Multiplicative statistics of Poissonized Plancherel random partitions

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Based on

- “Integrable equations associated with the finite-temperature deformation of the discrete Bessel point process” (with M. Cafasso — J. London Math. Soc. 2023)
- “Multiplicative Averages of Plancherel Random Partitions: Elliptic Functions, Phase Transitions, and Applications” (with M. Cafasso and M. Mucciconi — arXiv:2601.05164)

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(Université d’Angers)

Matteo Mucciconi
(National University of Singapore)

Overview

Part I: (Poissonized) Plancherel measure

Part II: Multiplicative statistics and asymptotics

Part III: Cylindrical Toda dynamics

Part IV: q-deformed PNG droplet

Part I

(Poissonized) Plancherel measure

Length of LIS in Uniform Random Permutations

Length of Longest Increasing Subsequence

Given π permutation of $\{1, \ldots, n\}$, define $L(\pi) =$ the largest $k \geq 1$ such that there exist $i_1 < i_2 < \cdots < i_k$ satisfying $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$.

Example

$\pi = (7\ \underline{2}\ 8\ 1\ \underline{3}\ \underline{4}\ 10\ \underline{6}\ \underline{9}\ 5) \Rightarrow L(\pi) = 5.$

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Baik–Deift–Johansson Theorem (1999)

Let L_n be the random variable $L(\pi)$ where π is a uniform random permutation of $\{1, \ldots, n\}$. Then,

$$\mathbb{P}\left(\frac{L_n - 2n^{1/2}}{n^{1/6}} \leq z\right) \to F_{\mathrm{TW}}(z) \quad (n \to +\infty)$$

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[*Ulam, Vershik–Kerov, Logan–Shepp, Borodin–Okounkov–Olshanski, ...*]

Partitions and Maya Diagrams

- Partitions of $n \in \mathbb{N}$:

$$\lambda = (\lambda_1, \lambda_2, \dots), \qquad \lambda_i \in \mathbb{Z}, \quad \lambda_i \geq \lambda_{i+1}, \quad n = |\lambda| = \sum_i \lambda_i$$

- Maya Diagram

$$\mathscr{D}(\lambda) = \left\{ \lambda_i - i + \frac{1}{2} \, : \, i \in \mathbb{N} \right\}$$

- Hook Lenghts

$$h_{i,j}(\lambda) = \lambda_i - i + \lambda'_j - j + 1$$

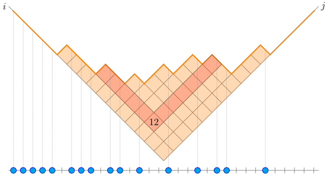

Figure: Young diagram (in Russian notation) of the partition $\lambda = (11, 8, 8, 7, 5, 3, 2, 2, 1, 1, 1)$ and the Maya diagram $\mathscr{D}(\lambda)$. The darker shaded cells represent the hook of the cell $(3, 2)$, whose length is 12.

Young tableaux

A *Standard Young tableau* (SYT) is a way of filling the boxes of a Young diagram λ with n boxes with

- numbers 1 to n with no repeats
- entries increase left to right in each row
- entries increase top to bottom in each column

λ is called *shape* of the standard Young tableaux.

Example

A SYT of shape $\lambda = (5, 4, 2, 1, 1)$ is

1	2	3	4	10
5	6	7	11	
8	12			
9				
13				

RSK correspondence

Number of SYT

Let f_λ = number of SYTs of shape λ. Then

$$f_\lambda = \frac{n!}{\prod_{i,j} h_{i,j}(\lambda)} \qquad \text{(Hook product formula)}$$

$$\sum_{\lambda\colon |\lambda|=n} f_\lambda^2 = n! \qquad \text{(Burnside identity)}$$

Well known from Representation Theory of S_n.

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Robinson–Schensted–Knuth (RSK) correspondence

It is a bijection between permutations $\pi \in S_n$ and pairs (P, Q) of SYTs of the same shape.

- Constructed using the *row insertion algorithm*.
- Widely used in Algebraic Combinatorics and Representation Theory.

