Local Unitary Invariant Polynomials in the Limit of Large Dimension

Characterization and Distinction of Quantum States

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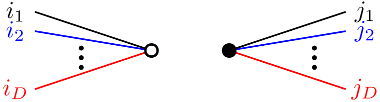
Main Objectives

- How efficiently can multipartite entanglement be characterize ?
- What kind of multipartite quantum information is captured by Local Unitary invariant polynomials in the limit of large dimension ?
- Are there universal properties of multipartite quantum states ?

Plan

Description

- **Multipartite system:** We consider the study of state vector in a Hilbert space
$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_D \,.$$
- **Representation:** Let $|\psi\rangle \in \mathcal{H}$, in a given basis $\{|i_c\rangle\}_{1 \leqslant i_c \leqslant N}$ of each N-dimensional $\mathcal{H}_c$, we have
$$|\psi\rangle = \sum_{i_1,\dots,i_D=1}^{N} \psi_{i_1 \dots i_D} \, |i_1\rangle \otimes \cdots \otimes |i_D\rangle \,.$$
The components $\psi_{i_1 \dots i_D}$ of $|\psi\rangle$ can be represented using a white vertex and colored-edges for each indices (resp. black vertex for $\langle\psi|$)



$= |\psi\rangle \langle\psi| = \rho$

Entanglement

- **Separable/Entangled states:** A pure state $|\psi\rangle$ is separable iff

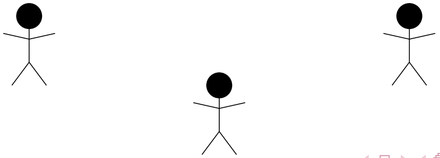
$$|\psi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \otimes \cdots \otimes |\varphi_D\rangle \, .$$

Then, a pure state is entangled iff it is not separable.

Example

$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ *is entangled* $\frac{|01\rangle - |11\rangle}{\sqrt{2}}$ *is separable.*

- **Classification of Entanglement:** [Walter, Gross, Eisert, 2016]



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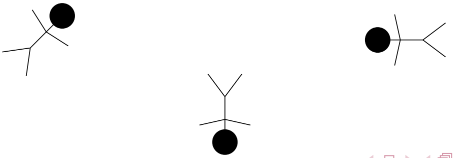
  Local Unitary (LU)



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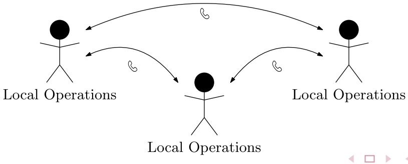
Local Operations and Classical Communication (LOCC)



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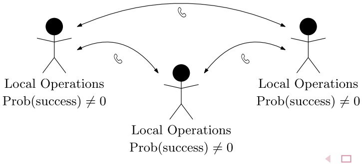
Stochastic LOCC (SLOCC)

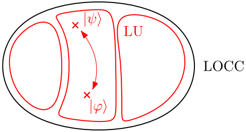


- **LU equivalence:**

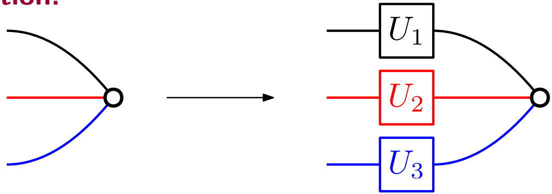
$$|\psi\rangle \sim_{\text{LU}} |\varphi\rangle \iff |\psi\rangle = (U_1 \otimes \cdots \otimes U_D)\,|\varphi\rangle\ ,$$

there exists $N \times N$ unitary matrices U_c (*i.e.* with $U_c \cdot U_c^\dagger = \text{id}$).

- **LU invariance:** The finest description of entanglement.



- **Representation:**



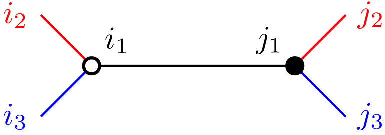
Why Trace-Invariants ?

- **Entanglement measure:** Should be LU-invariant.
- **Trace-invariants:** Defined as closed D-colored graphs G [Bonzom, Gurau, Rivasseau,..., 2009-...]

