Holographic Derivation of BMS Flux-Balance Laws

Adrien Fiorucci

École polytechnique, Centre de Physique Théorique CNRS

École Normale Supérieure, Lyon

24 June 2025

École Polytechnique | Institut Polytechnique de Paris | cnrs | CPHT Centre de Physique Théorique

Based on

A. Fiorucci, S. Pekar, P. M. Petropoulos and M. Vilatte,

Carrollian-holographic Derivation of BMS Flux-balance Laws

arXiv:2505.00077

Work supported by

Centre National de la Recherche Scientifique (France)

Outline of the talk

1 Flat-space holography

2 Aspects of the boundary geometry

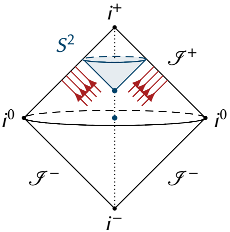
3 Carroll (hyper)momenta & dynamics

4 Summary & outlook

Holography

Promising route towards Quantum Gravity

- "Gravity in a given spacetime region can be encoded on the codimension-one boundary of that region." [t' Hooft 93, Susskind 95]
  - E_{grav} = boundary term [ADM 59], BH entropy $\propto$ area [Bekenstein 73].
  - Explicit realisation: AdS/CFT correspondence ($\Lambda < 0$) [Maldacena 97].
- Focus on realistic models:
  - Our Universe has $\Lambda \gtrsim 0$. $\Lambda = 0$ good approximation at astrophysical scale.
  - Asymptotically flat spacetimes ($\Lambda = 0$) cover large class of situations: collider physics, astrophysics, gravitational-wave astronomy, ...



Holographic energy-momentum tensor $(\Lambda = -3\ell^2,\ 16\pi G = 1)$

Bulk analysis in four dimensions

- Conformal compactification [Penrose 65]: spacelike infinity $(\mathscr{I}, \mathbf{g} \sim \mathcal{B}^2 \mathbf{g})$.
  - Gaussian normal coordinates [Fefferman-Graham 85] where $\mathscr{I} = \{\rho \longrightarrow 0^+\}$:

$$ds^2 = \frac{\ell^2}{\rho^2}\left(d\rho^2 + \gamma_{\mu\nu} dx^\mu dx^\nu\right), \qquad g_{\mu\nu} = \lim_{\rho \to 0^+} \gamma_{\mu\nu}.$$

  - Holographic renormalisation [de Haro et al. 00, Compère-Marolf 07]:

$$\delta S^{\text{ren}}_{\text{grav}} = \frac{1}{2} \int_{\mathscr{I}} d^3x \sqrt{-g}\, T^{\mu\nu} \delta g_{\mu\nu}, \qquad T_{\mu\nu} = \frac{1}{2\ell} \lim_{\rho \to 0^+} \partial^3_\rho \gamma_{\mu\nu}.$$

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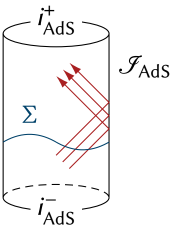
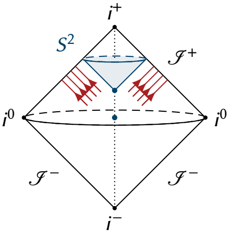
Boundary analysis in three dimensions

- Assume $\exists$ effective action $S_\mathrm{b}[\Phi, \mathbf{g}]$ s.t. $\delta S_\mathrm{b} = \delta S^{\mathrm{ren}}_{\mathrm{grav}}$ when Φ are on-shell.
  - Relation between grav. symplectic structure and vev/source in CFT_3.
- Diffeo. invariance: $\nabla_\mu T^{\mu\nu} = 0$. Weyl invariance: $g^{\mu\nu} T_{\mu\nu} = 0$.
  - Reproduce Einstein eq. without knowing the microscopic content in Φ.
  - Holo. dictionary: $T_{\mu\nu}$ = EM tensor of dual CFT_3. [Balasubramanian-Kraus 99]

Comparison AdS/flat-space holography

	AdS ($\Lambda < 0$)	**Flat** ($\Lambda = 0$)
Conformal boundary $\mathscr{I}$	Spatial infinity	Null infinity
Boundary metric	$\frac{\Lambda}{3}\mathrm{d}t^2 + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}$	$0\,\mathrm{d}t^2 + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}$
Boundary physics	Lorentzian	Carrollian ("$c \to 0$")
Grav. dynamics	$\nabla_\mu T^\mu{}_\nu = 0$, $T^\mu{}_\mu = 0$	BMS flux-balance laws
Compatible B.C.	Dirichlet ($\dot{Q} = 0$)	Leaky ($\dot{Q} \neq 0$)
Relevant symmetries	Conformal group	BMS group (∞-dim.)



