Lie Point Symmetries in Electrovacuum: New Type I spacetimes in Einstein theory

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Mostly based on our papers

⋆ Ehlers transformations as a tool for constructing accelerating NUT black holes, 2305.03765, J.Barrientos and A.C.

⋆ Plebański–Demiański à la Ehlers–Harrison: exact rotating and accelerating type I black holes, 2309.13656, J.Barrientos, A.C, K.Pallikaris.

⋆ Mixing "Magnetic" and "Electric" Ehlers–Harrison transformations: the electromagnetic swirling spacetime and novel type I backgrounds, 2401.02924, J.Barrientos, A.C, I.Kolář, K.Müller, M.Oyarzo, K.Pallikaris.

⋆ Revisiting Buchdahl transformations: new static and rotating black holes in vacuum, double copy, and hairy extensions, 2404.12194, J.Barrientos, A.C, M.Hassaine, J.Oliva.

⋆ A new exact rotating spacetime in vacuum: The Kerr–Levi-Civita Spacetime, 2506.07166, J.Barrientos, A.C, M.Hassaine, K.Müller, K.Pallikaris.

with some pertinent references from other authors

⋆ Black holes in a swirling universe, 2205.13548, M.Astorino, R.Martelli, A.Viganò.

⋆ Equivalence principle and generalized accelerating black holes from binary systems, 2312.00865, M. Astorino.

⋆ New improved form of black holes of type D, 2108.02239, J.Podolsky, A.Vratny.

Why spending time on looking for exact solutions?

Given any physical theory exact solutions play a major role:

⋆ Maxwell theory: Exact fields as Coulomb fields, dipoles, solenoids, etc, provide a fundamental toolkit for electromagnetism.

⋆ Quantum Mechanics: Exact solutions to Schrodinger equation (again linear) yield insights into the quantum world: Harmonic oscillator, various potential wells and other 3D problems provide glances into the ideas behind minimum and degenerated energy states, for example.

⋆ In nonlinear regimes the role is even more pertinent, grasping intuition in nonlinear systems is considerably more challenging: Fluid mechanics and its interplay with astrophysics.

⋆ Numerical computations are relevant for highly complex systems, however, having exact solutions offers a road map to compare with: Kerr metric.

Einstein equations: Represent a complex set of nonlinear coupled PDE's. They pose two main problems:

$\star$ Mathematical challenge: To find as many solutions as there are and/or to find a general solution as complete as possible.

$\star$ Physical challenge: To go beyond the local solution to the PDE's and proceed with a global analysis of the spacetime.

$\star$ Pure GR has been exhaustively studied during the last century: Black holes, cosmological spacetimes, gravitational radiation...

To find new exact solutions in GR is therefore very complicated. How do we face this:

⋆ Mathematically: Solution generating techniques, for example, Lie point symmetries.

⋆ Physically: Algebraic classification and global analysis of the spacetime.

Lie point symmetries are local group of transformations that map every solution of the system to another solution within the same system.

⋆ May appear straightforward, but their discovery is nontrivial, usually hidden in tensorial form.

⋆ Accurately recognising and assessing the applicability of a particular Lie point symmetry is crucial.

Outline

- ⋆ Circular, stationary and axisymmetric spacetimes.
- ⋆ CSA Einstein-Maxwell field equations and the Weyl problem
- ⋆ Ernst scheme and Lie Point Symmetries of Einstein-Maxwell theory.
- ⋆ Ehlers-Harrison transformations and accelerating spacetimes.
- ⋆ Completing and extending the expanding Plebański-Demiański class.
- ⋆ A new rotating spacetime in vacuum: The Kerr–Levi-civita spacetime.

Part I

Circular, stationary and axisymmetric spacetimes.

$\star$ Stationary and axisymmetric geometries in four dimensions are characterised by the action of a group $\mathbb{R} \times \mathbf{SO(2)}$ under which the spacetime metric remains invariant.

$\star$ If the orbits of $\mathbb{R} \times \mathbf{SO(2)}$ (surfaces of transitivity) are everywhere orthogonal to a family of hypersurfaces defined by the remaining, non-Killing coordinates of the spacetime (meridional surfaces), then the spacetime is said to be circular, and the action of the group is said to be orthogonally transitive.