RSK correspondence: Example

$\pi = (3\ 1\ 4\ 2\ 5)$

Insert 3: $P = \boxed{3}$ $Q = \boxed{1}$

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$$\boxed{L(\pi) = \lambda_1}$$

Plancherel Measure

Plancherel Measure: probability measure on partitions of n defined by

$$\mathbb{P}_{Pl(n)}(\lambda) = \frac{f_\lambda^2}{n!} = \frac{n!}{\prod_{i,j} h_{i,j}(\lambda)^2}$$

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- *Schensted '61, Logan–Shepp '77, Vershik–Kerov '77, Greene–Nijenhuis–Wilf '79, Baik–Deift–Johansson '99, ...*
- It is the push-forward of of the uniform measure on S_n by RSK correspondence;
  in particular, λ_1 has the same distribution as the length of the longest increasing subsequence in a random uniform permutation.

Poissonized Plancherel Measure

Poissonized Plancherel Measure: we also sample $n = |\lambda| \sim Pois(t^2)$,

$$\mathbb{P}_{PPl(t)}(\lambda) = \frac{e^{-t^2}\, t^{2|\lambda|}}{\prod_{i,j} h_{i,j}(\lambda)^2}$$

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- *Okounkov '99, Borodin–Okounkov–Olshanski '99, Johansson '00, ...*
- More tractable ($\mathscr{D}(\lambda)$ is a Determinantal Point Process)
- Large-n asymptotics of Plancherel can be recovered from large-t asymptotics of Poissonized Plancherel
- Directly related to the longest increasing subsequence in a uniform Poisson process (unit intensity) on the square $[0, t]^2$

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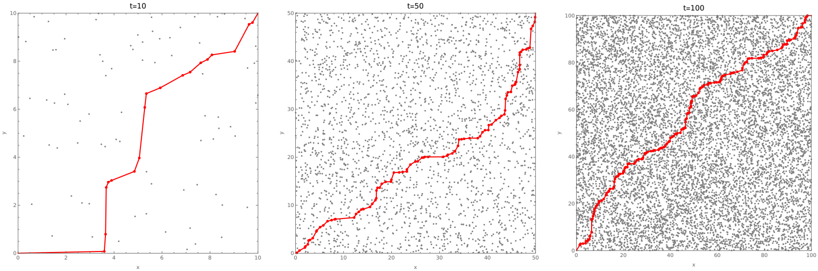

A determinantal point process and Bessel functions

Recall the Maya diagram $\mathscr{D}_\lambda = \{\lambda_i - i + \frac{1}{2}\}$ of a partition λ.

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Theorem (Okounkov, 1999)

If λ is distributed according to the Poissonized Plancherel measure, then

$$\mathbb{P}\Big(\{a_1, \ldots, a_m\} \subset \mathscr{D}_\lambda\Big) = \det_{i,j=1}^{m}\Big(K(a_i, a_j)\Big)$$

where ("discrete Bessel kernel")

$$K(a,b) = t\frac{j_{a-\frac{1}{2}}\, j_{b+\frac{1}{2}} - j_{a+\frac{1}{2}}\, j_{b-\frac{1}{2}}}{a-b}, \quad j_n = \mathrm{J}_n(2t)$$

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In other words, $\mathscr{D}_\lambda$ is a *determinantal point process* (DPP).
DPPs are the main algebraic structure in integrable probability

- very elegant and universal structure
- powerful for computation (various important statistics are Fredholm determinant built out of the kernel function $K(\cdot,\cdot)$)

Limit Shape

- (Re-scaled) Empirical Measure ρ:

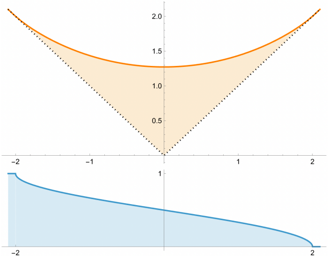
$$\rho(\theta) := \mathbf{1}_{\mathscr{D}(\lambda)}\left(\lfloor t\theta \rfloor + \tfrac{1}{2}\right)$$

$$\#\Big(\mathscr{D}(\lambda) \cap [ta, tb]\Big) = t\int_a^b \rho(\theta)\mathrm{d}\theta$$

- Concentration of measures (*Vershik–Kerov* and *Logan–Shepp*, '77)