Claim (Collins, Gurau, Lionni, 2024)

The trace-invariants asymptotically form a linearly independent generating set for the LU invariant polynomials. (Extends to any LU-invariant function as $N \to \infty$)

- **Graphical representation of this useful basis:**



$$= \quad \sum_{i_1, j_1} \psi_{i_1 i_2 i_3} \bar{\psi}_{j_1 j_2 j_3} \delta_{i_1 j_1}$$

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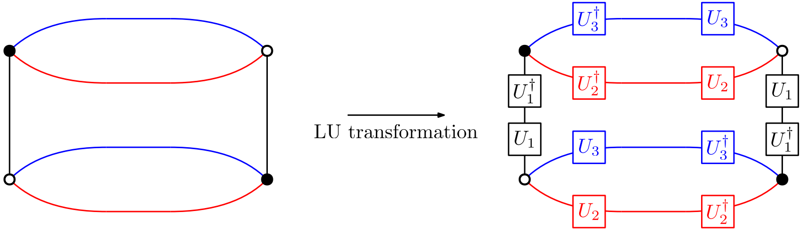
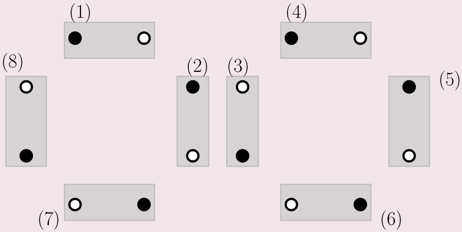


Colored Graph and Permutations

- A D-colored graph G can be described as a set of D permutations.

$$\mathrm{tr}_G(\psi) = \mathrm{tr}_{(\sigma_1,\dots,\sigma_D)}(\psi)\,.$$

Example (k=8, D=3)



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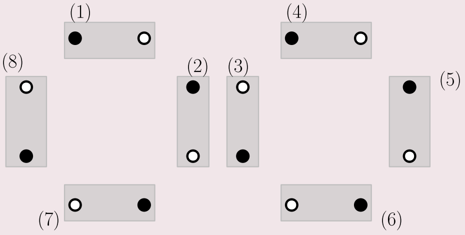

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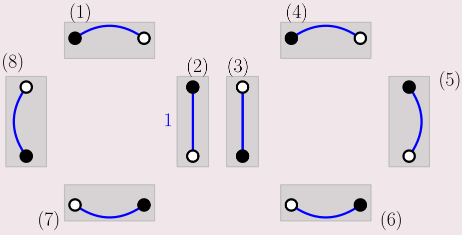

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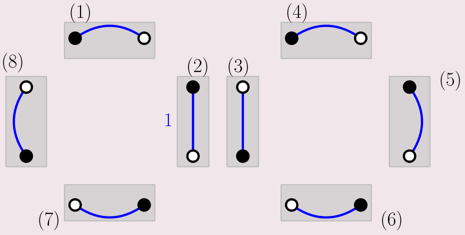

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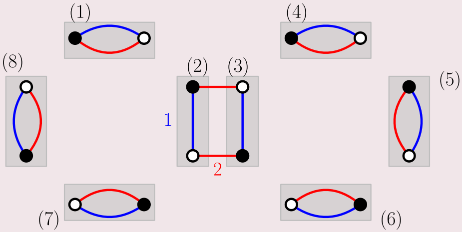

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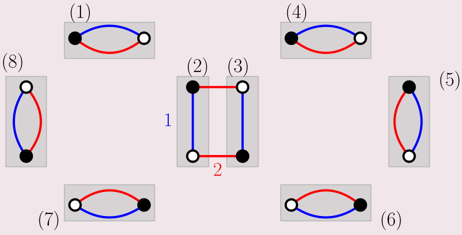

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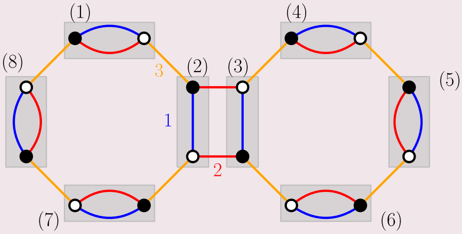

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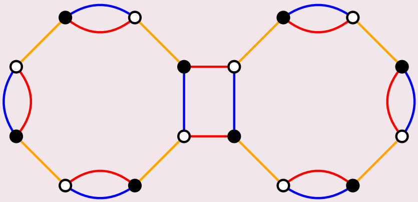

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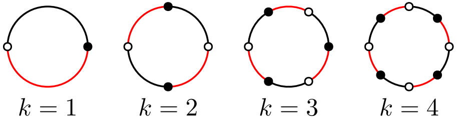
Summary

- LU equivalence class plays a key role in multipartite entanglement.
- Entanglement measures are LU invariant.
- Study of entanglement measures $\Leftrightarrow$ Study of trace-invariants.
- Trace-invariants $\Leftrightarrow$ Colored graphs $\Leftrightarrow$ Permutations.