$\star$ The most general class of circular, stationary, and axisymmetric (CSA) spacetimes is captured by the Weyl-Lewis-Papapetrou line element, which in canonical adapted coordinates $\{t, \rho, z, \varphi\}$ reads

$$ds^2 = -f(\rho, z)\,(dt - \omega(\rho, z)d\varphi)^2 + \frac{1}{f(\rho, z)}\left(e^{2\gamma(\rho,z)}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right),$$

and its magnetic counterpart, derived through a discrete double Wick rotation $t = i\varphi$ and $\varphi = it$

$$ds^2 = f(\rho, z)\left(d\varphi - \omega(\rho, z)dt\right)^2 + \frac{1}{f(\rho, z)}\left(e^{2\gamma(\rho,z)}(d\rho^2 + dz^2) - \rho^2 dt^2\right).$$

$\star$ The main effect of the Wick rotation is the change of the asymptotic behaviour of a given spacetime written in the magnetic form. It induces a Levi-Civita-like behaviour.

$\star$ The magnetic counterpart of a given solution is known as the conjugate, which is a non-equivalent solution to the same field equations.

The Weyl problem and CSA Einstein-Maxwell field equations

Let us consider Einstein field equations (vacuum) for a CSA spacetime in WLPe form. They are defined by the principal set of equations

$$
\begin{aligned}
f\left(f_{,\rho\rho}+f_{,zz}+\rho^{-1}f_{,\rho}\right)-f_{,\rho}^2-f_{,z}^2+\rho^{-2}f^4\left(\omega_{,\rho}^2+\omega_{,z}^2\right)=0,\\
\left(\rho^{-1}f^2\omega_{,\rho}\right)_{,\rho}+\left(\rho^{-1}f^2\omega_{,z}\right)_{,z}=0,
\end{aligned}
$$

supplement with the secondary equations

$$
\begin{aligned}
\gamma_{,z}&=\frac{1}{2}\rho f^{-2}f_{,\rho}f_{,z}-\frac{1}{2}\rho^{-1}f^2\omega_{,\rho}\omega_{,z}\quad,\\
\gamma_{,\rho}&=\frac{1}{4}\rho f^{-2}\left(f_{,\rho}^2-f_{,z}^2\right)-\frac{1}{4}\rho^{-1}f^2\left(\omega_{,\rho}^2-\omega_{,z}^2\right).
\end{aligned}
$$

Once the main set has been solved, the secondary equations are immediately given by a sort of quadratures. Notice that, however we have considered a relevant amount of symmetry and that we have restricted ourselves to the circular class, the field equations are highly envolved, with more non-linearity if the electromagnetic interaction is considered.

Let us start tackling them from the most simple subset: the vacuum static sector. This is known as the Weyl problem. For a Weyl metric ($f = e^{2U}$)

$$ds^2 = -e^{2U(\rho,z)}dt^2 + e^{-2U(\rho,z)}\left(e^{2\gamma(\rho,z)}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right),$$

Einstein equations $R_{\mu\nu} = 0$ heavily simplify

$$\begin{cases} R_{tt} = 0 \rightarrow \nabla^2_{\mathbb{E}^3} U = \dfrac{\partial^2 U}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial U}{\partial \rho} + \dfrac{\partial U}{\partial z} = 0, \\ R_{\rho z} = 0 \rightarrow \dfrac{\partial \gamma}{\partial z} = 2\rho \dfrac{\partial U}{\partial \rho}\dfrac{\partial U}{\partial z}, \\ R_{\rho\rho} - R_{zz} = 0 \rightarrow \dfrac{\partial \gamma}{\partial \rho} = \rho\left[\left(\dfrac{\partial U}{\partial \rho}\right)^2 - \left(\dfrac{\partial U}{\partial z}\right)^2\right]. \end{cases}$$

⋆ The gravitational potential U is given by a Laplace equation in a three-dimensional Euclidean space in cylindrical coordinates.

⋆ The metric function γ carries all the nonlinearity of Einstein equations.

⋆ This means that all static and axisymmetric solutions in vacuum are mathematically known.

Indeed, Laplace equation has a well-known solution in terms of Legendre polynomials given by the multipolar expansion

$$U = \frac{a_0}{\sqrt{\rho^2+z^2}} + \sum_{\ell=1}^{\infty} \left(\frac{a_\ell}{\left(\rho^2+z^2\right)^{\frac{(\ell+1)}{2}}} + b_\ell \left(\rho^2+z^2\right)^{\frac{\ell}{2}} \right) P_\ell \left[\frac{z}{\sqrt{\rho^2+z^2}} \right].$$

This establishes an interesting correspondence between Newtonian and relativistic solutions,

⋆ Take a solution for U, which represents the Newtonian potential outside a source.

⋆ Then solve for γ via the quadratures.

⋆ Plug U and γ on the WLP line element to have the relativistic counterpart.

Simple example: The Schwarzschild black hole is the relativistic analogue of a thin rod of mass M and large $2M$, hence with a linear density $\sigma = 1/2$

$$\mathrm{e}^{2U} = \frac{R_+ + R_- - 2m}{R_+ + R_- + 2m}, \quad \mathrm{e}^{2\gamma} = \frac{\left(R_+ + R_-\right)^2 - 4m^2}{4R_+R_-},$$

with $R_\pm^2 = \rho^2 + (z \pm m)^2$.