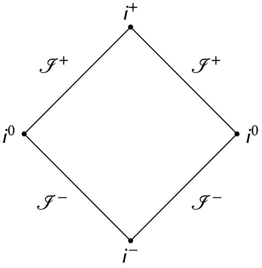


Null infinity

Conformal Carroll geometry

- Normal vector $\boldsymbol{\upsilon}$ is null at $\mathscr{I}$ $\Rightarrow$ degenerate boundary metric: $\mathbf{g}(\boldsymbol{\upsilon}, \cdot) = \mathbf{0}$.
- Repelled at ∞ $\Rightarrow$ meaning within *conformal compactification* [Penrose 65]

$$\mathbf{g} \sim \mathcal{B}^2 \mathbf{g}, \qquad \boldsymbol{\upsilon} \sim \mathcal{B}^{-1} \boldsymbol{\upsilon}.$$



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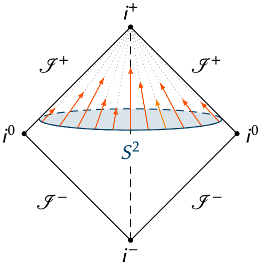
Symmetries [Bondi-van der Burg-Metzner 62, Sachs 62]

$$\mathscr{L}_{\boldsymbol{\xi}} \mathbf{g} = 2B(\boldsymbol{\xi})\mathbf{g}, \quad \mathscr{L}_{\boldsymbol{\xi}} \mathbf{\upsilon} = -B(\boldsymbol{\xi})\mathbf{\upsilon}.$$

- In coordinates (u, x^a) such that $\mathbf{\upsilon} = \boldsymbol{\partial}_u$,

$$\boldsymbol{\xi} = \left(T(x) + \tfrac{u}{2}\mathcal{D}_a Y^a\right)\boldsymbol{\partial}_u + Y^a(x)\boldsymbol{\partial}_a.$$

- BMS algebra: $\mathfrak{so}(3,1) \ltimes \mathfrak{s} \supset \mathfrak{iso}(3,1)$.
  - 3D conformal Carroll algebra [Duval et al. 14].
  - $\mathfrak{s}$ ∞-dim. abelian ideal because $\mathbf{\upsilon}$ is null.
  - $T(x)$: supertranslations. $\boldsymbol{Y}(x)$: C.K.V. on S^2.

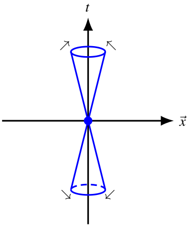


Carroll relativity

- Slow-boost limit of S.R. where $\Delta x \gg \Delta t$ [Lévy-Leblond 65]
  - Equivalent to $c \to 0 \Rightarrow$ opposite limit of Galilei (for which $\Delta x \ll \Delta t$).
  - Ultra-locality: lightcones close $\Leftrightarrow$ unique field-of-observers $\boldsymbol{\upsilon}$.
  - Space is absolute $x^a \mapsto x^a$. Boost in time direction: $t \mapsto t - \lambda_a x^a$.
- Geometry: $\boldsymbol{\upsilon}$ spans the 1D-radical of the metric $\mathbf{g}$. [Geroch 76, Henneaux 79]

$$\mathbf{g}(\boldsymbol{\upsilon}, \cdot) = \mathbf{0}, \qquad \mathbf{g}(\mathbf{V}, \mathbf{V}) \geq 0, \qquad \mathbf{g}(\mathbf{V}, \mathbf{V}) = 0 \Leftrightarrow \mathbf{V} \propto \boldsymbol{\upsilon}.$$

  - Ubiquitous to study null manifolds. [Jankiewicz 54, Vogel 65, Daŭtcourt 67]
  - Also useful for *e.g.* strings [Gomis et al. 16] or condensed matter [...].