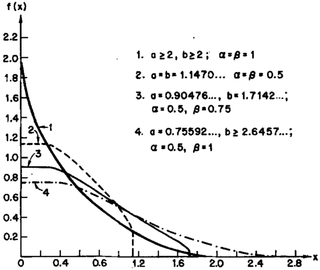
$$\rho \xrightarrow{t\to+\infty} \rho_{\mathrm{VKLS}}$$

$$\rho_{\mathrm{VKLS}}(\theta) = 1_{\theta \le -2} + 1_{|\theta|\le 2}\frac{\arccos(\theta/2)}{\pi}$$



Constrained Limit Shapes

- Logan–Shepp also computed limit shapes for Poissonized Plancherel random partitions λ with constraints on the first row or column of λ
- Implies a large-deviation principle for Poissonized Plancherel random partitions (*Seppäläinen, '98*)



Part II

Multiplicative Statistics and Asymptotics

The Multiplicative Statistics

- We study multiplicative statistics ("Laplace functionals") of the form

$$Q(t,s) = \mathbb{E}\left[\prod_{\xi \in \mathscr{D}(\lambda)} \frac{1}{1+e^{\eta(\xi - s)}}\right] = \mathbb{E}\left[\prod_{i \geq 1} \frac{1}{1+e^{\eta(\lambda_i - i + \frac{1}{2} - s)}}\right]$$

($\mathbb{E}$ with respect to $\mathbb{P}_{PPl(t)}$) for $\eta > 0$, $s \in \mathbb{R}$, and $t > 0$

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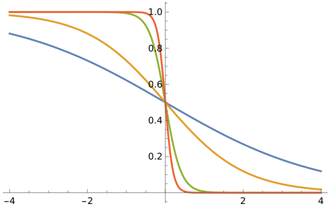

Figure: The function $\xi \mapsto \frac{1}{1+e^{\eta\xi}}$ for various values of η.

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  ($\mathbb{E}$ with respect to $\mathbb{P}_{PPl(t)}$) for $\eta > 0$, $s \in \mathbb{R}$, and $t > 0$
- Due to the determinantal structure, $Q(t,s)$ is a Fredholm determinant
- Why?
  1. When $\eta \to +\infty$, $Q(t,s) \to \mathbb{P}_{PPl(t)}(\lambda_1 < s + \frac{1}{2})$, natural generalization of Logan–Shepp analysis.
  2. Recent interest in such Laplace functionals as they describe marginals of solvable stochastic growth processes in the KPZ class (KPZ, ASEP, SSVM, PNG,...) and they solve integrable equations
  3. $Q(t,s)$ solves the (radially symmetric) 2D Toda eq. (*Cafasso–R* '23)
  4. Interesting asymptotic theory and phase transitions (reminiscent of *P. Zinn–Justin*'s analysis of the SSVM)

Rate Function

- Goal: asymptotics for $Q(t, xt)$ when $t \to +\infty$, $x = O(1)$
- "Rate Function"

$$\mathcal{F}(x) = -\lim_{t \to +\infty} \frac{1}{t^2} \ln Q(t, xt)$$

- From *Das–Liao–Mucciconi '25* we know that $\mathcal{F}(x)$ exists, it is $\mathscr{C}^1(\mathbb{R})$, and for some $x_* < 2$

$$\mathcal{F}(x) = \begin{cases} \frac{\eta}{2}x^2 + 1 - e^{-\eta} & \text{if } x \le x_*, \\ 0 & \text{if } x \ge 2, \end{cases}$$

Main Result

Theorem (*Cafasso–Mucciconi–R '26*) We have

$$x_* \,=\, -2\,(1-e^{-\eta})\,\eta^{-1}$$

$$\mathcal{F}(x) \,=\, \frac{\eta}{2}x^2 + 1 - e^{-\eta} + \eta\int_{x_*}^{x} \mathcal{L}(y)\mathrm{d}y\,, \qquad x_* < x < 2\,,$$

where

$$\mathcal{L}(x) \,=\, -\,\partial_K \mathcal{U}(K)\big|_{K=\mathcal{K}(x)}\,, \quad \mathcal{U}(K) \,=\, 2K e^{\frac{\eta}{2}\left(\frac{\eta}{2K}-1\right)} \frac{\vartheta_{11}(\frac{\eta}{2K}\,|\,\frac{i\pi}{K})}{\vartheta'_{11}(0|\frac{i\pi}{K})}$$

and $\mathcal{K}(x)$ is the unique solution to

$$(1-K\partial_K)\,\mathcal{U}(K) \,=\, -\eta x/2\,.$$

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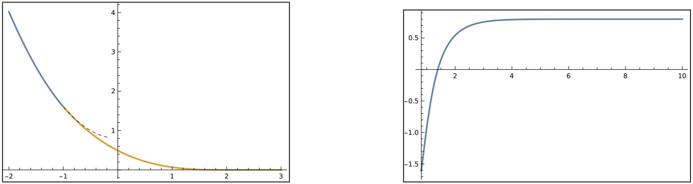