Same rod but with arbitrary linear density $\sigma = \delta/2$, with δ a deformation parameter is given by the Zipoy-Voorhees spacetime

$$\mathrm{e}^{2U} = \left(\frac{R_+ + R_- - 2\ell}{R_+ + R_- + 2\ell}\right)^{m/\ell}, \quad \mathrm{e}^{2\gamma} = \left(\frac{(R_+ + R_-)^2 - 4\ell^2}{4R_+R_-}\right)^{m^2/\ell^2},$$

with $R_\pm = \sqrt{\rho^2 + (z \pm \ell)^2}$. In the intuitive spherical coordinates

$$ds^2_{\mathrm{ZV}} = -f^\delta dt^2 + \frac{\left[\left(\frac{f}{g}\right)^{\delta^2} g\left(\frac{dr^2}{f} + r^2 d\theta^2\right) + fr^2 \sin^2\theta d\varphi^2\right]}{f^\delta},$$

$$f = \left(1 - \frac{2M}{r}\right), \quad g = \left(1 - \frac{2M}{r} + \frac{M^2 \sin^2\theta}{r^2}\right).$$

Ernst scheme and Lie Point Symmetries in Einstein-Maxwell

Ernst showed that the field equations of stationary and axisymmetric space-times (circular subset) in electrovacuum can be written as the pair of non-linear coupled equations

$$(\mathrm{Re}\,\mathcal{E} + \Phi\bar{\Phi})\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot(\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi),$$
$$(\mathrm{Re}\,\mathcal{E} + \Phi\bar{\Phi})\nabla^2\Phi = \nabla\Phi\cdot(\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi),$$

for the so-called Ernst complex potentials

$$\mathcal{E} = f - |\Phi|^2 + i\chi, \quad \Phi = A_t + i\bar{A}_\varphi,$$

which are defined in term of the characteristic functions of the WLP space-time and a Maxwell gauge field respecting the same symmetries. χ and $\bar{A}_\varphi$ are known as twist potentials and are given by the integrability conditions

$$\hat{\phi}\times\nabla\chi = -\rho^{-1}f^2\nabla\omega - 2\hat{\phi}\times\mathrm{Im}(\bar{\Phi}\nabla\Phi), \quad \hat{\phi}\times\nabla\widetilde{A}_\phi = \rho^{-1}f\left(\nabla A_\phi + \omega\nabla A_t\right),$$

being all vectorial quantities quantities understood as vectors in flat space-time with cylindrical coordinates (ρ, z, φ). The remaining function is always known via quadratures, $\gamma = \gamma(\mathcal{E}, \Phi)$.

Same field equations are achieved via the magnetic WLP configuration, now with re-definiton of the Ernst potentials

$$\mathcal{E} = -f - |\Phi|^2 - i\chi, \quad \Phi = A_\varphi - i\tilde{A}_t,$$

where twist equations are defined as

$$\hat{\varphi}\times\nabla\chi = -\frac{f^2}{\rho}\nabla\omega + 2\hat{\varphi}\times\mathrm{Im}\left(\Phi^*\nabla\Phi\right), \quad \hat{\varphi}\times\nabla\tilde{A}_t = -\frac{f}{\rho}\left(\nabla A_t + \omega\nabla A_\varphi\right).$$

⋆ The main advantages of the formalism are two: First, if dealing with direct integration of the equations the problem has been reduced to a system of PDE's to be solved in a Euclidean three dimensional space in cylindrical coordinates.

⋆ Second, Einstein(-Maxwell) equations can be shown to display a set of otherwise hidden Lie Point Symmetries.

These symmetries are

$$
\begin{aligned}
\mathrm{G}_1[a] : (\mathcal{E}_0, \Phi_0) &\mapsto (\mathcal{E}, \Phi) := (\mathcal{E} + ia, \Phi_0)\,, \\
\mathrm{G}_2[\alpha] : (\mathcal{E}_0, \Phi_0) &\mapsto (\mathcal{E}, \Phi) := (\mathcal{E}_0 - 2\bar{\alpha}\Phi_0 - \alpha\bar{\alpha}, \Phi_0 + \alpha)\,, \\
\mathrm{D}[\epsilon] : (\mathcal{E}_0, \Phi_0) &\mapsto (\mathcal{E}, \Phi) := (\epsilon\bar{\epsilon}\mathcal{E}_0, \epsilon\Phi_0)\,, \\
\mathrm{E}[c] : (\mathcal{E}_0, \Phi_0) &\mapsto (\mathcal{E}, \Phi) := \frac{(\mathcal{E}_0, \Phi_0)}{1 + ic\mathcal{E}_0}, \\
\mathrm{H}[\beta] : (\mathcal{E}_0, \Phi_0) &\mapsto (\mathcal{E}, \Phi) := \frac{(\mathcal{E}_0, \Phi_0 + \beta\mathcal{E}_0)}{1 - 2\bar{\beta}\Phi_0 - \beta\bar{\beta}\mathcal{E}_0},
\end{aligned}
$$