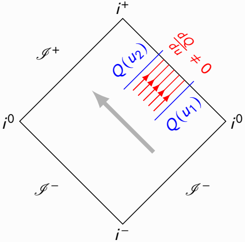
BMS flux-balance laws

Asymptotic Einstein eq. [BMS 62, Tamburino-Winicour 67, Barnich-Troessaert 10]

- Near $\mathscr{I}$, the gravitational field is described by
  1. M = *mass aspect* (distribution of grav. energy on celestial S^2).
  2. N_a = *angular-momentum aspect* (distribution of AM on celestial S^2).
  3. $\mathcal{C}_{ab}$ = shear of outgoing null vector. $\mathcal{N}_{ab} = \mathscr{L}_{\boldsymbol{\upsilon}} \mathcal{C}_{ab}$ (news: grav. radiation).
- Einstein eq. $\Rightarrow$ **flux-balance laws**: BMS charges not conserved along $\mathscr{I}$.

$$\boldsymbol{\upsilon}(M) = \frac{1}{4}\mathcal{D}_a\mathcal{D}_b\mathcal{N}^{ab} - \frac{1}{8}\mathcal{N}_{ab}\mathcal{N}^{ab},$$

$$\boldsymbol{\upsilon}(N_a) = \mathcal{D}_a M - \frac{1}{4}\mathcal{N}^{bc}\mathcal{D}_a\mathcal{C}_{bc} + \frac{1}{16}\mathcal{D}_a\left(\mathcal{C}_{bc}\mathcal{N}^{bc}\right) + \frac{1}{2}\mathcal{D}^b\left(\mathcal{C}^c{}_{[a}\mathcal{N}_{b]c} + \mathcal{D}_{[a}\mathcal{D}^c\mathcal{C}_{b]c}\right).$$



with **"hard" parts** = gravitons and **"soft" parts** = Goldstone/pure gauge.

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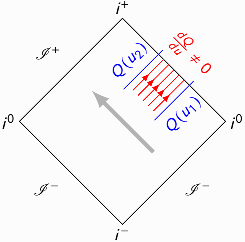

with **"hard" parts** = gravitons and **"soft" parts** = Goldstone/pure gauge.

Our goal: boundary derivation of these laws with Carroll-geometric tools.

Why do we care?

Test the limits of holographic paradigm

- Non-conserved charges $\Rightarrow$ sourced dual theory. Nature of the sources?
- Also applicable in AdS/CFT with leaky B.C. [*Fiorucci-Ruzziconi 21*]

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Building block of flat-space holography

- AdS/CFT: $T_{\mu\nu}$ provides the EM tensor of the dual CFT (Ward identities).
- Fundamental understanding of the status of flux-balance laws.
- Applicable to any null boundary, *e.g.*, BH horizon. [Ruzziconi-Zwikel 25]

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BUILDING BLOCK OF FLAT-SPACE HOLOGRAPHY

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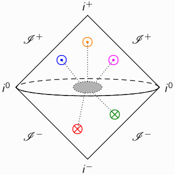
CARROLLIAN SCATTERING AMPLITUDES [*Fiorucci et al.* 22, Mason et al. 23,...]

- Massless scattering fully encoded at $\mathscr{I}$.
- Carroll-invariant effective generating functional [Adamo et al. 14, Kraus-Myers 25] and related EM tensor [Ruzziconi-Saha 25] ...



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Bulk gravitational field

Conformal compactification [Penrose 63-65, Geroch 77]

$\phi : \widetilde{\mathscr{M}} \to \mathscr{M}$ to remove the pole at infinity with

$$\phi^* \mathbf{G} = \Omega^2 \widetilde{\mathbf{G}}, \quad \Omega > 0, \quad \mathscr{I} \equiv \{\Omega \to 0^+\}.$$

- Bulk Weyl rescalings induced by the process:

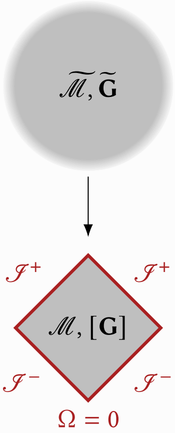
$$\Omega \mapsto \mathcal{B}^{-1}\Omega, \qquad \mathbf{G} \mapsto \mathcal{B}^{-2}\mathbf{G}.$$

  One physical $\widetilde{\mathbf{G}}$ ↔ “conformal” class of $\mathbf{G}$’s.

- Levi-Civita for $\widetilde{\mathbf{G}}$ induces *Weyl–Levi–Civita connection* [Weyl 18] for $\mathbf{G}$:

$$\boldsymbol{\nabla}\mathbf{G} - 2\mathbf{A} \otimes \mathbf{G} = \mathbf{0}.$$

- Weyl connection: $\mathbf{A}$ (*i.e.*, $\mathbf{A} \mapsto \mathbf{A} + \mathrm{d} \ln \mathcal{B}$). Additional background structure in normal one-form to $\mathscr{I}$ defined as $\boldsymbol{\nu} = \mathrm{d}\Omega + \mathbf{A}\Omega$.