Plan

What Does the Trace-Invariant Reveal? $(D = 2)$

- **Connected invariants G_k:** There exists a unique trace-invariant for each order k :



- **Meaning:** Thanks to the Schmidt decomposition, there exists (up to LU transf.) othonormal bases $\{|a_i\rangle\}$, $\{|b_i\rangle\}$ s.t.

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{p_i}\, |a_i b_i\rangle \longrightarrow \{p_i\}_i \text{ are LU invariant}\,,$$

$$\longrightarrow \{p_i\}_i \text{ are the spectrum of } \rho_1 = \mathrm{tr}_2\, \rho\,.$$

Evaluation from a $(D = 2)$ trace-invariant :

$$\mathrm{tr}_{G_k}(\psi) = \sum_{i=1}^{N} p_i^k \quad \Rightarrow \quad \text{Exactly } N \text{ is sufficient to recover the spectrum}\,.$$

Classification Problem and Entanglement Entropy

- **Classification:**

$$|\psi\rangle \sim_{\mathrm{LU}} |\varphi\rangle \Leftrightarrow \text{they have the same entanglement spectrum}\,,$$
$$\Leftrightarrow \mathrm{tr}_{G_k}(\psi) = \mathrm{tr}_{G_k}(\varphi) \text{ for } N \text{ trace-invariants}\,.$$

- **Entropy:** Since ρ is positive and has unit trace the p_i's defined a probability density.
  $\Rightarrow$ Define the Shannon entropy of the eigenvalues [von Neumann, 1927].

$$S_{\mathrm{ent.}}(\psi) = -\sum_{i=1}^{N} p_i \log p_i = \underbrace{-\,\mathrm{tr}\,\rho_1 \log \rho_1}_{\text{von Neumann entropy}}\,.$$

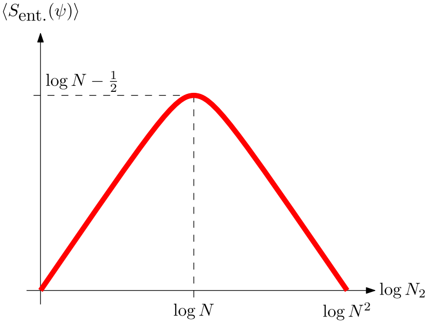
- **Trace-invariants and Entropy:** The replica trick yields [Rényi, 1961]

$$S_{\mathrm{ent.}}(\psi) = \lim_{k\to 1} \frac{1}{1-k} \log \mathrm{tr}_{G_k}(\psi)\,.$$

Universality

- **Page curve:** For any sufficiently large bipartite quantum system (meaning $N_1 N_2 = N^2 \gg 1$), the average entanglement entropy of a Haar-random pure state follows the Page curve.

  [Page, 1993]



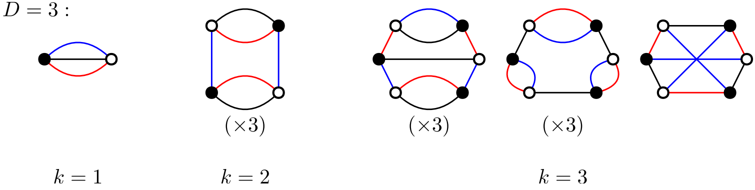
Summary

- N trace-invariants $\Leftrightarrow$ entanglement spectrum $\Rightarrow$ Characterize the LU equivalence class.
- Trace-invariants $\Rightarrow$ entanglement entropy.
- The average entanglement entropy follows a universal curve for uniform bipartite quantum state in the large dimension limit.

Plan

Local unitary invariants for $D > 2$

- **Connected invariants:** The number of trace-invariants grows super-exponentially with k [Ben Geloun, Ramgoolam, 2013].
  ($D = 3$: 1, 3, 7, 26, 97, 624, 4163 . . .)



  $\Rightarrow$ Which trace-invariants are the most relevant to characterize multipartite entanglement ?
- **Matrix → Tensor:** ~~Eigenvalues~~ ; ~~Singular Value Decomposition~~ ; . . .
  $\Rightarrow$ What kind of quantum information does theses relevant trace-invariants contain ?
- **Universality:** Can we extract universal properties in the multipartite context ?

Where do we start ?