Figure: Left: $\mathcal{F}(x)$. Right: $(1 - K\partial_K)\,\mathcal{U}(K)$

Refined Expansion

Theorem (*Cafasso–Mucciconi–R '26*) Fix $\eta > 0$. For all $x < 2$ there exists $\mathcal{C}(x) \in \mathbb{R}$ such that, when $t \to +\infty$,

$$Q(t, xt) = \mathcal{C}(x) e^{-t^2 \mathcal{F}(x)} \big(1 + O(t^{-1})\big), \qquad x < x_*$$

$$Q(t, xt) = \mathcal{C}(x) \vartheta\big(t\mathcal{L}(x) \big| \tfrac{i\pi}{\mathcal{K}(x)}\big) e^{-t^2 \mathcal{F}(x)}\, t^{\mathcal{A}} \big(1 + O(t^{-\frac{1}{2}})\big), \quad x_* < x < 2$$

where $\mathcal{A}$ is a constant depending only on η. These asymptotics are uniform for $x < 2$ away from 2 and x_*.

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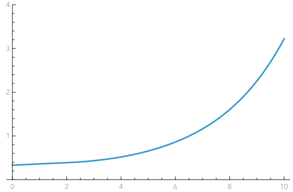

Figure: Plot of the function $\mathcal{A} = \mathcal{A}(\eta)$.

Idea of the Proof

- The Poissonized Plancherel measure is a discrete log-gas (*Vershik–Kerov, Logan–Shepp, '77*)

$$\mathbb{P}_{PPl(t)}\big(\lambda\big) = \exp\big(-t^2(1+\mathcal{E}[\mathfrak{h}]) + o(t^2)\big)$$

$$\mathcal{E}[\mathfrak{h}] = \iint \log\frac{1}{|\mu-\nu|}\mathfrak{h}(\mu)\mathfrak{h}(\nu)\mathrm{d}\mu\,\mathrm{d}\nu + \int 2\left(\mu\log|\mu| - \mu\right)\mathfrak{h}(\mu)\mathrm{d}\mu$$

$$\mathfrak{h}(\mu) = \rho(\mu) - \mathbf{1}_{(-\infty,0]}(\mu)$$

Idea of the Proof

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$$\mathbb{P}_{PPl(t)}\big(\lambda\big) = \exp\big(-t^2(1+\mathcal{E}[\mathfrak{h}]) + o(t^2)\big)$$
$$\mathcal{E}[\mathfrak{h}] = \iint \log\frac{1}{|\mu-\nu|}\mathfrak{h}(\mu)\mathfrak{h}(\nu)\mathrm{d}\mu\,\mathrm{d}\nu + \int 2\left(\mu\log|\mu| - \mu\right)\mathfrak{h}(\mu)\mathrm{d}\mu$$
$$\mathfrak{h}(\mu) = \rho(\mu) - \mathbf{1}_{(-\infty,0]}(\mu)$$

- Similarly,

$$Q(t,xt) = \exp\big(-t^2\mathcal{F}(x) + o(t^2)\big)\,, \qquad \text{as } t\to+\infty$$
$$\mathcal{F}(x) = 1 + \frac{\eta x^2}{2}\mathbf{1}_{(-\infty,0)}(x) + \min_{\mathfrak{h}\in\mathcal{H}}\mathcal{E}_{\eta,x}\left[\mathfrak{h}\right]\,,$$
$$\mathcal{E}_{\eta,x}\left[\mathfrak{h}\right] = \mathcal{E}[\mathfrak{h}] + \frac{\eta}{2}\int[\mu-x]_+\mathfrak{h}(\mu)\mathrm{d}\mu,$$

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- $\mathfrak{h}\in L^1(\mathbb{R})$, $1_{(-\infty,0]}+\mathfrak{h}\in[0,1]$, $\int\mathfrak{h}=0$.