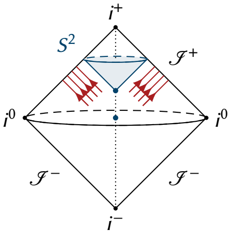
Boundary geometry (1)

Assuming $\widetilde{\mathbf{G}}$ solves Einstein equations on physical spacetime $\widetilde{\mathscr{M}}$,

$\mathscr{I}$ is a codimension-one null hypersurface

$$\mathbf{g}(\boldsymbol{\upsilon}, \cdot) = \mathbf{0} \quad \text{where } \mathbf{g} = \iota^* \mathbf{G}, \quad \iota_* \boldsymbol{\upsilon} = \mathbf{G}^{-1}(\boldsymbol{\nu}, \cdot) \quad \text{for } \iota : \mathscr{I} \hookrightarrow \mathscr{M}.$$

- Conformal Carroll geometry [Geroch 76, Henneaux 79]
- Field-of-observers $\boldsymbol{\upsilon}$ (null generators of $\mathscr{I}$); (dual) clock form $\boldsymbol{\tau}(\boldsymbol{\upsilon}) = 1$.
- "Cut" = horizontal space defined as $\boldsymbol{\tau}(\mathbf{V}) = 0$ with topology of S^2.
- Orthonormal basis $\mathbf{e}_a$ and dual cobasis $\theta^a \quad \Rightarrow \quad \mathbf{g} = \delta_{ab} \theta^a \odot \theta^b$.



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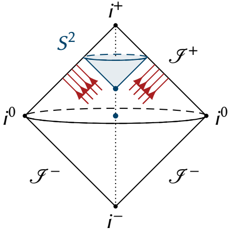
Notations

Carroll–Cartan frame $\{\mathbf{e}_A\}$ & dual coframe $\{\theta^A\}$ $(A = 0, a)$ with $\mathbf{e}_0 \equiv \boldsymbol{\upsilon}$, $\theta^0 \equiv \boldsymbol{\tau}$ and $a = 1, 2$.



Boundary geometry (2)

Assuming $\widetilde{\mathbf{G}}$ solves Einstein equations on physical spacetime $\widetilde{\mathscr{M}}$,

Weingarten map of $\mathscr{I}$ proportional to identity.

- Shear of $\boldsymbol{\upsilon}$ vanishes. [Penrose 65, Geroch 76, Ashtekar 81]
- Bulk Weyl–Levi–Civita connection induces boundary *torsion-free* and *Weyl–Carroll compatible* connection [Petropoulos et al. 18-20]

$$\nabla \mathbf{g} - 2\boldsymbol{\alpha} \otimes \mathbf{g} = \mathbf{0}, \quad \nabla \boldsymbol{\upsilon} + \boldsymbol{\alpha} \otimes \boldsymbol{\upsilon} = \mathbf{0},$$

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- Connection one-forms: $\nabla_{\mathbf{e}_A} \mathbf{e}_B \equiv \omega^C{}_{AB} \mathbf{e}_C$ and $\boldsymbol{\omega}^A{}_B = \omega^A{}_{CB} \theta^C$.
- Weyl-covariant derivative $\boldsymbol{\nabla} \to \boldsymbol{\mathfrak{D}} \equiv \boldsymbol{\nabla} - w\boldsymbol{\alpha}$ on weight-w fields:

$$\Phi \mapsto \mathcal{B}^w \Phi \quad \Rightarrow \quad \boldsymbol{\mathfrak{D}} \Phi \mapsto \mathcal{B}^w \boldsymbol{\mathfrak{D}} \Phi$$

Symmetries = $\mathfrak{carr}(3) \oplus \mathbb{R}_{\text{Weyl}} \oplus \mathfrak{diff}(\mathscr{I})$

- Spatial rotations $(r_{ab} \in \mathfrak{so}(2))$.

$$\delta_r \boldsymbol{\upsilon} = \mathbf{0}, \quad \delta_r \mathbf{e}_a = r_a{}^b \mathbf{e}_b, \quad \delta_r \boldsymbol{\tau} = \mathbf{0}, \quad \delta_r \theta^a = r^a{}_b \theta^b,$$
$$\delta_r \boldsymbol{\omega}^0{}_0 = \mathbf{0}, \quad \delta_r \boldsymbol{\omega}^a{}_0 = r^a{}_b \boldsymbol{\omega}^b{}_0, \quad \delta_r \boldsymbol{\omega}^0{}_a = r_a{}^b \boldsymbol{\omega}^0{}_b, \quad \delta_r \boldsymbol{\omega}^a{}_b = -\mathfrak{D} r^a{}_b.$$

- Carroll boosts.

