

Noncommutative geometry and deformed Minkowski space-time

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(based on 2503.07176 with F.Pozar and J-C.Wallet)

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Summary

- ⇒ Introduction to Noncommutative geometry
- ⇒ Deformed Minkowski space-time
- ⇒ Nonunimodular group and KMS weight

Introduction

Quantum gravity

Quantum Field Theory \hbar

c General relativity,
 G Cosmology

Quantum gravity:

$$\Rightarrow \ell_{\text{Planck}} = \sqrt{\frac{G\hbar}{c^3}} \sim 10^{-35} m$$

$$\Rightarrow E_{\text{Planck}} = \sqrt{\frac{\hbar c^5}{G}} \sim 10^{28} \text{eV}$$

- Introduction of a minimal length Addazi and all.
 - (DSR): Doubly/Deformed special relativity: two observer-independent quantities
 - (LIV): Lorentz invariance violation (broken or deformed)
 - (GUP): Generalised uncertainty principle $\Delta x \Delta p \geq \frac{\hbar}{2} (1 + \ell_P^2 p^2 + \mathcal{O}(p^4))$

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Quantization of space-time

- ➡ General relativity: Space-time is a dynamic quantity
- ➡ Quantum mechanics: Dynamic quantities must be quantize

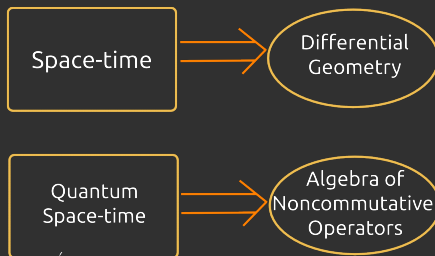
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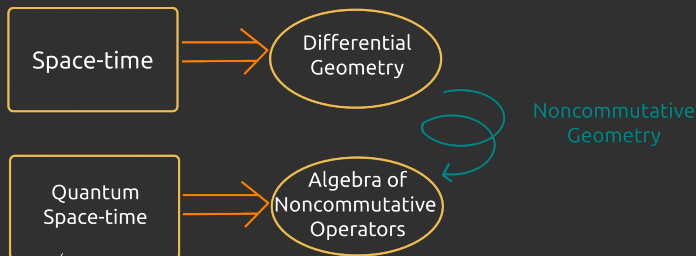
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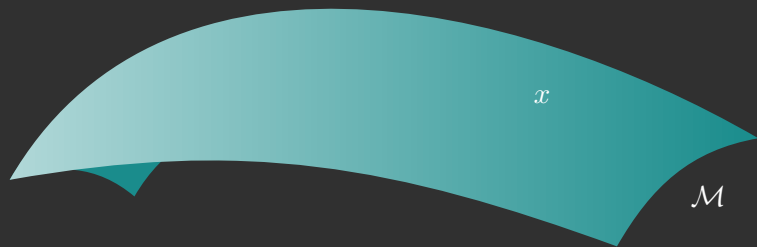
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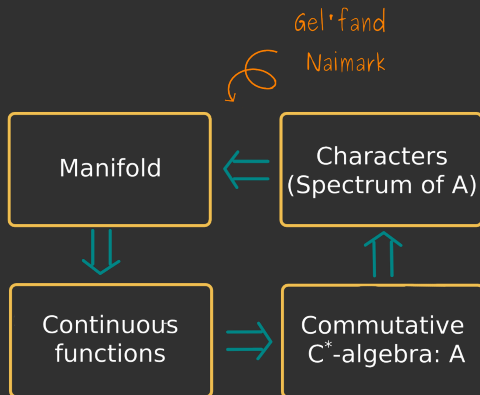
- Algebraization of geometry

Generalisation of differential geometry: Topology



How to find an equivalent of a manifold in an algebraic language?

