Test-Field *vs* Physical Quasi-Normal Modes of a modified BH solution

Supervised by Etera LIVINE (LPENSL) and Alexandre ARBEY (IP2I)

Clara MONTAGNON

PhD student in Laboratoire de Physique de l'ENS de Lyon (LPENSL, France)

Mathematical Physics of Gravity and Symmetry 2025, June 23-26, 2025

Motivations

- Since the first detection of GWs ten years ago by LIGO, the QNMs regained a lot of interest and are now viewed as a direct way to study the physic of BHs.
- The QNMs are allowing to characterise each theory and BH model.

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- Two main different directions are considered for the search of an alternative theory of gravity: modifying GR or developing new microscopic space-time structures.
- Despite the lack of an equivalent of the Einstein equations for the latter case, the study of test-field perturbations on effective models of BHs via the QNMs has become a common practice.
- However, the spin 2 test-field perturbations are derived from the Einstein equations and thus do not take into account the modified dynamic.

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- Despite the lack of an equivalent of the Einstein equations for the latter case, the study of test-field perturbations on effective models of BHs via the QNMs has become a common practice.
- However, the spin 2 test-field perturbations are derived from the Einstein equations and thus do not take into account the modified dynamic.

- **How good approximation are the spin $s = 2$ test-field perturbations for the description of the proper physical gravitational perturbations?**
- **In other words, we want to question the relative importance of BH metrics corrections *versus* space-time dynamics corrections.**

We consider here the quadratic shift-symmetric Horndeski theories described by the action

$$S\left[g_{\mu\nu},\phi\right] = \int \mathrm{d}^4x\sqrt{-g}\left[F(X)R + P(X) + Q(X)\Box\phi + 2\frac{\partial F}{\partial X}\left(\phi_{\mu\nu}\phi^{\mu\nu} - (\Box\phi)^2\right)\right],$$

where $X = \nabla^\mu\phi\nabla_\mu\phi$ and $\phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$.
Here we will restrict to a special case where

$$F(X) = F_0 + F_1\sqrt{X}, \quad P(X) = -P_1 X, \quad Q(X) = 0,$$

with F_0, F_1, P_1 being constant.
The motivation is the existence of a BH solution, the **BCL BH**:

$$\mathrm{d}s^2 = f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + H(r)\mathrm{d}\Omega^2, \quad \text{with } \boxed{f(r) = \left(1 - \frac{r_+}{r}\right)\left(1 + \frac{r_-}{r}\right), \quad H(r) = r^2.}$$

We have: $r_+r_- = \frac{F_1^2}{2F_0P_1}, \quad r_+ - r_- = r_s, \quad r+ > r_- > 0.$

The BCL BH solution

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- Axial gravitational perturbations equations ✓
- Corresponding QNM spectrum ✓

Test-field *vs* Physical perturbations

"Physical" perturbations:

- Substitute in the quadratic shift-symmetric Horndeski action the perturbed metric along with the perturbed scalar field:
  $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad \phi = \bar{\phi} + \delta\phi.$
- Expand the action up to the quadratic order in $h_{\mu\nu}$ and $\delta\phi$.
- Apply the Euler-Lagrange equations on $S_{\text{quad}}\left[h_{\mu\nu}, \delta\phi\right]$ to obtain the equations of motion for the linear perturbations.

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- Use the Teukolsky equations for the propagation of spin 2 test-field perturbations on a static and spherically symmetric BH metric.
- It is equivalent to using the Einstein-Hilbert action ➔ the modified dynamic is not taken into account.
- In the case of BH effective solutions, it is the only way to study gravitational-type perturbations.

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How does the difference between those two types of perturbation manifest itself in QNM spectra?

Test-field *vs* Physical effective potential

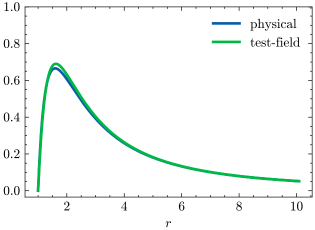
Both types of perturbation's equations can be casted into:

$$\boxed{\frac{d^2\Psi}{dx^2} + [\omega^2 - V(r)]\Psi = 0}$$

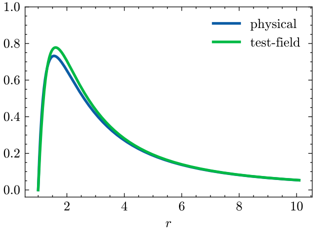
where:

- $V_F(r) = \frac{f(r)}{r}\left[\frac{\lambda}{r} - \frac{3(r_+ - r_-)}{r^2} - 4\frac{r_- r_+}{r^3}\right].$
- $V_P(r) = \frac{f(r)}{r\left(1+2\frac{r_- r_+}{r^2}\right)^3}\left[\frac{\lambda}{r} - \frac{3(r_+ - r_-)}{r^2} - \frac{r_- r_+(6\lambda - 15)}{r^3} + \frac{4r_- r_+(r_- - r_+)}{r^4} + \frac{r_-^2 r_+^2(12\lambda - 30)}{r^5} + \frac{3r_-^2 r_+^2(r_- - r_+)}{r^6} + \frac{r_-^3 r_+^3(8\lambda - 21)}{r^7}\right].$

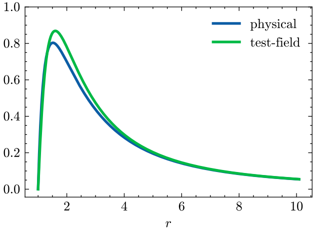
Test-field *vs* Physical effective potential



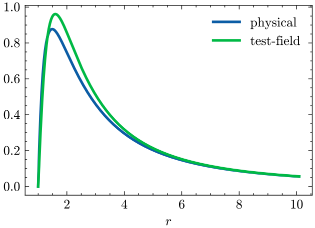
(a) $r_- = 0.2$



(b) $r_- = 0.4$



(c) $r_- = 0.6$



(d) $r_- = 0.8$

Test-field *vs* Physical effective potential

At infinity:

$$V_F \overset{\infty}{\sim} \frac{\lambda}{r^2} - (3+\lambda)\frac{r_+ - r_-}{r^3} + \frac{3(r_-^2 + r_+^2) - (\lambda+10)r_- r_+}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right)$$

$$V_P \overset{\infty}{\sim} \frac{\lambda}{r^2} - (3+\lambda)\frac{r_+ - r_-}{r^3} + \frac{3(r_-^2 + r_+^2) - (\lambda+21)r_- r_+}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right)$$

At the horizon:

$$V_F \overset{r_+}{\sim} \frac{(r-r_+)(r_- + r_+)}{r_+^5}\left[(\lambda-3)r_+ - r_-\right] + \mathcal{O}\left((r-r_+)^2\right)$$

$$V_P \overset{r_+}{\sim} \frac{(r-r_+)(r_- + r_+)}{r_+^4(2r_- + r_+)^2}\left[(\lambda-3)r_+^2 + (4\lambda-9)r_-^2 + (4\lambda-10)r_- r_+\right] + \mathcal{O}\left((r-r_+)^2\right)$$

- ➔ Important disparities directly related to the modified dynamics
- ➔ Discrepancies are also expected on QNM spectra

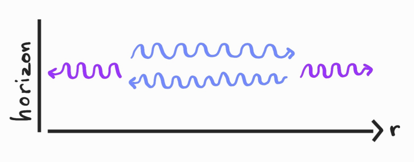
Continued fraction method

The Continued fraction method is a numerical method allowing for a high-accuracy computation of a large number of QNMs.

- Goal: find the complex frequencies ω for which:

$$\frac{d^2\Psi}{dx^2} + [\omega^2 - V(r)]\Psi = 0$$

+



Continued fraction method

The tortoise coordinate defined by $\mathrm{d}x = f^{-1}(r)\,\mathrm{d}r$ reads:

$$x(r) = r + \frac{r_+^2 \ln(r-r_+) - r_-^2 \ln(r+r_-)}{r_+ + r_-} + \text{cste}.$$

The boundary conditions for QNMs are then:

$$\Psi \sim e^{i\omega x} \sim e^{i\omega r} r^{i\omega(r_+ - r_-)}, \qquad (r \to +\infty)$$

$$\Psi \sim e^{-i\omega x} \sim (r - r_+)^{-i\omega \frac{r_+^2}{r_+ + r_-}}. \qquad (r \to r_+)$$

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- Leaver: Formulate an ansatz as a power series which satisfies the QNMs boundary conditions

→ $\Psi(r) = e^{i\omega(r-r_+)} r^{i\omega(r_+ - r_-)} \left(\frac{r-r_+}{r+r_-}\right)^{-i\omega \frac{r_+^2}{r_+ + r_-}} \sum_{n=0}^{\infty} a_n \left(\frac{r-r_+}{r+r_-}\right)^n.$

Continued fraction method

The QNM boundary conditions directly translate into conditions on the coefficient sequence $\{a_n\}_{n\in\mathbb{N}}$:

- QNM boundary condition at **horizon**: we require that $a_0 \neq 0$ and that there is no negative power terms, $a_n = 0$ for $n < 0$.
- QNM boundary condition at **infinity**: As $(r - r_+)/(r + r_-) \sim 1$ as $r \to +\infty$, we require that the series $\sum a_n$ is convergent.