where a and c are real and α , β and ϵ are complex. They are associated with 8 Killing vectors whose linear representation is a representation of $SU(2,1)$. The last two symmetries, known as Ehlers and Harrison have a non-trivial effects.

⋆ When applying Ehlers or Harrison on a seed spacetime written in electric WLP form the resulting spacetime must be at least asymptotically locally flat.

⋆ The effect of the transformation must be the addition of a feature that respect such restriction.

$\star$ Ehlers add NUT parameter while Harrison monopole electromagnetic charges.

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$\star$ Now, let us notice the following: when dealing with a seed in a magnetic WLP form the asymptotic behaviour of the seed is naturally changed, it changes to a Levi-Civita-like spacetime

$$ds^2_{LC} = -\rho^{4\sigma} dt^2 + k^2 \rho^{4\sigma(2\sigma-1)}(d\rho^2 + dz^2) + \rho^{2(1-2\sigma)} d\varphi^2,$$

specifically the one defined by $\sigma = 1$.

$\star$ Hence the effect of Ehlers and Harrison cannot be the addition of a point like source as otherwise the nontrivial LC-asymptotic would not be possible.

$\star$ Magnetic Ehlers and Harrison transformations add background-like properties on the seed spacetime, swirling background rotation and electromagnetic background fields.

$$E_c|_{WLP_e} \to \text{ NUT } ,\, E_c|_{WLP_m} \to \text{ swirling } ,$$
$$H_\beta|_{WLP_e} \to (Q_e, Q_m)\; H_\beta|_{WLP_m} \to \text{ Melvin } - \text{ Bonnor } .$$

⋆ Ehlers transformations form a one-parameter subgroup

$$E_b \circ E_c = E_{b+c},$$

in addition they commute with Harrison transformations, $E_c \circ H_\beta = H_\beta \circ E_c$.
⋆ Harrison transformations, generically, fail to form a subgroup

$$H_\alpha \circ H_\beta = E_{i(\alpha\beta^* - \beta\alpha^*)} \circ H_{\alpha+\beta},$$

unless $\alpha\beta^* - \beta\alpha^* = 0$. This implies that H_α can produce stationarity when combined with itself.

⋆ Finally, notice that all these composition have been performed on the same WLP line element, either electric or magnetic. However, we could explore the mixing of the ansatz. In general we will have 8 possible mixed compositions

$$U^\alpha_\beta \circ U^\gamma{}_\delta,$$

with $\beta \neq \gamma$ and $, \alpha \ldots \delta = 0, 1$ and

$$U^\alpha_\beta = \begin{pmatrix} E_e & E_m \\ H_e & H_m \end{pmatrix}$$

⋆ Commutativity is completely lost, and there is no clear way to show if the continuos use of the symmetry finally closes.

$\star$ Finally, an interesting discrete symmetry arises by properly combining gravitational gauge transformations, a Weyl rescaling and an Ehlers symmetry in the limit where the Ehlers parameter is taken to infinity

$$\mathrm{I} : (\mathcal{E}_0, \Phi_0) \mapsto (\mathcal{E}, \Phi) := \frac{(1, \Phi_0)}{\mathcal{E}_0}.$$

$\star$ An intuitive version of this symmetry has been given long ago by Buchdahl in metric form, but constrained to the static case. The rotating inversion has been completely overlooked.

$\star$ Given a vacuum seed the entire inversion (magnetic in this case) is given by

$$\mathcal{E} = -f - i\chi \quad \rightarrow \quad f = \frac{f_0}{f_0^2 + \chi_0^2}, \qquad \chi = -\frac{\chi_0}{f_0^2 + \chi_0^2}.$$

$\star$ It can be proven that asymptotically flat solutions are inmune to the inversion. This is consistent with uniqueness theorems.

$\star$ Then it is necessary to use the inversion on a magnetic WLP line element.