New limit shapes

Theorem (*Cafasso–Mucciconi–R '26*) Let $\eta > 0$ and $x \in \mathbb{R}$. The minimizer $\rho_{\eta,x}$ of the logarithmic energy $\mathcal{E}_{\eta,x}$ is given explicitly as follows.

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If $x \leq x_*$, $\rho_{\eta,x}(\mu) = \rho_{\mathrm{VKLS}}\big(e^{\eta/2}\mu\big)$ and if $x \geq 2$, $\rho_{\eta,x}(\mu) = \rho_{\mathrm{VKLS}}(\mu)$.

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If $x_* < x < 2$,

$$\begin{aligned}\rho_{\eta,x}(\mu) &= \mathbf{1}_{(-\infty,a)\cup(b,c)}(\mu) \\ &\quad + \mathbf{1}_{(a,b)}(\mu)\left[1 + \frac{R(\mu)}{\pi}\left(\int_d^{+\infty} \frac{\mathrm{d}\nu}{R(\nu)(\nu-\mu)} - \frac{\eta}{2\pi}\int_c^d \frac{\mathrm{d}\nu}{R(\nu)(\nu-\mu)}\right)\right] \\ &\quad + \mathbf{1}_{(c,d)}(\mu)\left[1 - \frac{R(\mu)}{\pi}\left(\int_d^{+\infty} \frac{\mathrm{d}\nu}{R(\nu)(\nu-\mu)} - \frac{\eta}{2\pi}\,\mathrm{p.v.}\int_c^d \frac{\mathrm{d}\nu}{R(\nu)(\nu-\mu)}\right)\right]\end{aligned}$$

where

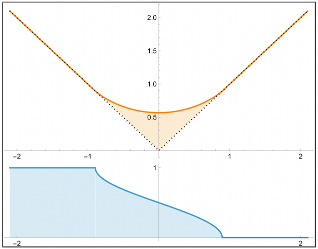
$$R(\mu) = \big|(\mu-a)(\mu-b)(\mu-c)(\mu-d)\big|^{1/2}.$$

The endpoints $a = a(\eta,x)$, $b = b(\eta,x)$, $c = c(\eta,x)$, and $d = d(\eta,x)$ are given by

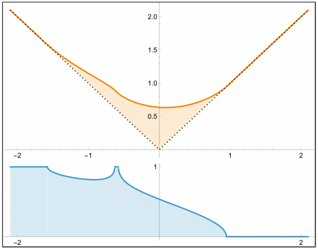
$$a = \mathcal{T}\left(\frac{\vartheta_{01}'\left(\frac{\eta}{4\mathcal{K}}\,\middle|\,\frac{i\pi}{\mathcal{K}}\right)}{\vartheta_{01}\left(\frac{\eta}{4\mathcal{K}}\,\middle|\,\frac{i\pi}{\mathcal{K}}\right)}\right),\ b = \mathcal{T}\left(\frac{\vartheta'\left(\frac{\eta}{4\mathcal{K}}\,\middle|\,\frac{i\pi}{\mathcal{K}}\right)}{\vartheta\left(\frac{\eta}{4\mathcal{K}}\,\middle|\,\frac{i\pi}{\mathcal{K}}\right)}\right),\ c = \mathcal{T}\left(\frac{\vartheta_{10}'\left(\frac{\eta}{4\mathcal{K}}\,\middle|\,\frac{i\pi}{\mathcal{K}}\right)}{\vartheta_{10}\left(\frac{\eta}{4\mathcal{K}}\,\middle|\,\frac{i\pi}{\mathcal{K}}\right)}\right),\ d = \mathcal{T}\left(\frac{\vartheta_{11}'\left(\frac{\eta}{4\mathcal{K}}\,\middle|\,\frac{i\pi}{\mathcal{K}}\right)}{\vartheta_{11}\left(\frac{\eta}{4\mathcal{K}}\,\middle|\,\frac{i\pi}{\mathcal{K}}\right)}\right),$$

$$\mathcal{T}(z) = \frac{\mathcal{U}(\mathcal{K})}{\mathcal{K}}\left(z - \frac{\eta}{2} - \frac{\vartheta_{11}'\left(\frac{\eta}{2\mathcal{K}}\,\middle|\,\frac{i\pi}{\mathcal{K}}\right)}{\vartheta_{11}\left(\frac{\eta}{2\mathcal{K}}\,\middle|\,\frac{i\pi}{\mathcal{K}}\right)}\right).$$

