Asymptotically-flat spacetimes with twist

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work with Marc Geiller and Pujian Mao (to appear soon)

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Contents

Abstract

- Generalization of Bondi solution space for asymptotically-flat spacetimes
- Both in Newman-Penrose and metric formalism
- Well-suited to study algebraically special solutions

Outline

Asymptotically-flat spacetimes

Motivations and interesting features

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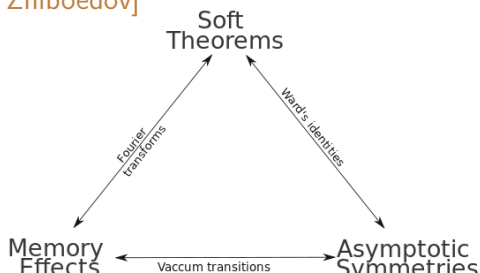

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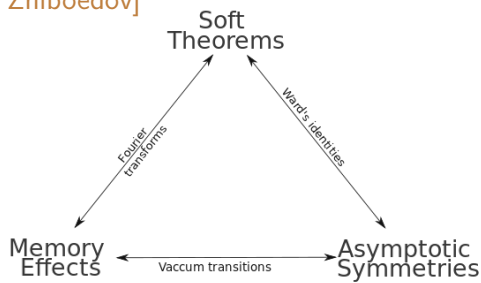

- Black hole soft hair and the information paradox [Hawking, Perry, Strominger]

Asymptotically-flat spacetimes

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5. Compute the symplectic structure and study the charges

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  The metric can be written in Bondi form

$$\mathrm{d}s^2 = -UW\mathrm{d}u^2 - 2W\mathrm{d}u\mathrm{d}r + g_{ab}(\mathrm{d}x^a - X^a\mathrm{d}u)(\mathrm{d}x^b - X^b\mathrm{d}u)$$

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- Upon choosing radial expansions for the free functions, *eg.* $g_{ab} = r^2 q_{ab} + rC_{ab} + \dots$, this *partial* Bondi gauge is enough to solve Einstein equations order by order in $1/r$ [Geiller, Zwikel]

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- But usually the gauge and the fall-offs are further restricted...

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Historical treatment and generalizations

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- Completing the gauge fixing: choice of radial coordinate
  - Bondi-Sachs: $\det g_{ab} = r^4 \det q_{ab} \implies r$ is the areal distance
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  - Introduction of logarithmic terms in the radial expansion [Bieri, Chruściel, Christodoulou, Damour, MacCallum, Friedrich, Gajic, Geiller, Kehrberger, Kroon, Laddha, Masaood, Singleton, Winicour, Zwikel,...]

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  Brief recall: NP formalism
  - *Type of tetrad formalism with null tetrad* $e_i = (\ell, n, m, \bar{m})$
  - *Spin coefficients* α, β, γ, ϵ, π, ρ, σ,...
  - *Weyl scalars* $\Psi_{0,\ldots,4}$
  - *Algebraic speciality:* $\Psi_0 = \Psi_1 = 0$ *in some tetrad*

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  If one wants to connect Bondi treatment with Newman-Penrose (NP) formalism:
  - Find a 'Bondi' tetrad $e_i = (\ell, n, m, \bar{m})$ such that $g_{\mu\nu} = e^i_\mu e^j_\nu \eta_{ij}$ is of Bondi form

$$\ell = \partial_r, \qquad n = W\partial_u + U\partial_r + X^a\partial_a, \qquad m = 0\partial_u + \omega\partial_r + m^a\partial_a$$

  - Then for *eg.* Kerr: $\Psi_0 \neq 0$ and $\Psi_1 \neq 0$ (Ψ_1 encodes the angular momentum...)
  - Algebraic speciality is not manifest...