Part II

Accelerating spacetimes and Ehlers-Harrison transformations

The line element of an accelerating black hole, also known as the c-metric, is given by

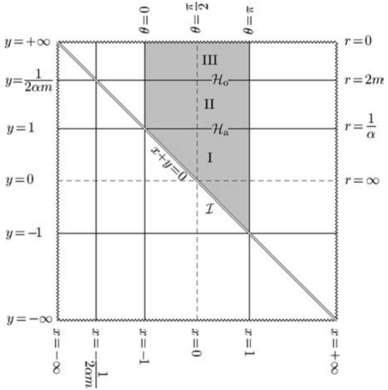
$$ds^2 = \frac{1}{(1 - Ar\cos\theta)^2}\left[-F(r)dt^2 + \frac{dr^2}{F(r)} + r^2\frac{d\theta^2}{P(\theta)} + r^2\sin^2\theta P(\theta)d\varphi^2\right],$$

$$F(r) = (1 - A^2r^2)\left(1 - \frac{2m}{r}\right), \quad P(\theta) = 1 - 2Am\cos\theta.$$

⋆ It represents an asymptotically locally flat spacetime of two accelerating black holes pull or pushed by a string/strut (conical defect).

⋆ In the simplest case it can be seen as a Schwarzschild black hole embedded on a Rindler blackground. Conformal infinity is now a region.

⋆ An interesting manner to visualise the construction of a c-metric (to be use below) is the following:



$\star$ Notice that for Weyl metrics the Newtonian potential U satisfies a linear equation, $\nabla^2 U = 0$.

$\star$ Hence, a partial superposition principle applies. However, the quadratures for non-Killing metric component γ are nonlinear.

$\star$ The superposition of Newtonian potentials is doable, with a price: the function γ exhibits conical defects, defects that are responsible for the stability of the superposed system.

$\star$ The superposition of two Schwarzschild black holes is known as the Bach-Weyl binary which in canonical coordinates reads

$$ds^2 = -\frac{\mu_1\mu_3}{\mu_2\mu_4}dt^2 + \frac{16 C_f \mu_1^3\mu_2^5\mu_3^3\mu_4^5\left(d\rho^2 + dz^2\right)}{\mu_{12}^2\mu_{14}^2\mu_{23}^2\mu_{34}^2 W_{13}^2 W_{24}^2 W_{11} W_{22} W_{33} W_{44}} + \rho^2\frac{\mu_2\mu_4}{\mu_1\mu_3}d\varphi^2,$$

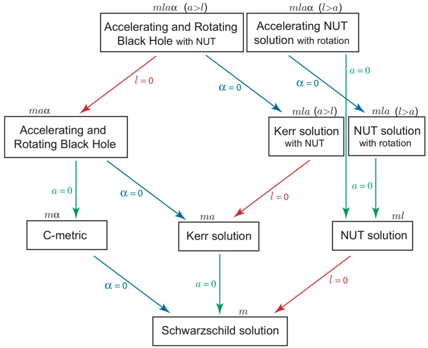
where

$$\mu_i = w_i - z + \sqrt{\rho^2 + (z - w_i)^2}, \qquad \mu_{ij} = (\mu_i - \mu_j)^2, \qquad W_{ij} = \rho^2 + \mu_i\mu_j.$$

$\star$ The c-metric line element arises in the limit in which one of the black hole horizons grows indefinitely while keeping the distance between the sources fixed. The origin of the conical defect of the c-metric is then clear.

⋆ The c-metric, along with the Taub-NUT spacetime, complement the Kerr-Newman family of black holes that altogether form the most general class of type D expanding metrics in electrovacuum$-\Lambda$, the Plebański-Demiański spacetime.

⋆ Up to not so long ago the expanding Plebański-Demiański spacetime was schematically presented as $\rightarrow$



⋆ It was conjecture the absence of a pure accelerating-NUT spacetime.

$$ds^2 = \frac{1}{\Omega^2}\left(-\frac{Q}{\rho^2}\left[dt - \left(a\sin^2\theta + 4l\sin^2\frac{1}{2}\theta\right)d\varphi\right]^2 + \frac{\rho^2}{Q}dr^2 + \frac{\rho^2}{P}d\theta^2 + \frac{P}{\rho^2}\sin^2\theta\left[adt - \left(r^2 + (a+l)^2\right)d\varphi\right]^2\right),$$

where

$$\Omega = 1 - \frac{\alpha a}{a^2+l^2}r(l + a\cos\theta), \quad r_\pm = m \pm \sqrt{m^2 + l^2 - a^2 - e^2 - g^2},$$

$$\rho^2 = r^2 + (l + a\cos\theta)^2$$

$$P(\theta) = \left(1 - \frac{\alpha a}{a^2+l^2}r_+(l + a\cos\theta)\right)\left(1 - \frac{\alpha a}{a^2+l^2}r_-(l + a\cos\theta)\right)$$

$$Q(r) = (r - r_+)(r - r_-)\left(1 + \alpha a\frac{a-l}{a^2+l^2}r\right)\left(1 - \alpha a\frac{a+l}{a^2+l^2}r\right)$$

⋆ No good zero angular momentum limit. Acceleration vanishes in the limiting process.