What do trace-invariants typically encode ?

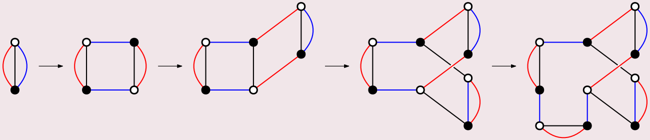
What Trace-Invariants Typically Encode ?

- **Typical value of trace-invariants:** $\psi_{i_1 \dots i_D}$ is a Gaussian random tensor

$$-\langle \log \mathrm{tr}_G(\psi) \rangle \underset{N\to\infty}{\sim} \left(k - 1 + \frac{\omega(G) + 2\Delta(G)}{D-1} \right) \log N \,.$$

- **Gurau degree $\omega \in \mathbb{N}$:** Vanishes for melonic graphs ($D = 3$: planar) [Gurau, 2011].

Example (Melonic graphs)

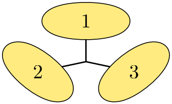


- **Degree of compatibility $\Delta \in \mathbb{N}$:** For $d(\cdot,\cdot)$ the Cayley distance [Collins, Gurau, Lionni, 2024]

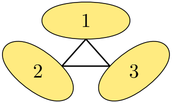
$$\Delta(G) = \min_{\pi \in S_k} \sum_{i<j} \frac{1}{2} \left(d(\sigma_i, \pi) + d(\pi, \sigma_j) - d(\sigma_i, \sigma_j) \right) .$$

Characterization of Trace-Invariants and Quantum Distinction

- **Reference states:** Gives "quantum interpretation" for ω



$= |\mathrm{GHZ}\rangle \longrightarrow -\log \mathrm{tr}_G(\mathrm{GHZ}) = (k-1)\log N$



$= |\phi_{\mathrm{f}}\rangle \longrightarrow -\log \mathrm{tr}_G(\phi_{\mathrm{f}}) = \left(k - 1 + \frac{\omega(G)}{D-1}\right)\log N$

$\Rightarrow \Delta$ is a contribution arising from randomness.

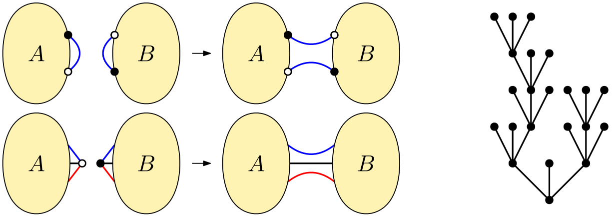
- **Characterization:** Use the ability to distinguish states

ω	Δ	Distinction
$= 0$	$= 0$	$\mathrm{GHZ} = \phi_{\mathrm{f}} = \psi_{\mathrm{random}}$
> 0	$= 0$	$\mathrm{GHZ} = \phi_{\mathrm{f}} \neq \psi_{\mathrm{random}}$
> 0	> 0	$\mathrm{GHZ} \neq \phi_{\mathrm{f}} \neq \psi_{\mathrm{random}}$

Plan

1. How can we construct trace-invariants with non-zero Δ ? How can we track Δ ?
   ⇒ Binary operations $\mathcal{O}$ and tree-based construction

$$\Delta(A\mathcal{O}B) = \Delta(A) + \Delta(B)\,.$$

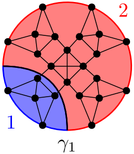


2. Asymmetrize $\mathcal{H}$.
   ⇒ $\dim \mathcal{H}_i = N \to \dim \mathcal{H}_i = N_i$
3. Explore invariants from the literature ?
   ⇒ Compute Δ and the Page "surface" of the Multi-entropy. . .

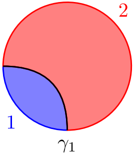
Random Tensor Network

- **Gaussian tensor → Random Tensor Network:**

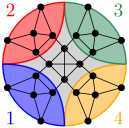
- **Random Tensor Network → AdS / CFT:**



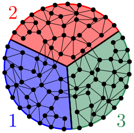
$$S_{vN} = \gamma_1 \log N \to S_{\mathrm{RT}} = \frac{\text{area } \gamma_1}{4G}$$



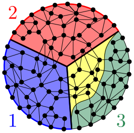
- **Partial results:** [Pennington et al, 2023] [Dong et al, 2021] [Akers et al, 2022-2024]



$\Delta = 0 \ \forall D$



$\Delta > 0 \ (D = 3)$
For specific examples



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