{\mathsf{x},\mathsf{y}}(u) = \prod_{j=0}^{n-1} g_{\mathsf{x}_j,\mathsf{x}_{j+1}}(u)$.



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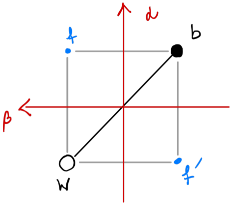
$$(\mathsf{x}, \mathsf{y}, u) \mapsto g_{\mathsf{x},\mathsf{y}}(u)$$

$$\text{Path } \mathsf{x} = \mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_n = \mathsf{y} \text{ in } \mathsf{A}_n^\diamond : g_{\mathsf{x},\mathsf{y}}(u) = \prod_{j=0}^{n-1} g_{\mathsf{x}_j,\mathsf{x}_{j+1}}(u).$$

$$g_{\mathsf{f},\mathsf{w}}(u) = \frac{\theta(u + t + \mathbf{d}(\mathsf{w}))}{E(u,\beta)} = g_{\mathsf{w},\mathsf{f}}(u)^{-1}$$

$$g_{\mathsf{b},\mathsf{f}}(u) = \frac{\theta(u - t - \mathbf{d}(\mathsf{b}))}{E(u,\alpha)} = g_{\mathsf{f},\mathsf{b}}(u)^{-1}.$$



EXPLICIT EXPRESSION FOR K^{-1}

PROPOSITION (FOCK'15)

For any vertex x *of* $\mathsf{A}_n^{\diamond}$, $(g_{\mathsf{b},\mathsf{x}}(u))_{\mathsf{b}\in B}$ *is in the kernel of* K, *in the bulk.*

PROOF.

Consequence of Fay's trisecant identity.

$$\frac{\theta(s+u-\alpha-\beta)}{E(\alpha,u)E(\beta,u)}\frac{E(\alpha,\beta)}{\theta(s-\alpha)\theta(s-\beta)} = $$

$$= \frac{\theta(s+u-\beta-\gamma)}{E(\beta,u)E(\gamma,u)}\frac{E(\gamma,\beta)}{\theta(s-\beta)\theta(s-\gamma)} - \frac{\theta(s+u-\alpha-\gamma)}{E(\alpha,u)E(\gamma,u)}\frac{E(\gamma,\alpha)}{\theta(s-\alpha)\theta(s-\gamma)}$$

□

Explicit expression for K^{-1}

Theorem (Boutillier-dT)

Consider the dimer model on the Aztec diamond with Fock's weights. Then, for every pair (b,w) *of black and white vertices of* A_n*, the coefficient* (b,w) *of the inverse Kasteleyn matrix is explicitly given by*

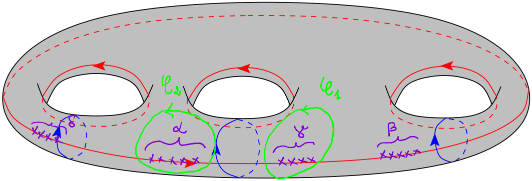
$$\mathsf{K}^{-1}_{\mathsf{b},\mathsf{w}} = \frac{1}{(2\pi i)^2}\frac{1}{\theta(p)}\int_{\mathcal{C}_2}\int_{\mathcal{C}_1}\frac{\theta(p+(v-u))}{E(v,u)}g_{\mathsf{b},0}(u)g_{0,\mathsf{w}}(v)\prod_{j=1}^{n}\frac{E(\beta_j,u)}{E(\delta_j,u)}\frac{E(\delta_j,v)}{E(\beta_j,v)}+$$

$$-\mathbb{I}_{\{\mathsf{b}\ \textit{right of}\ \mathsf{w}\}}\frac{1}{2\pi i}\int_{\mathcal{C}_2}g_{\mathsf{b},\mathsf{w}}(v),$$

where $\mathcal{C}_1$*, resp.* $\mathcal{C}_2$*, is a closed contour used to integrate over* u*, resp.* v*, and* $p = \sum_{j=1}^{n}(\delta_j-\beta_j) - t - \mathbf{d}(0)$.