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Next step: plugging the ansatz into the master equation

→ $\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} + \delta_n a_{n-2} + \epsilon_n a_{n-3} = 0$

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➔ $\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} + \delta_n a_{n-2} + \epsilon_n a_{n-3} = 0$

Problem: a three-terms recurrence relation equation is needed to follow Leaver's procedure.

Solution: use the Gaussian reduction procedure

➔ $\boxed{\overline{\alpha}_n a_{n+1} + \overline{\beta}_n a_n + \overline{\gamma}_n a_{n-1} = 0}$

The three-terms recurrence relation equation can be transformed into a continued fraction equation.

Define $\boxed{R_n \equiv \frac{a_n}{a_{n-1}}}$.

$$\alpha_n R_{n+1} + \beta_n + \gamma_n \frac{1}{R_n} = 0 \quad \Rightarrow \quad R_n = \frac{-\gamma_n}{\beta_n + \alpha_n R_{n+1}} = \frac{-\gamma_n}{\beta_n - \frac{\alpha_n \gamma_{n+1}}{\beta_{n+1} + \alpha_{n+1} R_{n+2}}} = \dots$$

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$$\alpha_0 a_1 + \beta_0 a_0 = 0 \Rightarrow 0 = \beta_0 + \alpha_0 R_1 = \beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 + \alpha_1 R_2} = \boxed{\beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \frac{\alpha_2 \gamma_3}{\beta_3 -} \dots = 0.}$$

➔ The **QNM frequencies** are the **roots** of this continued fraction equation and can be computed numerically.

Results: Test-field *vs* Physical QNMs

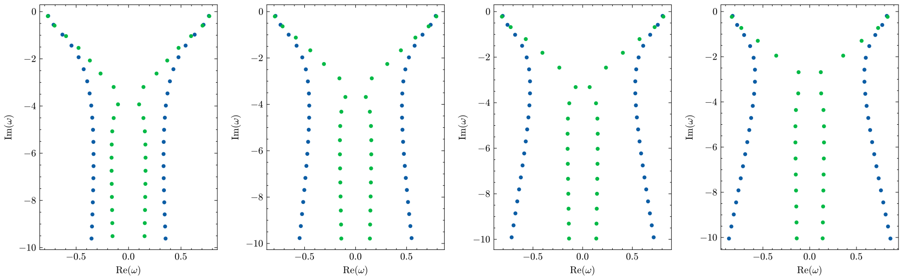
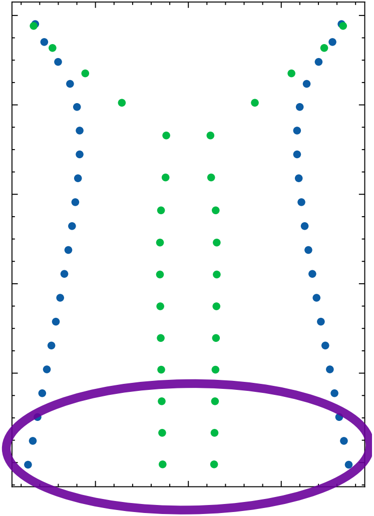


Figure: (Test-field QNMs) *vs* (Physical QNMs) for the BCL BH.
(Here r_- goes from 0.1 to 0.4 and $r_+ = 1$.)

- The shape of the green spectrum is quite different from the blue one.
- This disparity becomes more pronounced as r_- increases.

Let us zoom in...

Test-field *vs* Physical QNMs



The asymptotic: highly-damped QNMs

- No hope of detection by BH spectroscopy.
- Asymptotic Test-field QNMs converge to a constant real part and a constant imaginary gap, as for the Schwarzschild BH.
- An analytic analysis confirmed this result.

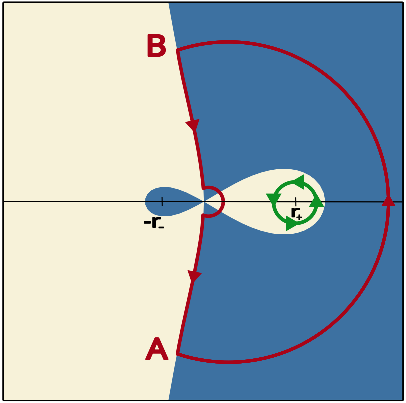
Monodromy technique

Analytical computation of highly-damped QNMs:

- Solve at infinity (plane-waves)
- Solve around $r = 0$ with $V_F \sim (3s - 2)\frac{r_-^2 r_+^2}{r^6}$ (Bessel functions)
- Match along the Stokes lines
- Equal the monodromy around the red contour with the one around the green contour