⋆ Initial evidence for an accelerating-TaubNUT black hole was provided soon after the conjecture was given: A sort of accelerating-TaubNUT Zipoy-Voorhees spacetime with a Misner string.

⋆ Later on recognised as a truly accelerating-TaubNUT black hole: however, algebraically general, so could not be part of the Plebański-Demiański class.

⋆ A simple method to obtain the spacetime: Acting on a c-metric seed with an electric Ehlers transformation:

$$ds^2 = \frac{1}{(1 + A(r - r_-)\cos\theta)^2}\left[-\frac{\mathcal{Q}(r)}{\mathcal{R}^2(r,\theta)}\left[d\tau - 2l\left(\cos\theta + A\mathcal{T}(r,\theta)\sin^2\theta\right)d\varphi\right]^2 \right.$$
$$\left. + \mathcal{R}^2(r,\theta)\left(\mathcal{P}(\theta)\sin^2\theta d\varphi^2 + \frac{dr^2}{\mathcal{Q}(r)} + \frac{d\theta^2}{\mathcal{P}(\theta)}\right)\right.,$$

where

$$\mathcal{Q}(r) = \left(1 - A^2(r - r_-)^2\right)(r - r_-)(r - r_+), \quad \mathcal{P}(\theta) = 1 + A(r_+ - r_-)\cos\theta,$$
$$\mathcal{T}(r,\theta) = \frac{P(\theta)(r - r_-)^2}{(r_+ - r_-)\Omega(r,\theta)^2}, \quad r_\pm = m \pm \sqrt{m^2 + n^2},$$
$$\mathcal{R}^2(r,\theta) = \frac{1}{r_+^2 + l^2}\left[r_+^2(r - r_-)^2 + l^2\frac{\left[1 - A^2(r - r_-)^2\right]^2}{\Omega(r,\theta)^4}(r - r_+)^2\right].$$

⋆ Main observation: the NUT parameter l enters not only in the black hole horizons, but on the accelerating horizons as well.

⋆ Indeed, the solution is centred around $r = r_-$.

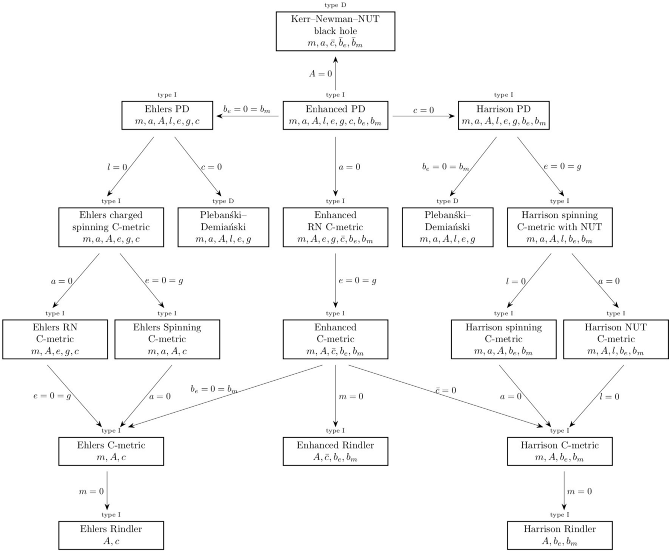
⋆ As a consequence the Rindler background acquires a NUT charge, it is not conformally flat any longer but algebraically special, type D.

⋆ If acceleration is removed we recover the classic TaubNUT spacetime (type D).

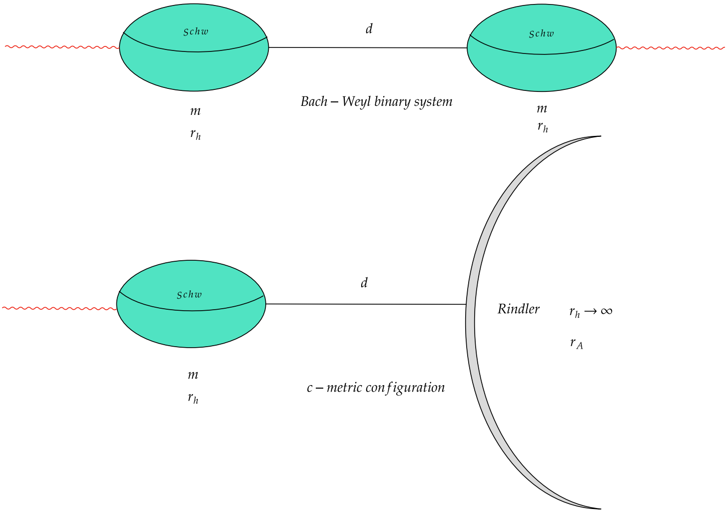
⋆ It is the interaction with accelerating horizons what makes the solution algebraically general.