Explicit expression for K^{-1}

- Contours of integration



- Proof: Show that $\mathsf{K}\,\mathsf{K}^{-1} = \mathrm{Id}$.
  - Forms g in the kernel of K
  - Choice of contours $\mathcal{C}_1, \mathcal{C}_2$
  - Complex integration.

Explicit expression for K^{-1}: example

- Assuming that : $\boldsymbol{\alpha} \equiv \alpha,\ \boldsymbol{\beta} \equiv \beta,\ \boldsymbol{\gamma} \equiv \gamma,\ \boldsymbol{\delta} \equiv \delta, \quad \alpha < \gamma < \beta < \delta,$
- Using natural coordinates $\mathsf{w} = (\mathsf{w}_x, \mathsf{w}_y)$, $\mathsf{b} = (\mathsf{b}_x, \mathsf{b}_y)$, we have

$$
\begin{aligned}
\mathsf{K}^{-1}_{\mathsf{b},\mathsf{w}} = {} & \frac{1}{(2\pi i)^2} \frac{1}{\theta(p)} \int_{\mathcal{C}_2} \int_{\mathcal{C}_1} \frac{\theta(p + (v-u))}{E(u,v)} \cdot \\
& \cdot\ \theta(-t + u - \mathbf{d}(\mathsf{b})) \frac{E(\alpha,u)^{\frac{\mathsf{b}_y}{2}} E(\beta,u)^{n-\frac{\mathsf{b}_y}{2}}}{E(\gamma,u)^{\frac{\mathsf{b}_x+1}{2}} E(\delta,u)^{n-\frac{\mathsf{b}_x-1}{2}}} \theta(t + v + \mathbf{d}(\mathsf{w})) \frac{E(\gamma,v)^{\frac{\mathsf{w}_x}{2}} E(\delta,v)^{n-\frac{\mathsf{w}_x}{2}}}{E(\beta,v)^{n-\frac{\mathsf{w}_y-1}{2}} E(\alpha,v)^{\frac{\mathsf{w}_y+1}{2}}} \\
& - \mathbb{I}_{\{\mathsf{b} \text{ right of } \mathsf{w}\}} \frac{1}{2\pi i} \int_{\mathcal{C}_2} \theta(-t+v-\mathbf{d}(\mathsf{b}))\theta(t+v+\mathbf{d}(\mathsf{w})) \frac{E(\alpha,v)^{\frac{\mathsf{b}_y-\mathsf{w}_y-1}{2}}}{E(\beta,v)^{\frac{\mathsf{b}_y-\mathsf{w}_y+1}{2}}} \frac{E(\delta,v)^{\frac{\mathsf{b}_x-\mathsf{w}_x-1}{2}}}{E(\gamma,v)^{\frac{\mathsf{b}_x-\mathsf{w}_x+1}{2}}},
\end{aligned}
$$

RECURRENCE FORMULA FOR THE PARTITION FUNCTION

- Aztec diamond A_n with Fock's weights, $d = \mathbf{d}((0,0))$.

$$Z_n(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta};d) = Z_n((\alpha_j)_{j=1}^n,(\beta_j)_{j=1}^n,(\gamma_j)_{j=1}^n,(\delta_j)_{j=1}^n;d) = \sum_{\mathsf{M}\in\mathcal{M}(\mathsf{A}_n)}\prod_{\mathsf{e}\in\mathsf{M}}|\mathsf{K}_{\mathsf{w},\mathsf{b}}|.$$

THEOREM (BOUTILLIER - DT)

For every $n \geq 1$, the partition function of the Aztec diamond A_n with Fock's weights satisfies the following recurrence:

$$Z_n(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta};d)\cdot\prod_{f\in\mathrm{odd}_n}\frac{\theta(t+\mathbf{d}(\mathsf{f}))}{\theta(t+\mathbf{d}(\mathsf{f})+\alpha+\beta-\gamma-\delta)}\prod_{\mathsf{f}\in\mathsf{bry}_n}\theta(t+\mathbf{d}(\mathsf{f})) =$$

$$= Z_{n-1}((\alpha_j)_{j=1}^{n-1},(\beta_j)_{j=2}^n,(\gamma_j)_{j=1}^{n-1},(\delta_j)_{j=2}^n;d+\beta_1-\delta_1)\cdot\Big[\prod_{j=1}^n|E(\alpha_j,\beta_j)E(\gamma_j,\delta_j)|\Big],$$

with the convention that $Z_0 = 1$.