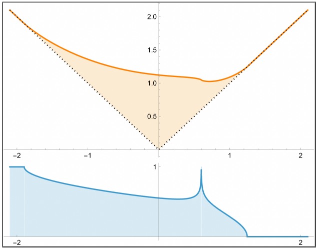
New limit shapes



$x \leq x_* = -0.994136\ldots$



$x = -0.6$



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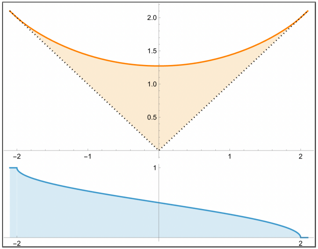

$x \geq 2$

Refining the Asymptotics via RH Problems

- Although the leading order term $\mathcal{F}(x)$ can be extracted in this way, to refine the asymptotics we employ the Deift–Zhou non-abelian steepest descent method to analyze asymptotically a Riemann–Hilbert problem.
- Riemann–Hilbert problem characterization of the Fredholm determinant $Q(t, s)$ (*Cafasso–R '23*) obtained using the theory of integrable ("IIKS") operators (continuous case *Its–Izergin–Korepin–Slavnov '90*; discrete case, *Borodin '00*)
- Analysis bears similarities with the RH asymptotic analysis of discrete OPs (*Baik–Deift–Kriecherbauer–McLaughlin–Miller*) and of the SSVM (*Bleher–Liechty–Bothner*, in turn inspired by *P. Zinn-Justin*)
- The "g-function" is a crucial input in the asymptotic analysis of the Riemann–Hilbert problem; it is the anti-derivative of the Cauchy transform of the equilibrium measure.

Phase Transitions

Theorem (*Cafasso–Mucciconi–R '26*)
We have a Tracy–Widom type third-order phase transition at $x = 2$:

$$\mathcal{F}(x) = \frac{1}{12}(2-x)^3 + O\big((2-x)^4\big) \quad x \uparrow 2$$

We have a "birth-of-a-cut" type third order phase transition at $x = x_*$

$$\begin{aligned}\mathcal{F}(x) = {} & \mathcal{F}(x_*) - 2(1-e^{-\eta})(x-x_*) \\ & + \frac{\eta}{2}(x-x_*)^2 + O((x-x_*)^2/\log(x-x_*)), \quad x \downarrow x_*\end{aligned}$$

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Discreteness-induced third-order phase transition (new?).
Cf. birth of a cut in RMT (*Eynard '06, Claeys '08, Bertola–Lee–Mo '09*)

The large$-\eta$ limit

As $\eta, K \to +\infty$ with $K > \eta/2$, we have

$$\mathcal{U}(K) \sim 1 - e^{\eta - 2K}$$

$\Rightarrow$ for all $x \in (0, 2)$, as $\eta \to +\infty$ we have

$$\mathcal{K}(x) = \frac{\eta}{2} + \frac{1}{2}\log(2/x) + O(\eta^{-1})$$
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We obtain

$$\lim_{\eta \to +\infty} \mathcal{F}(x) = \begin{cases} +\infty & \text{if } x \leq 0, \\ \frac{1}{2}x^2 \log \frac{2}{x} + \frac{3}{4}x^2 - 2x + 1 & \text{if } 0 < x < 2, \\ 0 & \text{if } x \geq 2. \end{cases}$$

which recovers the lower-tail large deviation of the length of the LIS in a Poisson random environment (*Seppäläinen* '98)

Part III

Cylindrical Toda dynamics

Cylindrical Toda

Theorem (*Cafasso–R '23 / Matetski–Quastel–Remenik '23*)
The function

$$y(t,s) \,=\, \log \frac{Q(t,s)}{Q(t,s-1)} \qquad (t>0,\ s\in\mathbb{Z})$$

satisfies the *cylindrical Toda equation*

$$\partial_t^2 y(t,s) \,+\, t^{-1}\partial_t y(t,s) \,=\, 4\big(e^{y(t,s+1)-y(t,s)} \,-\, e^{y(t,s)-y(t,s-1)}\big),$$

with initial conditions

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The cylindrical Toda equation is the radial reduction $Y(T,\overline{T},s) = y(T\overline{T},s)$ of the celebrated *2D Toda equation*

$$\partial_T\partial_{\overline{T}} Y(T,\overline{T},s) = e^{Y(T,\overline{T},s+1)-Y(T,\overline{T},s)} \,-\, e^{Y(T,\overline{T},s)-Y(T,\overline{T},s-1)}.$$