⋆ Does a similar effect occurs if we instead add electromagnetic monopole charges with a Harrison transformations? Yes, same interaction with the acceleration is found. Again, all solutions are algebraically general (type I).

⋆ Due to the commutativity of Ehlers and Harrison transformations both can be simultaneously applied on the entire Plebański-Demiański class providing a wide class of exact spacetimes featuring many interesting features.

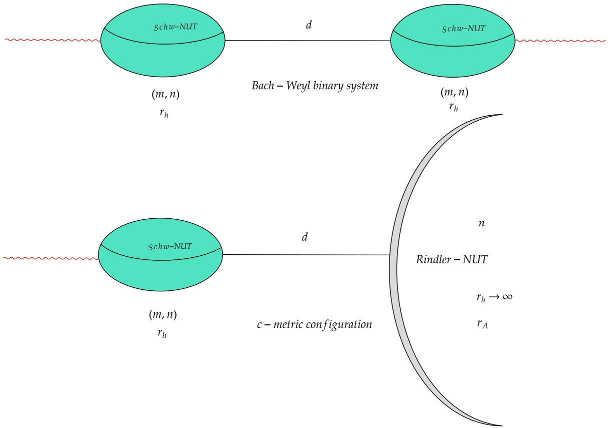


⋆ To understand this interaction between Rindler horizons and NUT and electromagnetic charges let us look at the Bach-Weyl binary again.



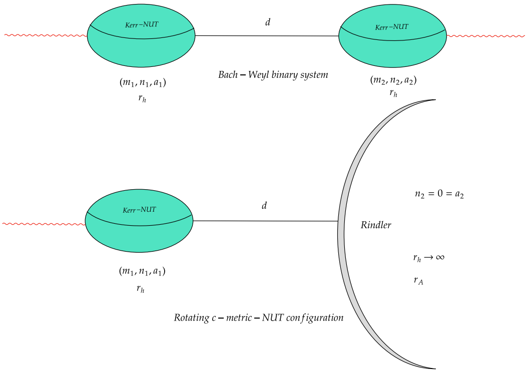
⋆ Now, observe what happens if initially our binary included a NUT charge.

⋆ The original NUT charge get stuck on the newly formed Rindler horizon, providing the type I nature of the final spacetime. Similarly occurs with electromagnetic charges instead of NUT.



⋆ Now, this process opens the door to complete the Plebański-Demiański family.

⋆ Using other solution generating techniques a binary composed of two Kerr-NUT black holes has been constructed. If the limit acts on one of them, where the NUT charge and rotation have been eliminated, this Schwarzschild black hole under the limiting process will provide a pure Rindler horizon for the companion and in this way provide a type D accelerating-NUT geometry.



Part III

The Kerr–Levi-Civita spacetime

⋆ Let us start by considering probably the oldest Lie point symmetry of vacuum Einstein equations, written in its original (metric) form as given by Buchdahl.

In its original formulation, Buchdahl theorem (of the first kind) can be presented as follows: Consider any d-dimensional vacuum solution of the Einstein field equations

$$ds_0^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{aa}\left(x^k\right)\left(dx^a\right)^2 + g_{ij}\left(x^k\right) dx^i dx^j,$$

which is said to be static with respect to a coordinate " a ", namely, that it satisfies $g_{ai} = 0 = \partial_a g_{\mu\nu}$. Here, a represents a cyclic coordinate, and i, j run from 1 to $d-1$. It is proven that a new static and vacuum solution ($d \geq 4$) is automatically given by

$$ds^2 = \left(g_{aa}\right)^{-1}\left(dx^a\right)^2 + \left(g_{aa}\right)^{\frac{2}{d-3}} g_{ij} dx^i dx^j,$$

which is said to be reciprocal to the original seed.

$\star$ Intuitively if the ignorable coordinate is the time t, then by means of Birkhoff's theorem we simply get Schwarzschild in a different set of coordinates.

$\star$ In the context of static axisymmetric spacetime results natural to explore the case in which the ignorable coordinate is the azimuthal angle φ. If this is the case the asymptotic of the spacetime is changed and the solution is known as the Schwarzschild–Levi-Civita spacetime

$$ds^2 = \frac{d\varphi^2}{r^2 \sin^2\theta} + r^4 \sin^4\theta \left[-\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\theta^2 \right],$$

named in this way as it lands asymptotically and in the massless case in the Levi-Civita background (cylindrical coordinates)

$$ds^2 = \rho^4 \left(-dt^2 + d\rho^2 + dz^2\right) + \frac{d\varphi^2}{\rho^2}.$$

$\star$ But can we go to stationary cases? Certainly the theorem conditions forbids it.