COROLLARY: GENUS 0

Assume that the underlying M-curve Σ is the Riemann sphere $\hat{\mathbb{C}}$, *i.e.*, weights are given by Kenyon's critical weights. Then,

$$Z_n(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta}) = 2^{n(n+1)} \prod_{\ell=0}^{n-1} \prod_{j=1}^{n-\ell} |\sin(\bar\beta_{j+\ell} - \bar\alpha_j)\sin(\bar\delta_{j+\ell} - \bar\gamma_j)|$$

$$=2^{n(n+1)} \prod_{\ell=0}^{n-1} \prod_{j=1}^{n-\ell} |\sin(\bar\delta_{j+\ell} - \bar\beta_{j+\ell})\sin(\bar\gamma_j - \bar\alpha_j) + \sin(\bar\beta_{j+\ell} - \bar\gamma_j)\sin(\bar\delta_{j+\ell} - \bar\alpha_j)|.$$

- If furthermore, $\boldsymbol{\alpha} \equiv \alpha$, $\boldsymbol{\beta} \equiv \beta$, $\boldsymbol{\gamma} \equiv \gamma$, $\boldsymbol{\delta} \equiv \delta$ for some $\alpha < \gamma < \beta < \delta \in A_0 = S^1$, then

$$Z_n(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta}) = 2^{n(n+1)}|\sin(\bar\beta - \bar\alpha)\sin(\bar\delta - \bar\gamma)|^{\frac{n(n+1)}{2}}.$$

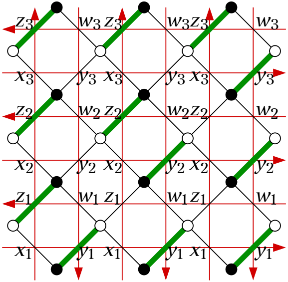
- If furthermore, $\bar\beta - \bar\alpha = \bar\gamma - \bar\delta = \frac{1}{2}$, then $\bar\gamma - \bar\alpha = \bar\delta - \bar\beta := \rho \in (0, \frac{1}{2})$, $\bar\beta - \bar\gamma = \bar\alpha + 1 - \bar\delta = \frac{1}{2} - \rho \in (0, \frac{\pi}{2})$, and

$$Z_n(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta}) = 2^{n(n+1)}.$$

COROLLARY: GENUS 0

Suppose that edges are assigned weights

$\mathbf{x} = (x_j)_{j=1}^n, \mathbf{y} = (y_j)_{j=1}^n, \mathbf{w} = (w_j)_{j=1}^n, \mathbf{z} = (z_j)_{j=1}^n \rightsquigarrow Z(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w})$



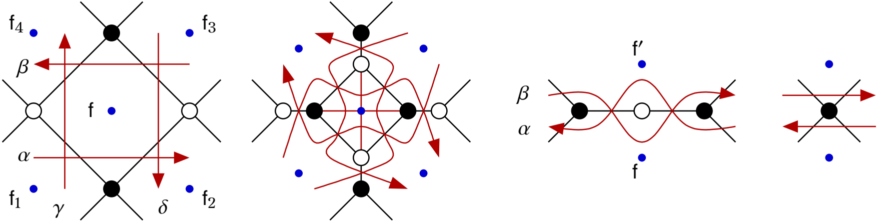
COROLLARY (STANLEY'S FORMULA)

The partition function of the A_n *with weights* $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$ *is equal to*

$$Z(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) = \prod_{\ell=0}^{n-1} \prod_{j=1}^{n-\ell} (x_j w_{j+\ell} + y_j z_{j+\ell}).$$

PROOF (IDEA OF PROPP)

- Elementary moves



Spider move and contraction of a degree two vertex.