$$\delta_\lambda \boldsymbol{\upsilon} = \mathbf{0}, \quad \delta_\lambda \mathbf{e}_a = \lambda_a \boldsymbol{\upsilon}, \quad \delta_\lambda \boldsymbol{\tau} = -\lambda_a \theta^a, \quad \delta_\lambda \theta^a = \mathbf{0},$$
$$\delta_\lambda \boldsymbol{\omega}^0{}_0 = -\lambda_a \boldsymbol{\omega}^a{}_0, \quad \delta_\lambda \boldsymbol{\omega}^a{}_0 = \mathbf{0}, \quad \delta_\lambda \boldsymbol{\omega}^0{}_a = \mathfrak{D}\lambda_a, \quad \delta_\lambda \boldsymbol{\omega}^a{}_b = \lambda_b \boldsymbol{\omega}^a{}_0.$$

- Weyl rescalings.

$$\delta_B \boldsymbol{\upsilon} = B\boldsymbol{\upsilon}, \quad \delta_B \mathbf{e}_a = B\mathbf{e}_a, \quad \delta_B \boldsymbol{\tau} = -B\boldsymbol{\tau}, \quad \delta_B \theta^a = -B\theta^a,$$
$$\delta_B \boldsymbol{\omega}^0{}_0 = \mathrm{d}B, \quad \delta_B \boldsymbol{\omega}^a{}_0 = \mathbf{0}, \quad \delta_B \boldsymbol{\omega}^0{}_a = \mathbf{0}, \quad \delta_B \boldsymbol{\omega}^a{}_b = \delta^a{}_b \mathrm{d}B.$$

- Diffeomorphisms: $\delta_\xi \mathbf{e}_A = \mathscr{L}_\xi \mathbf{e}_A$, $\delta_\xi \theta^A = \mathscr{L}_\xi \theta^A$, $\delta_\xi \boldsymbol{\omega}^A{}_B = \mathscr{L}_\xi \boldsymbol{\omega}^A{}_B$.
- **Note:** $\boldsymbol{\omega}^a{}_0$ (co. for Galilean boosts) & $\boldsymbol{\omega}_{\langle ab \rangle}$ covariant $\forall$ transformations.

Bondi shear

No Levi-Civita connection for Carroll geometries

Are $\boldsymbol{\nabla}\mathbf{g}$, $\boldsymbol{\nabla}\boldsymbol{\upsilon}$ and the torsion Θ^A sufficient input data to determine the connection ∇? No, because $\nexists\, \mathbf{g}^{-1}$, which implies [e.g. Bekaert-Morand 15]

$$\boldsymbol{\Theta}_{(a}(\boldsymbol{\upsilon}, \mathbf{e}_{b)}) = \boxed{\frac{1}{2}(\mathscr{L}_{\boldsymbol{\upsilon}}\mathbf{g})_{ab}} + \theta^c(\nabla_{\mathbf{e}_{(a}}\boldsymbol{\upsilon})\delta_{b)c} - \frac{1}{2}\boldsymbol{\nabla}_{\boldsymbol{\upsilon}}\mathbf{g}(\mathbf{e}_a, \mathbf{e}_b).$$

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Radiative contributions to ∇

- Symmetric transverse part of torsion (3 components) cannot be fixed independently from geometry.
- Therefore, 3 d.o.f. encoded into symmetric transverse tensor $\omega^0{}_{(ab)}$ (part of Carroll-boost connection).
- Bulk analysis: $\omega^0{}_{ab}$ corresponds to deviation of outgoing null geodesics arriving at $\mathscr{I}$ ($\omega^0{}_{\langle ab\rangle}$ *Bondi shear*). [Geroch 77, Ashtekar 81]

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Lessons

1. Work in conformally compactified spacetime

- $\mathscr{I}$ does not exist outside conformal compactification. [Penrose 65]
- Weyl geometry on the unphysical spacetime $\longrightarrow$ Weyl geometry on $\mathscr{I}$.
- BMS only emerges after boundary gauge fixing.