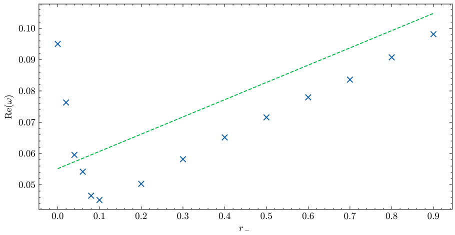
→ $$4\pi R\omega = \log(2) - i(2n+1)\pi$$

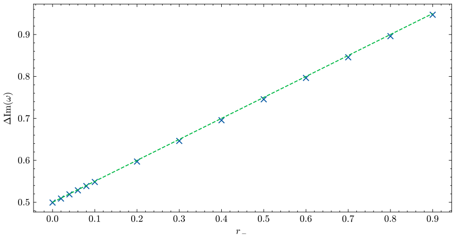
where $R = \frac{r_+^2}{r_- + r_+}$.



Contour for the calculation of QNM frequencies in the complex r plane.

Monodromy results



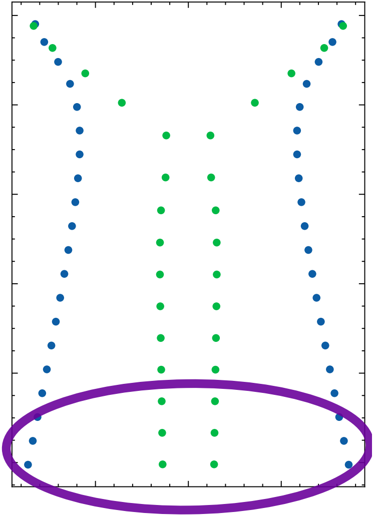


Evolution of asymptotic value $s = 2$ test-field QNMs with r_-.

The blue crosses correspond to the constant value fitted from the asymptotic part of the QNM spectra obtained via the Leaver method (the first 20 QNMs were removed for this asymptotic analysis).

The green line is the monodromy prediction for $r_- > 0$.

Test-field *vs* Physical QNMs



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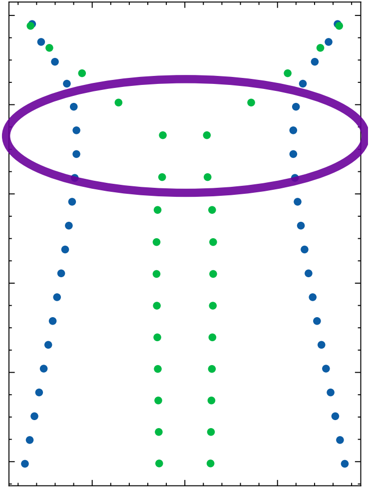
- An analytic analysis confirmed this result.
- The Physical QNMs present a very different asymptotic behaviour, with non-vertical branches.
- An analytical analysis is very challenging and has not been made possible.

Test-field *vs* Physical QNMs



Overtones: mid-damped QNMs

- Very low hope of detection by BH spectroscopy.
- The crossing of branches still occurs for Test-field QNMs but not for the Physical ones.
- It is a strong feature of the Schwarzschild BH.

→ How to interpret the loss of this feature?

Test-field *vs* Physical QNMs

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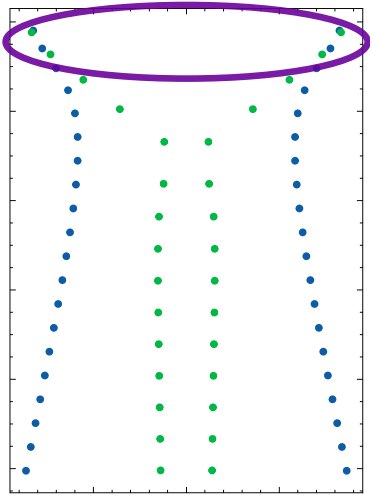

The fundamental mode: low-damped QNM

- Detectable by BH spectroscopy.
- For the BCL BH, the Test-field fundamental mode is a very good approximation for the Physical one.
- There are also hopes of detecting the first overtones with the future GW detectors (LISA, ET).

→ The approximation becomes less and less good though as we go up in the overtones.

Conclusion

- For the BCL BH, the Test-field perturbations are a good approximation of the Physical ones for the first QNMs.
- The larger the imaginary part is, the less good the approximation becomes. In other words, the modified dynamic have a bigger effect on highly-damped QNMs, for the BCL BH.
- There is more to be understood about the general impact of the modified BH solution *vs* the modified dynamic, and also about the symmetries ruling the QNMs spectra.
- Test-field perturbations are still relevant for the study of BH solutions for which there is no access to modified Einstein equations.

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Thank you !

More details in arXiv:2505.16883 :)