

Symmetries of Discrete and UltraDiscrete Integrable Systems

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Plan

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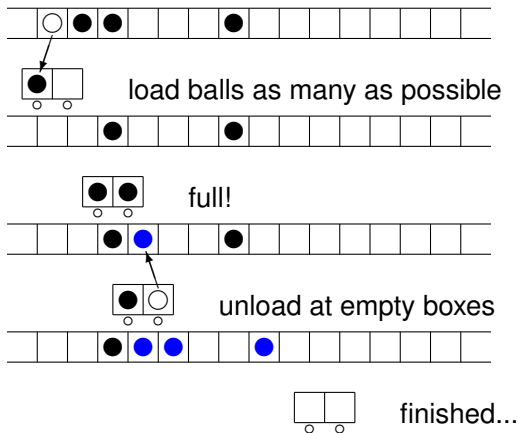
References:

I-Kuniba-Takagi: [arXiv:1109.5349](https://arxiv.org/abs/1109.5349) (review)

§1 Various BBS and descriptions

- Carrier [Takahashi-Matsukidaira 1997]

m -carrier: a carrier of capacity $m (= 2) \rightsquigarrow$ time evolution T_m



$$T_2: \quad t = \begin{array}{l} 0 \quad \dots 12221112111111111111 \dots \\ 1 \quad \dots 11122211211111111111 \dots \\ 2 \quad \dots 11111222121111111111 \dots \\ 3 \quad \dots 11111112212211111111 \dots \\ 4 \quad \dots 11111111121122211111 \dots \end{array}$$

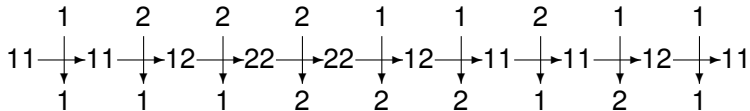
$$T_1: \quad t = \begin{array}{l} 0 \quad \dots 12221112111111111111 \dots \\ 1 \quad \dots 11222111211111111111 \dots \\ 2 \quad \dots 11122211121111111111 \dots \end{array}$$

$$T_{m \geq 3}: \quad t = \begin{array}{l} 0 \quad \dots 12221112111111111111 \dots \\ 1 \quad \dots 11112221211111111111 \dots \\ 2 \quad \dots 11111112122211111111 \dots \\ 3 \quad \dots 11111111211122211111 \dots \end{array}$$

velocity is m at most; T_m recovers the original BBS when $m \rightarrow \infty$
 soliton, scattering rule, IM are the same as in case of T_∞

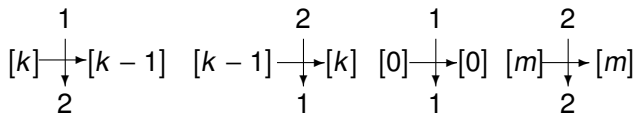
Vertex model description [Hikami-I-Komori 99][Fukuda-Okado-Yamada 00]

T_2 :



\leadsto the rule is localized:

$[k]$: a carrier stacking k balls; $1 \leq k \leq m$



These are nothing but **combinatorial R** of $\hat{\mathfrak{sl}}_2$ -crystal (\rightarrow §2)

Piecewise-linear eq.

(i) Eulerian description

$U_k^t \in \{0, 1\}$ (# balls in k -th box at time t).

$$U_k^{t+1} = \min \left[1 - U_k^t, \sum_{j=-\infty}^{k-1} (U_j^t - U_j^{t+1}) \right] \quad [\text{Tokihiro et al 1996}]$$

↙ # of balls in ∞ -carrier

(ii) Lagrangian description

$Q_k^t, W_k^t \in \mathbb{Z}_{\geq 0}$: the length of the k -th series of 2, 1 at time t

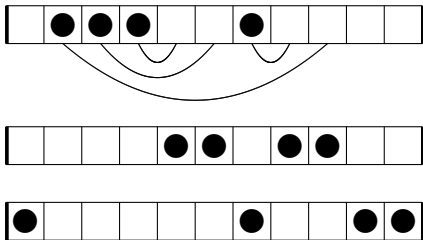
g : # of solitons $\rightsquigarrow W_0^t = W_g^t = \infty$

$$\begin{cases} Q_k^{t+1} = \min \left[\sum_{i=1}^k Q_i^t - \sum_{i=1}^{k-1} Q_i^{t+1}, W_k^t \right] \\ W_k^{t+1} = Q_{k+1}^t + W_k^t - Q_k^{t+1} \end{cases}$$

They are the **ultradiscretization (tropicalizaion)** of discrete KdV and Toda lattice (\rightarrow §4)

• Periodic BBS [Yura-Tokihiro 2002]