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2. Do not insist on Lorentzian intuition

- Non-conservation of the charges $\Rightarrow$ notion of EM tensor elusive.
  [Wald-Zoupas 99, Barnich-Troessaert 11, Petropoulos et al. 18-20]
- $\Lambda \to 0$ limit of AdS (Brown–York) energy-momentum tensor ill-defined.
  [*Compère et al. 19*, Petropoulos et al. 23]
- Carroll connections are not entirely fixed by background geometry
  [Jankiewicz 54, Vogel 65, Daŭtcourt 67, Ashtekar 81, Hartong 15, Bekaert-Morand 15…]
- Equivalent statement: $\exists$ external sources on top of geometry at $\mathscr{I}$.
  [Troessaert 15, Wieland 20, *Fiorucci et al. 22*]

Carrollian (hyper)momenta

Boundary variational principle

Assuming ∃ microscopic description of the boundary theory in terms of fields Φ coupled to geometry $\{\theta^A\}$ and connection $\{\omega^A{}_B\}$.

$$\delta S = \int_{\mathscr{I}} \mu \left(\mathcal{E}_\Phi \delta\Phi + \delta\theta^A (\mathbf{T}_A) + \delta\omega^A{}_B (\Omega_A{}^B) \right).$$

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Carrollian momenta [Petropoulos et al. 18-22]

- $\mathbf{T}_0 = \Pi\upsilon + \Pi^a \mathbf{e}_a$ where Π is the *energy* and Π^a is the *energy flux*.
- $\mathbf{T}_a = P_a\upsilon + \Pi^b{}_a \mathbf{e}_b$ where P_a is the *momentum* and $\Pi^b{}_a$ the *stress tensor*.

Carrollian (hyper)momenta

Boundary variational principle

Assuming ∃ microscopic description of the boundary theory in terms of fields Φ coupled to geometry $\{\theta^A\}$ and connection $\{\omega^A{}_B\}$.

$$\delta S = \int_{\mathscr{I}} \mu \left(\mathcal{E}_\Phi \delta\Phi + \delta\theta^A(\mathbf{T}_A) + \delta\omega^A{}_B(\Omega_A{}^B) \right).$$

Carrollian momenta [Petropoulos et al. 18-22]

- $\mathbf{T}_0 = \Pi\upsilon + \Pi^a \mathbf{e}_a$ where Π is the *energy* and Π^a is the *energy flux*.
- $\mathbf{T}_a = P_a\upsilon + \Pi^b{}_a \mathbf{e}_b$ where P_a is the *momentum* and $\Pi^b{}_a$ the *stress tensor*.

Hypermomenta [Hehl et al. 76, Iosifidis 19-21, Iosifidis et al. 25]

- Vector fields $\Omega_A{}^B \equiv \Omega_A{}^{CB}\mathbf{e}_C$ reacting to variations of the connection.
- Unavoidable since part of ∇ is *extrinsic data*.
- Geometric implementation of *radiative sources*. [*Fiorucci et al. 22*]

Constraints and dynamical equations

Carroll-boost invariance

$$\delta_\lambda S = 0 \quad \Longrightarrow \quad \Pi^a + \mathfrak{D} \cdot \Omega_0{}^a = 0 \text{ on-shell.}$$

Carroll-boost invariance forbids matter energy flux but allows for "redshift."

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Rotation and Weyl invariance

$$\delta_r S = 0 \quad \Rightarrow \quad \Pi_{[ab]} = \mathfrak{D} \cdot \Omega_{[ab]} \text{ on-shell,}$$
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Diffeomorphism invariance $\Rightarrow$ reflection of "$\nabla_\mu T^\mu{}_\nu = 0$"

$$\mathfrak{D} \cdot \mathbf{T}_0 = \mathcal{R}^A{}_B(\boldsymbol{\upsilon}, \Omega_A{}^B), \quad \mathfrak{D} \cdot \mathbf{T}_a = \mathcal{R}^B{}_C(\mathbf{e}_a, \Omega_B{}^C) \quad \text{on-shell.}$$

If $\boldsymbol{\omega}^A{}_B$ is not completely determined by the geometry, the R.H.S. cannot be recast as divergences $\Rightarrow$ **genuine flux-balance laws!** [Duval-Kunzle 74]

Gravitational flux-balance laws

Our main result

- Upon providing the right holographic dictionary between *gravitational momenta* and *Carroll (hyper)momenta* [*Fiorucci et al. 22*], the Carrollian evolution equations

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  describe the Bondi mass and AM loss formulas.

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Minimal holographic dictionary

Boundary (3D) $\mathscr{I}$	$\Leftrightarrow$	Shear-free conformal Carroll manifold.
Carrollian momenta	$\Leftrightarrow$	$\mathcal{O}(\Omega^3)$ in the metric.
Hypermomenta	$\Leftrightarrow$	turned on by $\omega^0{}_{\langle ab \rangle} = \mathcal{C}_{ab}$,

Flat-space holographic dictionary (BMS gauge)

- Carrollian momenta are Barnich-Troessaert's BMS momenta [BT 11].