Flaschka Variables for Cylindrical Toda

The Flaschka variables

$$\mathfrak{a}(t,s) = \exp\left(\frac{y(t,s+1) - y(t,s)}{2}\right), \qquad \mathfrak{b}(t,s) = \frac{1}{2}\frac{\partial}{\partial t} y(t,s)$$

satisfy

$$\dot{\mathfrak{a}}(t,s) = \mathfrak{a}(t,s)\Big(\mathfrak{b}(t,s+1) - \mathfrak{b}(t,s)\Big), \quad \dot{\mathfrak{b}}(t,s) = 2\Big(\mathfrak{a}(t,s)^2 - \mathfrak{a}(t,s-1)^2\Big) - \frac{\mathfrak{b}(t,s)}{t},$$

with initial conditions

$$\mathfrak{a}(0,s) = \left(\frac{1 + e^{-\eta(s+1)}}{1 + e^{-\eta s}}\right)^{1/2}, \qquad \mathfrak{b}(0,s) = 0.$$

Note that $\lim_{s\to+\infty} \mathfrak{a}(0,s) = 1$ and $\lim_{s\to-\infty} \mathfrak{a}(0,s) = e^{-\eta/2}$.

$\Rightarrow$ step-like initial conditions

Asymptotics for the Flaschka Variables

We obtain asymptotics for $\mathfrak{a}(t, xt)$ and $\mathfrak{b}(t, xt)$ as $t \to +\infty$ and $x = O(1)$ (away from x_* and 2):

$$\mathfrak{a}(t, xt) = \mathfrak{a}_0(t, x) + O(t^{-1}), \qquad \mathfrak{b}(t, xt) = \mathfrak{b}_0(t, x) + O(t^{-1})$$

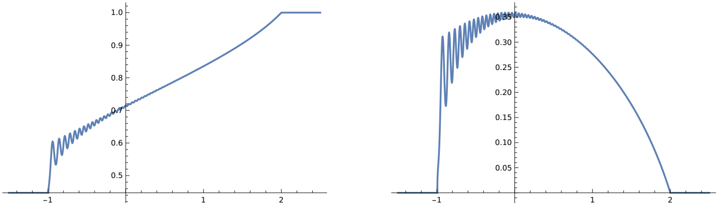

Figure: Plots of $\mathfrak{a}_0(t, x)$ and $\mathfrak{b}_0(t, x)$ (left and right, respectively) with $\eta = \log 5$ and $t = 40$.

Part IV

q-deformed PNG (polynuclear growth) droplet

Polynuclear Growth (PNG) Model

A simple stochastic growth model for random interfaces where random nucleation events expand and merge, modeling interface dynamics. Famously analyzed by M. Prahöfer and H. Spohn (KPZ).

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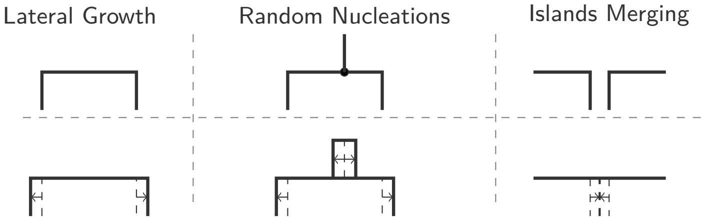
A simple stochastic growth model for random interfaces where random nucleation events expand and merge, modeling interface dynamics. Famously analyzed by M. Prahöfer and H. Spohn (KPZ).