Algebraization of geometry: Commutative picture



*-algebra

Algebra $(\mathcal{A}, +, \star)$ equipped with an antihomomorphism $\dagger : \mathcal{A} \rightarrow \mathcal{A}$,

$$(zf + wg)^\dagger = \bar{z}f^\dagger + \bar{w}g^\dagger \quad (\text{antilinear})$$

$$(f \star g)^\dagger = g^\dagger \star f^\dagger \quad (\text{antimorphism})$$

$$(f^\dagger)^\dagger = f \quad (\text{idempotent})$$

$$\forall f, g \in \mathcal{A}, z, w \in \mathbb{C}$$

C^* -algebra

A Banach space $(\mathcal{A}, \|\cdot\|)$ (i.e. a complete normed vector space) such that

- ↪ It is an $*$ -algebra for a given product $*$ and involution \dagger
- ↪ Its norm satisfies $\|fg\| \leq \|f\| \|g\|, \quad \forall f, g \in \mathcal{B}$
- ↪ \dagger is an isometry for the norm: $\|f^\dagger\| = \|f\|$

A Banach $*$ -algebra is called a C^* -algebra if

$$\|ff^\dagger\| = \|f\|^2, \quad \forall f \in \mathcal{A}$$

Examples

Let \mathcal{M} be a locally compact Hausdorff space and consider $(\mathcal{C}_0(\mathcal{M}, \mathbb{C}), +, \cdot, \dagger, \|\cdot\|)$ with

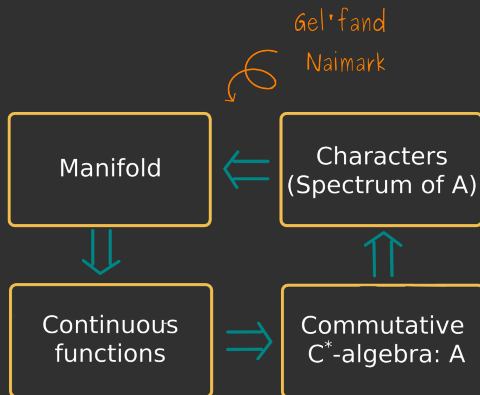
⇒ \cdot the pointwise multiplication, \dagger complex conjugation

⇒ $\|f\| = \sup_{x \in \mathcal{M}} |f(x)|$

For example with $\mathcal{M} = [0, 1]$:



Algebraization of geometry: Commutative picture



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Spectrum/Characters of the algebra

$\Delta(\mathcal{A}) =$ Non zero $*$ -homomorphism $\chi : \mathcal{A} \rightarrow \mathbb{C}$

For $\mathcal{A} = \mathcal{C}_0(\mathcal{M})$, each $x \in \mathcal{M}$ define a linear map $\chi_x : \mathcal{A} \rightarrow \mathbb{C}$,
$$\chi_x(f) := f(x)$$

Gelfand Naimark theorem

There is a one to one correspondence between the spectrum $\Delta(\mathcal{A})$ and \mathcal{M} a topological locally compact Hausdorff space.

Every characters of $\Delta(\mathcal{A})$ are of the form χ_x

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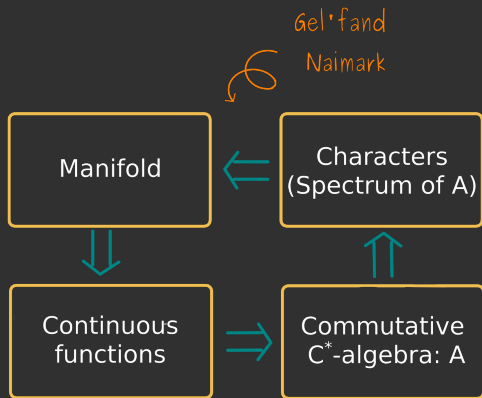
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Algebraization of geometry: Commutative picture



Manifold, \mathcal{M} \iff Commutative algebra, $\mathcal{A} = (\mathcal{C}^\infty(\mathcal{M}), \cdot, \dagger, \|\cdot\|)$
Points, $x \in \mathcal{M}$, \iff Characters, $\Psi \in \Delta(\mathcal{A})$

Algebraization of geometry: Noncommutative picture

Quantum version

→ Commutative algebra \implies Noncommutative algebra $(\mathcal{A}, \star, \dagger, \|\cdot\|)$

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be Hilbert space and consider $(\mathcal{B}(\mathcal{H}), +, \circ, \dagger, \|\cdot\|_{\langle \cdot, \cdot \rangle})$ the set of bounded linear operators on \mathcal{H} .

→ $\Delta(\mathcal{A}) = \{ \text{(pure) states} \} = \{ \text{positive linear form of norm one} \}$

Any $\psi \in \mathcal{H}$, $\|\psi\| = 1$, give state: $\omega_\psi(f) = \langle \psi | f \psi \rangle$, $\forall f \in \mathcal{B}(\mathcal{H})$

→ Elements $a \in \mathcal{A}$ are quantum observables.