PROOF

LEMMA (BOUTILLIER, DT)

Evolution of partition under:

1. *a spider move*

$$Z = \frac{\prod_{j=1}^{4} \theta(t + \mathbf{d}(\mathsf{f}_j))}{|E(\alpha,\beta)E(\gamma,\delta)|} \frac{\theta(t + \mathbf{d}(\mathsf{f}) + \alpha + \beta - \gamma - \delta)}{\theta(t + \mathbf{d}(\mathsf{f}))} Z',$$

   where $\mathsf{f}, \mathsf{f}_1, \ldots, \mathsf{f}_4$ *are as in the figure,*

2. *a contraction of a degree two vertex*

$$Z = \frac{|E(\alpha,\beta)|}{\theta(t + \mathbf{d}(\mathsf{f}))\theta(t + \mathbf{d}(\mathsf{f}'))} Z'',$$

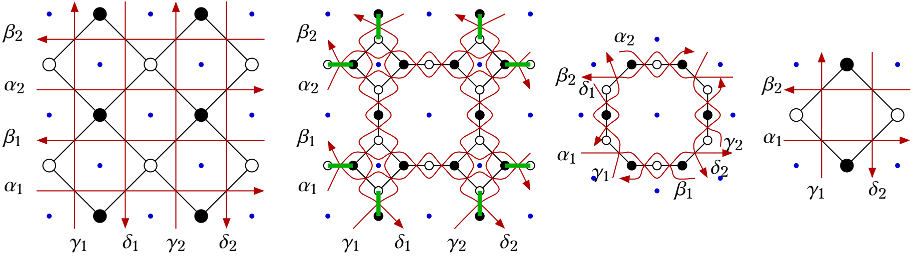
   where f, f' *are as in the figure.*

PROOF.

Invariance of the partition function when considered as model with face weights [Fock'15], [Boutillier, Cimasoni, dT]. ☐

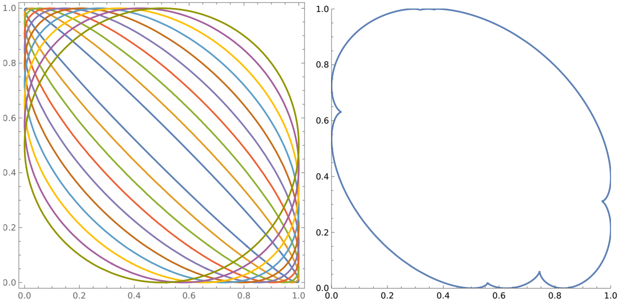
PROOF (IDEA OF PROPP)

- Induction



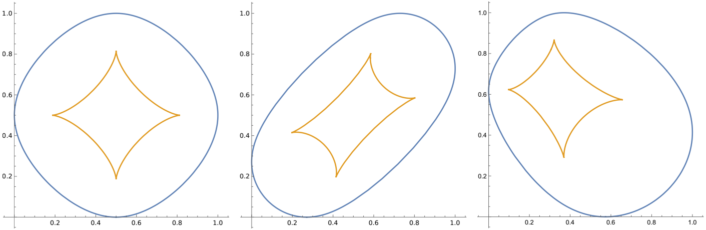
LIMIT SHAPES

- Single edge probability explicit formula
- Saddle point analysis [Okounkov, Reshetikhin '03] [Boutillier, Bouttier, Chapuy, Corteel, Ramassamy '17] [Berggren, Borodin '23]
- Recover known results [Jokusch, Propp, Shor '98] [Johansson '02] [Chhita, Johansson '16] [Di Francesco, Soto-Garrido] [BB'23] … , and obtain new