$$\Pi = 4M, \quad P_a = 2N_a + \frac{1}{16}\mathcal{D}_a\big(\mathcal{C}^{bc}\mathcal{C}_{bc}\big), \quad \Pi^a = -\mathcal{D}_b\mathcal{N}^{ab} - \frac{1}{2}\mathcal{D}^a\mathcal{R},$$
$$\Pi_{ab} = \mathcal{D}^c\mathcal{D}_{[a}\mathcal{C}_{b]c} - \frac{1}{2}\mathcal{N}_{[a}{}^c\mathcal{C}_{b]c} - \frac{1}{4}\mathcal{R}\mathcal{C}_{ab} - 2M\delta_{ab}.$$

- Hypermomenta are driven by radiative d.o.f.

$$\Omega_0{}^0 = \mathbf{0}, \quad \Omega_{ab} = \Omega_{[ab]} = \mathcal{D}_{[a}\mathcal{C}^c{}_{b]}\mathbf{e}_c, \quad \Omega_0{}^a = \big(\mathcal{N}^{ab} + \frac{1}{2}\mathcal{R}\delta^{ab}\big)\mathbf{e}_b.$$

Note. Constraints under local symmetries fix the "soft" parts.

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Back on the notion of energy-momentum tensor

- Flux-balance laws $\Rightarrow$ notion of energy-momentum tensor ellusive as $\exists$ ambiguity in separing charge/flux. This is a well-known fact!
[Wald-Zoupas 99, Barnich-Troessaert 11, *Compère et al. 18-20*, …]

$$\nabla_\mu T^\mu{}_\nu = \widehat{F}_\nu - \nabla_\mu H^\mu{}_\nu \Rightarrow \nabla_\mu \widehat{T}^\mu{}_\nu = \widehat{F}_\nu, \quad \widehat{T}^\mu{}_\nu = T^\mu{}_\nu + H^\mu{}_\nu.$$

Outline of the talk

1 Flat-space holography

2 Aspects of the boundary geometry

3 Carroll (hyper)momenta & dynamics

4 Summary & outlook

Summary & perspectives

Main punchlines

1. First purely boundary-intrinsic derivation of **BMS flux-balance laws**.
2. Introduction of **hypermomenta** (“*sources*”) is unavoidable.
3. There is generally **non-vanishing local energy flux** in Carroll theories. This is ***not*** in contradiction with Carroll-boost invariance.
4. Form of the hypermomenta $\longrightarrow$ hints for the microscopic theory for the radiative sources $\Longrightarrow$ relevant for **Carrollian amplitudes**.

Summary & perspectives

Main punchlines

1. First purely boundary-intrinsic derivation of **BMS flux-balance laws**.
2. Introduction of **hypermomenta** ("*sources*") is unavoidable.
3. There is generally **non-vanishing local energy flux** in Carroll theories. This is ***not*** in contradiction with Carroll-boost invariance.
4. Form of the hypermomenta $\longrightarrow$ hints for the microscopic theory for the radiative sources $\Longrightarrow$ relevant for **Carrollian amplitudes**.

Open avenues

- Link with $\Lambda \longrightarrow 0$ limit of AdS/CFT. [*Fiorucci-Ruzziconi 21*, Petropoulos et al. 18-23]
- Link with Brown-York EM tensor at $\mathscr{I}$ [Chandrasekaran et al. 23] and holographic renormalisation [de Haro et al. 00] adapted to Carroll/null geometries [*Compère et al. 18*, Hartong et al. 25].
- Microscopic theory for the d.o.f. in connection.

Merci pour votre attention!

"*The Mad Hatter teaching Carroll holography*, drawn in the style of Disney Studios."

Generated by **DALL·E**.

Appendices

BMS boundary gauge fixing

- Adapted coordinates (u, x^i). Writing $\mathbf{g} = \delta_{ab}\theta^a{}_i\theta^b{}_j \mathrm{d}x^i \mathrm{d}x^j$, we impose

$$\boldsymbol{\tau} = \mathrm{d}u, \qquad \boldsymbol{\upsilon} = \partial_u, \qquad \boldsymbol{\theta}^a = \theta^a{}_i(x^j)\mathrm{d}x^i, \qquad \mathbf{e}_a = \theta^i{}_a(x^j)\partial_i.$$

- Gauge-fixing variation: action of diffeo compensated by local transfos.

$$\delta_\xi \equiv \mathscr{L}_\xi + \delta_{\lambda(\xi)} + \delta_{r(\xi)} + \delta_{B(\xi)} \qquad \text{s.t.} \qquad \delta_\xi \boldsymbol{\tau} = \mathbf{0} = \delta_\xi(\mathscr{L}_{\boldsymbol{\upsilon}}\boldsymbol{\theta}^a).$$