→ Elements $\Psi \in \Delta(\mathcal{A})$ are quantum states.

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Noncommutative geometry and link with canonical quantization

- ➔ Generalise coordinate to noncommutative functions $x^\mu(x) = x^\mu$:

$$[x^\mu, x^\nu] = x^\mu \star x^\nu - x^\nu \star x^\mu \neq 0$$

- ➔ Noncommutative geometry as canonical quantization of phase space
(Weyl [1927], von Neumann [1931])

Quantum mechanic with Moyal product

$$(f \star g)(q, p) = \frac{1}{(\pi \hbar)^2} \int du dv ds dt f(q+u, p+v) g(q+s, p+t) e^{\frac{2i}{\hbar}(ut - vs)}$$

$$[q(q, p), p(q, p)]_\star = q \star p - p \star q = i\hbar$$

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Noncommutative spaces

- Deformed Minkowski space-time (and associated Poincaré symmetry (Hopf) algebra)
- Well known examples: κ/ρ -Minkowski [Lukierski, Ruegg, Nowicki, Tolstoy,1991],[Dimitrijevic, Konjik,Samsarov,2018]
- Classification of Poisson structures of the Poincaré group Zakrzewski [1997],
- Deformed Minkowski space-time Lukierski,Woronowicz [2005][Mercati [2024]
- Lie algebraic deformations:

$$[x^\mu, x^\nu] = i\zeta^\mu(\eta^{\mu\beta}x^\nu - \eta^{\nu\beta}x^\mu) - i\zeta^\nu(\eta^{\mu\beta}x^\nu - \eta^{\nu\beta}x^\mu), \quad (1)$$

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Construction of $*$ -algebra [Maris,Pozar,Wallet,2025]

- ⇒ Convolution algebra and Weyl quantization map: [von Neumann,1931], [Hersent,Mathieu,Wallet,2018]

$$\mathfrak{g} \Rightarrow G \Rightarrow (L^1(G), \circ, *) \Rightarrow Q(f) = \pi(Ff) \Rightarrow \begin{aligned} Q(f \star g) &= Q(f)Q(g) \\ Q(f^\dagger) &= Q(f)^* \end{aligned}$$

where \mathfrak{g} is a lie algebra, G is locally compact lie group, $\pi : L^1(G) \rightarrow \mathcal{B}(H)$ representation of the convolution algebra and F Fourier transform of $f \in L^1(G)$

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Convolution algebra and Weyl quantization map (I)

⇒ General structure of Lie groups :

$$\mathcal{G} = H \ltimes \mathbb{R}^n, \quad n \geq 1, \quad (2)$$

where $H \subset GL(n, \mathbb{R})$ one parameter abelian subgroup.

⇒ $(a, \vec{p}) \in \mathcal{G} = H \ltimes \mathbb{R}^n \implies (p^M, \vec{p}) \in \mathbb{R} \ltimes \mathbb{R}^n$ such that $a := a(p^M)$

$$(a_1, \vec{p}_1)(a_2, \vec{p}_2) = (a_1 a_2, \vec{p}_1 + a_1 \vec{p}_2), \quad (3)$$

$$(a, \vec{p})^{-1} = (a^{-1}, -a^{-1} \vec{p}), \quad \mathbb{I}_{\mathcal{G}} = (\mathbb{I}_H, 0) \quad (4)$$

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Convolution algebra and Weyl quantization map (II)

Right and left Haar measures and modular function for $s \in \mathcal{G}$:

$$d\nu_{\mathcal{G}}(s) = \Delta_{\mathcal{G}}(s^{-1})d\mu_{\mathcal{G}}(s) \quad \left\{ \begin{array}{l} d\mu_{\mathcal{G}}((a, \vec{p})) = d^n \vec{p} d\mu_H(a) |\det(a)|^{-1} \\ \Delta_{\mathcal{G}}((a, \vec{p})) = \underbrace{\Delta_H(a)}_{=1} |\det(a)|^{-1} \end{array} \right. \quad (5)$$