Genus 0: 1-periodic, 3 × 2-periodic

LIMIT SHAPES

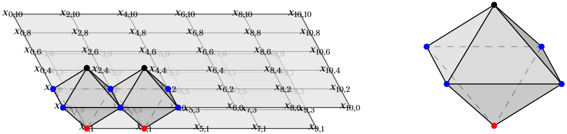


Genus 1: 2-periodic weights unbiased and biased, 2-periodic angles with non-periodic weights

SPEYER GRAPHS

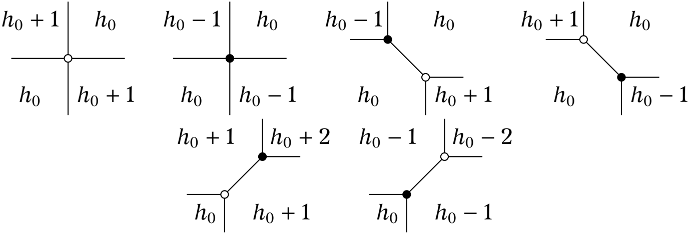
Work in progress with Cédric Boutillier and Bishal Deb.

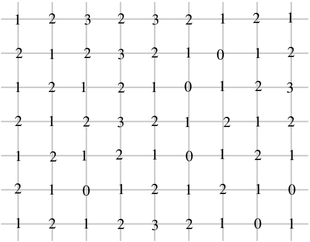
- Speyer's height function [Speyer '06]. Function $h : \mathbb{Z}^2 \to \mathbb{Z}$, such that
  - $h(i,j) \equiv i + j \mod 2$
  - if $(i,j) \sim (i',j')$, then $|h(i,j) - h(i',j')| = 1$
  - $\lim_{|i|+|j|\to\infty} h(i,j) + |i| + |j| = \infty$
- Defines a discrete surface in the octahedral-tetrahedral lattice $\mathcal{L} = \{(i,j,k) \in \mathbb{Z}^3 : \ i + j + k \in 2\mathbb{Z}\}$.



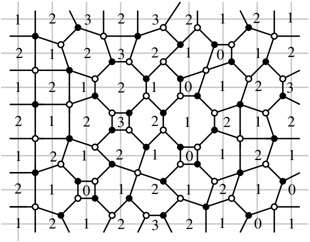
CROSSES AND WRENCHES GRAPH (SPEYER)

The infinite crosses-and-wrenches graph is obtained through the following local configurations



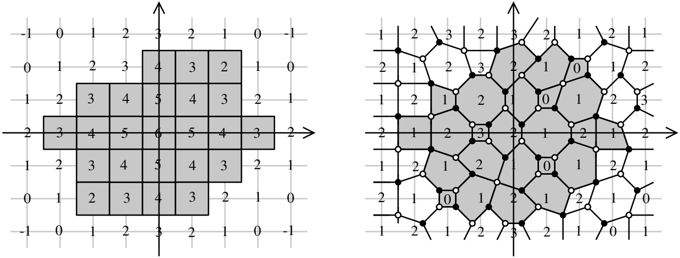


1	2	3	2	3	2	1	2	1
2	1	2	3	2	1	0	1	2
1	2	1	2	1	0	1	2	3
2	1	2	3	2	1	2	1	2
1	2	1	2	1	0	1	2	1
2	1	0	1	2	1	2	1	0
1	2	1	2	3	2	1	0	1



CROSSES AND WRENCHES GRAPH (SPEYER)

The finite crosses-and-wrenches graph is obtained by taking the set of faces satisfying $\{(i,j) : |i| + |j| < h(i,j)\}$ and all incident edges.



- Theorem [Speyer'07]: the dimer partition function is a combinatorial interpretation for the "octahedron recurrence".
- Examples : Aztec diamond, tower graphs, finite railyard graphs, pine cone graphs [Bousquet-Mélou, Propp, and West'09] [Borodin-Ferrari'18] [Di-Francesco-Vu'24], [Nicoletti'25], [Keating Vu'25].

See also the new paper "Dimer models on astroidal zig-zag graphs" [Berggren, Borodin, George'26]