L : period, $M < \frac{L}{2}$: # of balls

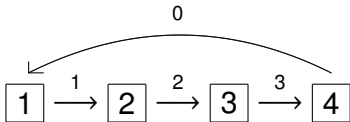


$t =$	0	1	1	2	2	1	1	2	1	1	1	
	1	1	1	1	1	2	2	1	2	2	1	
	2	2	1	1	1	1	1	2	1	1	2	$\rightsquigarrow \mu = (2, 1, 1)$
	3	1	2	2	2	1	1	1	2	1	1	

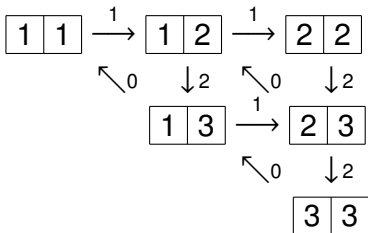
$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_g) : \text{IM}$ $\lambda = (3, 1) = \mu^t$

§2 Crystal

$\hat{\mathfrak{sl}}_4$ -crystal B_1 :



$\hat{\mathfrak{sl}}_3$ -crystal B_2 :

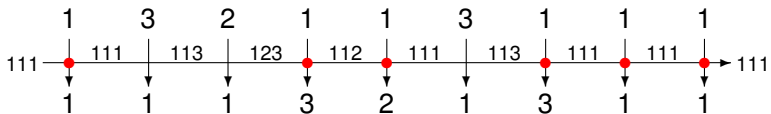


$\hat{\mathfrak{sl}}_2$ -crystals

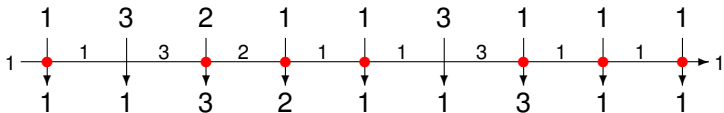
$$\begin{array}{ccccc}
 B_2 \otimes B_1 & & & & \\
 & & \begin{array}{c} \uparrow_0 \\ \downarrow_1 \end{array} & & \\
 \begin{array}{c} \boxed{1} \boxed{1} \otimes \boxed{1} \\ \boxed{1} \boxed{1} \otimes \boxed{2} \end{array} & \xrightarrow{1} & \begin{array}{c} \boxed{1} \boxed{2} \otimes \boxed{1} \\ \boxed{1} \boxed{2} \otimes \boxed{2} \end{array} & \xrightleftharpoons[0]{1} & \begin{array}{c} \boxed{2} \boxed{2} \otimes \boxed{1} \\ \boxed{2} \boxed{2} \otimes \boxed{2} \end{array}
 \end{array}$$

$$\begin{array}{ccccc}
 B_1 \otimes B_2 & & & & \\
 & & \begin{array}{c} \uparrow_0 \\ \downarrow_1 \end{array} & & \\
 \begin{array}{c} \boxed{1} \otimes \boxed{1} \boxed{1} \\ \boxed{1} \otimes \boxed{1} \boxed{2} \end{array} & \xrightarrow{1} & \begin{array}{c} \boxed{2} \otimes \boxed{1} \boxed{1} \\ \boxed{1} \otimes \boxed{2} \boxed{2} \end{array} & \xrightleftharpoons[0]{1} & \begin{array}{c} \boxed{2} \otimes \boxed{1} \boxed{2} \\ \boxed{2} \otimes \boxed{2} \boxed{2} \end{array}
 \end{array}$$

(Ex 3)



$$\leadsto E_3 = \sum_{k=1}^L (1 - H(v_k \otimes b_k)) = 3$$



$$\leadsto E_1 = 2$$

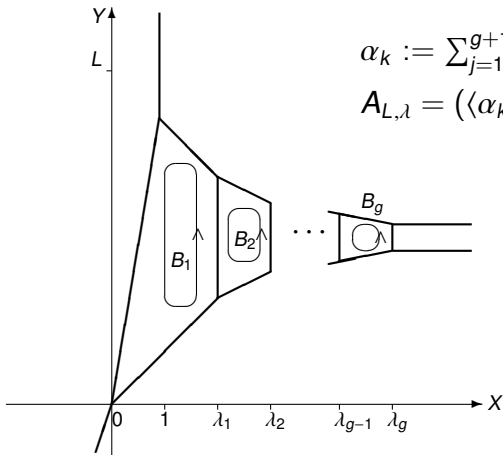
$$\leadsto (E_m)_{m \geq 1} = (2, 3, 3, \dots)$$

$$\leadsto (N_\ell)_{\ell \geq 1} = (1, 1, 0, \dots)$$

§4 Tropicalization

$$F = \min[2Y, Y + \min[gX, (g-1)X + C_{g-1}, \dots, X + C_1, C_0], XL, L]$$

$$C_j = \sum_{i=j+1}^g \lambda_i \quad (j = 0, \dots, g-1).$$



$$\alpha_k := \sum_{j=1}^{g+1-k} B_j$$

$$A_{L,\lambda} = (\langle \alpha_k, \alpha_j \rangle)_{k,j=1,\dots,g}$$