- Height function $h(\cdot, t)$ piecewise constant, integer-valued, jump discontinuities of unit size.
- Evolution in time t:
  - random *nucleations* form Poisson point process on $\mathbb{R} \times \mathbb{R}_+$ with intensity 1 and generate an island of height 1, on top of the profile
  - islands grow laterally at constant speed 1, merging with others upon contact



PNG model

Variants

- *Flat PNG*: nucleations occur uniformly in space simulation
- *Droplet PNG*: nucleations only occur inside the forward light cone of the origin, producing a curved ("droplet") shape simulation

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Johansson, '00: Let $h(0,t)$ denote the height at the origin at time t for the droplet PNG. Then,

$$\mathbb{P}\left(\frac{h(0,t)-2t}{t^{1/3}} \le z\right) \to F_{\mathrm{TW}}(z) \quad (t \to +\infty)$$

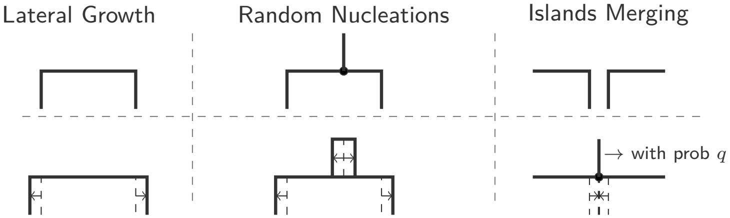
For the droplet PNG, $h(x,t) = h(0, \sqrt{t^2 - x^2})$, in distribution.

q-deformed PNG model

The q-deformed PNG model is a deformation of the PNG model (retrieved when $q = 0$).

Namely, for a parameter $q \in [0, 1)$, the basic mechanisms of PNG evolution are modified only as far as merging of islands is concerned:

- with probability $1 - q$, islands merge (as in the undeformed PNG);
- with probability q, merging generates a nucleation.



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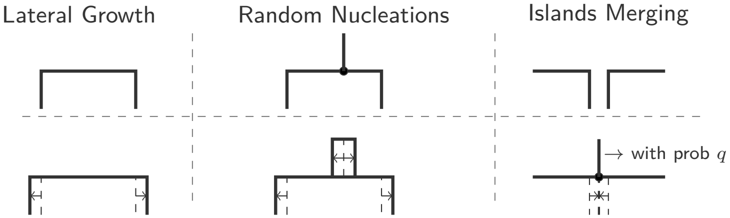
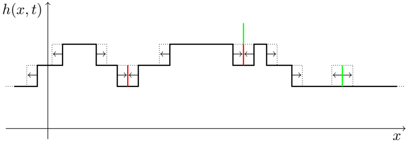


Introduced by Aggarwal–Borodin–Wheeler (2022). They also prove Tracy–Widom fluctuations for this deformation.

Relation between $Q(s,t)$ and q-PNG droplet



Theorem (*Borodin '18, Imamura–Sasamoto–Mucciconi '22*) Let $h(x,t)$ be the height of the droplet q-deformed PNG. Then,

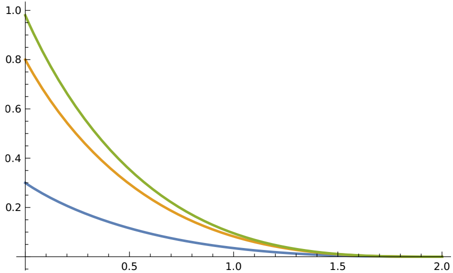
$$\mathbb{P}\big(h(0,t) + S + \chi \leq s\big) = Q(s,t)$$

where $\mathbb{P}(S = k) = \frac{q^{k^2/2}}{\sum_{l \geq 0} q^{l^2/2}}$, $\mathbb{P}(\chi = k) = q^k \prod_{j>k}(1 - q^j)$.

Lower Tail Rate Function of the q-PNG droplet

Theorem (*Cafasso–Mucciconi–R '26*, after *Das–Liao–Mucciconi '25*)
Fix $q \in (0,1)$ and let $\eta = -\log q$. Let $h(x,t)$ be the height function of the q-PNG with intensity $\Lambda = 2(1-q)$ and droplet initial conditions. Then, for $\mu \in [0,2]$ we have

$$\lim_{t \to +\infty} t^{-2} \log \mathbb{P}[h(0,t) \leq \mu t] = \Phi_-(\mu) = \max_{y \in \mathbb{R}} \left\{ \mathcal{F}(y) - \frac{\eta}{2}(\mu - y)^2 \right\}.$$



Comments and Open Questions

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Open directions

- More general discrete determinantal point processes with log-gas structure. Natural and interesting first candidates are other instances of Okounkov's Schur measure (e.g., Meixner ensembles).
- Finer analysis at critical points and constant problem. Nature of the logarithmic term.
- Witham theory for cylindrical Toda.