Twisting asymptotically-flat spacetimes

Introducing the twist

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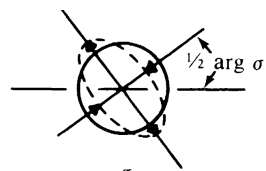
- To achieve $\Psi_0 = \Psi_1 = 0$, ℓ must be a degenerated *principal null direction*
  For Kerr, one finds that it has *rotation* ($\mathrm{Im}\rho \neq 0$), the source of which is the **twist** L

$$\ell = \partial_r, \qquad n = W\partial_u + U\partial_r + X^a\partial_a, \qquad m = L\partial_u + \omega\partial_r + m^a\partial_a$$



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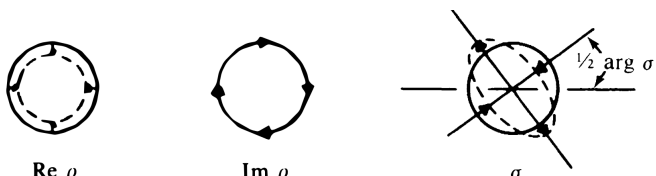

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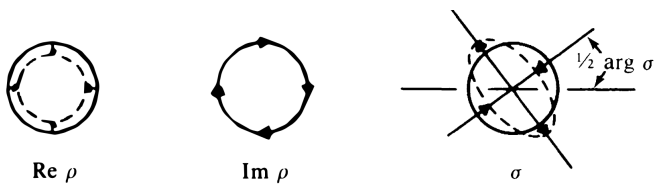

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- **Motivation**: merge Bondi formalism with [Chandrasekhar] [Stephani, Kramer, MacCallum, Hoenselaers, Herlt]

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- Such relaxations of the Bondi gauge were already studied in 4d, mainly for Carrollian physics purposes [Campoleoni, Delfante, Pekar, Petropoulos, Rivera-Betancour, Vilatte], [Hartong, Have, Nenmeli, Oling]

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- **Our goal:** obtain the complete picture in the presence of twist
  - NP solution space
  - Metric solution space
  - Dictionary NP-metric
  - Algebraically special solution space
  - Asymptotic symmetries
  - Charges
  - Flux balance laws

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  - Tetrad components *eg.* $W = 1 - \frac{2\mathrm{Re}(L\bar{\tau}_1)}{r} + \frac{2\mathrm{Re}\left(L(\bar{\sigma}_2\tau_1 - i\Sigma\bar{\tau}_1)\right)}{r^2} + O(r^{-3})$

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- Parameterized by free (σ_2, γ_0, L) and u-constrained data ($\mathrm{Re}(\Psi_2^0)$, Ψ_1^0, Ψ_0^n)

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- However the metric resolution is made possible by starting from the metric induced by the tetrad (and not simply relaxing $g_{ra} = 0$)

$$g_{\mu\nu} = \begin{pmatrix} \left(2U + X_a X^a\right) & -1 & -X_a - g_{uu} L_a \\ -1 & 0 & L_a \\ -X_b - g_{uu} L_b & L_b & \gamma_{ab} + g_{uu} L_a L_b - 2X_{(a} L_{b)} \end{pmatrix}$$

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- The gauge conditions when translated to metric formalism are now $g_{rr} = 0$, $g_{ur} = -1$ and $\partial_r L_a = 0$ (geometrically: $\ell^\nu \nabla_\nu \ell^\mu = \Gamma^\mu_{rr} = 0$)

Twisting asymptotically-flat spacetimes

Metric solution space [Geiller, Mao, AV]

- NP resolution is powerful but obscure: what are the corresponding metric gauge conditions? what is the no-log condition? what is the Bondi hierarchy?
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- Bondi hierachy (Bianchi identities + specific form of the metric)
  - radial equations: $G_{r\mu} = 0$
  - evolution equations: $\widetilde{G}^{\mathsf{TF}}_{ab} = 0$, $\widetilde{G}_{ua|\mathcal{O}(r^{-2})} = 0$, $\widetilde{G}_{uu|\mathcal{O}(r^{-2})} = 0$
  - trivial equations: all the rest!

Twisting asymptotically-flat spacetimes

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  eg. $W = 1 - 2\mathrm{Re}(L\bar{\tau}), \quad \rho = -\frac{1}{r+i\Sigma}, \quad \Psi_2 = \frac{\Psi_2^0}{(r+i\Sigma)^3}, \ldots$
- $\delta_\xi \Psi_1^0 = \pounds_\xi \Psi_1^0 + \Upsilon \Psi_2^0$
  Algebraic speciality preserved if $\Upsilon^a = 0$, and the vector fields also truncate!

Twisting asymptotically-flat spacetimes

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  - Recover r-independant charge with non-zero superrotation [Compère, Long]:

$$Q = \oint_S dS^2\, M(T + Y^a \partial_a C)$$

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Thank you!