- Impose usual boundary conditions $\delta\mathbf{g} = \mathbf{0}$, *i.e.* $\delta_\xi \boldsymbol{\theta}^a = \mathbf{0}$. Then:
  1. $\xi = f\partial_u + Y^i\partial_i$ is a CKV on $\mathscr{I}$ ($\in \mathfrak{bms}_4$ algebra),

$$\partial_u f = \tfrac{1}{2}\mathcal{D}_i Y^i, \qquad \partial_u Y^i = 0, \qquad \mathcal{D}_{(i}Y_{j)} = \delta_{ij}\mathcal{D}_k Y^k.$$

  2. Local parameters fixed as $\lambda_a = \mathbf{e}_a(f)$, $r_{ab} = \theta^i{}_{[a}\theta^j{}_{b]}\mathcal{D}_i Y_j$ & $B = \boldsymbol{\upsilon}(f)$.
  3. Recover usual phase-space transformations from Carroll-geometric considerations, *i.e.* for the shear $\mathcal{C}_{ij} = -2\theta^a{}_{\langle i}\theta^b{}_{j\rangle}\omega^0{}_{ab}$ [Barnich-Troessaert 10]

$$\delta_\xi \mathcal{C}_{ij} = f\partial_u \mathcal{C}_{ij} + \mathscr{L}_Y \mathcal{C}_{ij} - \tfrac{1}{2}\mathcal{D}_k Y^k \mathcal{C}_{ij} - 2\mathcal{D}_{\langle i}\mathcal{D}_{j\rangle} f.$$

Derivation of generic flux-balance laws (1)

- Contract δS with a diffeomorphism $(\delta \to \mathscr{L}_\xi)$ and set Φ on-shell.

$$\mathscr{L}_\xi S = \int_{\mathscr{I}} \mu\big(\mathscr{L}_\xi \theta^A(\mathbf{T}_A) + \mathscr{L}_\xi \omega^A{}_B(\Omega_A{}^B)\big) = 0.$$

- Expand Lie derivatives and use Cartan's magic formula to integrate by parts, as the identity

$$\mu f_1 \mathbf{e}_A(f_2) = \mathrm{d}(\dots) - \mu\big(f_2 \mathbf{e}_A(f_1) - f_1 f_2 C^B{}_{AB}\big)$$

can be checked **separately** for $A = 0$ and $A = a$ where $[\mathbf{e}_A, \mathbf{e}_B] \equiv C^C{}_{AB}\mathbf{e}_C$.

- Use the fact that ∇ is torsion-free and Weyl–Carroll compatible to get

$$\int_{\mathscr{I}} \mu\, \xi^A \left(\mathfrak{D} \cdot \mathbf{T}_A - \mathcal{R}^C{}_{BAD}\Omega_C{}^{DB} + \omega^C{}_{AB}\Delta^B{}_C\right) = 0,$$

where

$$\Delta^B{}_C \equiv T^B{}_C + \mathfrak{D} \cdot \Omega_C{}^B.$$

Derivation of generic flux-balance laws (2)

- Finally impose the constraints imposed by local transformations:
  1. Carroll-boost invariance: $\Delta^a{}_0 \equiv \Pi^a + \mathfrak{D} \cdot \Omega_0{}^a = 0$.
  2. Rotation invariance: $\Delta_{[ab]} = \Pi_{[ab]} + \mathfrak{D} \cdot \Omega_{[ba]} = 0$.
  3. Weyl invariance: $\Delta^A{}_A = \Pi + \Pi^a{}_a + \mathfrak{D} \cdot \left(\Omega_0{}^0 + \Omega_a{}^a\right) = 0$.
- The term with the bare connection therefore vanishes:

$$\begin{aligned}\omega^C{}_{AB}\Delta^B{}_C &= \alpha_A \Delta^0{}_0 + \omega^0{}_{Ab}\Delta^b{}_0 + \omega^c{}_{Ab}\Delta^b{}_c \\ &= \alpha_A\left(\Delta^0{}_0 + \Delta^a{}_a\right) + \omega_{cAb}\Delta^{[bc]} = 0\end{aligned}$$

  recalling that $\omega_{(a|B|c)} = \alpha_B$, by Carroll–Weyl compatibility.
- It then remains

$$\int_{\mathscr{I}} \boldsymbol{\mu}\, \xi^A \left(\mathfrak{D} \cdot \mathbf{T}_A - \mathfrak{R}^C{}_B(\mathbf{e}_A, \Omega_C{}^B)\right) = 0$$

  for arbitrary ξ^A, hence the result. □