Convolution algebra, $(L^1(\mathcal{G}), \circ, *)$

$$(F \circ G)(s) = \int_{\mathcal{G}} d\nu_{\mathcal{G}}(t) F(st^{-1})G(t) \xrightleftharpoons[\text{Inverse}]{\text{Fourier}} f * g = \mathcal{F}^{-1}(\mathcal{F}f \circ \mathcal{F}g) \quad (6)$$

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$$F, G \in L^1(\mathcal{G}), \quad \mathcal{F}f := F$$

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Explicit star product

Star product and involution:

$$(f \star g)(x^M, \vec{x}) = \int \frac{dp^M dy^M}{(2\pi)} e^{-ip^M y^M} f(x^M + y^M, \vec{x}) g(x^M, a(p^M)x)$$
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where $x = (x^M, \vec{x}) \in \mathbb{R} \times \mathbb{R}^3$

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Commutators

Commutators of lie algebra of coordinates \mathfrak{g} reads:

$$[x^M, x^\mu]_\star = x^M \star x^\mu - x^\mu \star x^M = -i \left[\partial_{p^M} a(p^M) \Big|_{p^M=0} \right]^\mu{}_\sigma x^\sigma, \quad (9)$$

x^M can be a:

- ⇒ timelike coordinate
- ⇒ spacelike coordinate
- ⇒ lightlike coordinate

Rescaling $p^M \rightarrow \lambda p^M$ and thus (9) become

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Nonunimodular group and twisted trace

$$\mathcal{G} \text{ nonunimodular} \implies \begin{cases} \Delta(p) = \det(a(\lambda p_0)) = e^{-n\lambda p_0} & d\nu = \Delta d\mu \\ x^M = x^0 \\ \text{tr}(fg) = \text{tr}((\sigma \triangleright g)f), \quad \text{tr} := \int d\nu \end{cases} \quad (10)$$

where the twist $\sigma \in \text{Aut}(\mathcal{M}_\lambda)$ reads:

$$(\sigma \triangleright g)(x^0, \vec{x}) = (e^{in\lambda \partial_0} \triangleright g)(x^0, \vec{x}) = g(x^0 + in\lambda, \vec{x}) \quad (11)$$

note that such twist already appear in κ -Minkowski (Poulain, Wallet 1994)

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KMS weight

With (11) we can define a one-parameter group of $*$ -automorphisms of \mathcal{M}_λ :

$$\{\sigma_t := e^{itn\lambda\partial_0}\}_{t \in \mathbb{R}}, \quad (12)$$

$\Rightarrow \{\sigma_t\}_{t \in \mathbb{R}}$ modular group for the KMS weight given by the twisted trace

KMS weight

A KMS weight ω on a $*$ algebra \mathcal{A} for a modular group of $*$ -automorphism $\{\sigma_t\}_{t \in \mathbb{R}}$

A weight $\omega : \mathcal{A} \rightarrow \mathbb{R}$ densely define and lower semi-continuous

$\{\sigma_t\}_{t \in \mathbb{R}}$ normed-continuous and admit analytic extension $\{\sigma_z\}_{z \in \mathbb{C}}$

KMS conditions

$$\omega \circ \sigma = \omega, \quad \omega(f^\dagger * f) = \omega(\sigma_{i/2}(f) * (\sigma_{i/2}(f))^\dagger)$$

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- $\Rightarrow \{\sigma_t\}_{t \in \mathbb{R}}$ normed-continuous and admit analytic extension $\{\sigma_z\}_{z \in \mathbb{C}}$
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With (11) we can define a one-parameter group of $*$ -automorphisms of \mathcal{M}_λ :

$$\{\sigma_t := e^{tn\lambda\partial_0}\}_{t \in \mathbb{R}}, \quad (13)$$

- ⇒ $\{\sigma_t\}$ modular group for the KMS weight given by the twisted trace
- ⇒ Link with Tomita-Takesaki modular theory with Tomita operator given by $\Delta_T = e^{-in\lambda\partial_0}$
- ⇒ Tomita flow can be used to define a global time which can be interpreted as the "physical time" ⇒ **Thermal time hypothesis**
[Rovelli,1992] [Connes,Rovelli,1994]

Conclusion and Outlook

- ⇒ Construction of several $*$ -algebras
- ⇒ KMS weight appears for only timelike deformed space-times and gives a natural time flow.
- ⇒ Next: Construct an associated symmetry Poincaré deformed algebra.