$\star$ The existence of a circular cross term will natural imply the presence of a $g_{ia} \neq 0$ term.

⋆ But let us first note: the theorem conditions are nothing else than the imposition of staticity and of axisymmetry,

⋆ In the known static case the net effect of the transformations is nothing else than the inversion of the metric function g_{aa}.

⋆ It is therefore easy to recognise that this is nothing else than the inversion symmetry in vacuum under the staticity assumption

$$\mathrm{I} : \mathcal{E}_0 \mapsto \mathcal{E} := \frac{1}{\mathcal{E}_0}, \quad \mathcal{E} = -f \quad \rightarrow \quad f = \frac{1}{f_0}.$$

⋆ What is happening is that Buchdahl theorem was nothing but a static inversion, that when applied with respect to the azimuthal coordinates was equivalent to an inversion in a magnetic WLP fashion, therefore the Levi-Civita character of the solution.

⋆ Then, room for the exploration of inverted rotating solutions has been opened.

⋆ First, the inversion of an asymptotically flat stationary Ernst potential provides another asymptotically flat potential. Hence, the inversion of rotating solutions in electric WLP form does not produce novel solutions due to the uniqueness theorems applying thereof.

⋆ Then, it is necessary to move to an asymptotically Levi-Civita background, meaning, to the use of a magnetic WLP line element.

⋆ We start with Kerr metric written in a magnetic WLP form (conjugate Kerr) in spherical-like coordinates

$$ds_0^2 = f_0(d\varphi - \omega_0 dt)^2 - \frac{\Delta_r \Delta_x}{f_0} dt^2 + \frac{e^{2\gamma_0}}{f_0}\left(\frac{dr^2}{\Delta_r} + \frac{dx^2}{\Delta_x}\right),$$

where $x := \cos\theta$ and

$$\begin{aligned}
f_0(r,x) &= -\frac{\Delta_x\left[a^2\Delta_r\Delta_x - (r^2+a^2)^2\right]}{\varrho^2}\\
\omega_0(r,x) &= -\frac{a\left[\Delta_r - (r^2+a^2)\right]}{\left[(r^2+a^2)^2 - a^2\Delta_r\Delta_x\right]}\\
\gamma_0(r,x) &= \frac{1}{2}\ln\left(\Delta_x\left[(r^2+a^2)^2 - a^2\Delta_r\Delta_x\right]\right)\\
\Delta_r(r) &= r^2 - 2mr + a^2, \quad \Delta_x(x) = 1 - x^2.
\end{aligned}$$

We proceed as follows:

$\star$ Using the seed rotation ω_0 on the twist equation provides us with the seed imaginary part of the seed Ernst potential χ_0.

$\star$ The metric function f is readily obtained. Rotation, however, comes from:

$\star$ Using χ_0 the new imaginary part χ is obtained.

$\star$ Then, using the twist equation once again, we get the final rotation function ω .

$\star$ Thus, we get the Kerr-Levi-Civita spacetime

$$ds^2_{Kerr-LC} = f(d\varphi - \omega dt)^2 - \frac{\Delta_r \Delta_x}{f} dt^2 + \frac{e^{2\gamma_0}}{f} \left(\frac{dr^2}{\Delta_r} + \frac{dx^2}{\Delta_x} \right),$$

with

$$f(r,x) = \Delta_x \varrho^2 \frac{(r^2+a^2)^2 - a^2 \Delta_r \Delta_x}{4m^2 a^2 x^2 \left[a^2 \Delta_x^2 + \varrho^2(\Delta_x + 2)\right]^2 + \Delta_x^2 \left[(r^2+a^2)^2 - a^2 \Delta_r \Delta_x\right]^2},$$

$$\omega(r,x) = -2ma \frac{(2a^2 m - 3a^2 r + r^3)\Delta_r x^4 - 6r(a^2+r^2)\Delta_r x^2 + (2a^2 m + a^2 r + r^3)(a^2 - 6mr - 3r^2)}{\Delta_r a^2 x^2 + r(2a^2 m + a^2 r + r^3)}.$$

where as usual $\varrho^2(r,x) = r^2 + a^2 x^2$. Some comments are in order:

$\star$ The azimuthal component of the metric is $g_{\varphi\varphi} = f \geq 0$ and therefore the Killing vector ∂_ϕ is everywhere positive in $r > 0$, except at the axis of symmetry $x = \pm 1$ where it becomes null. Hence, there are no close timelike curves.

Using some modified canonical coordinates $(t, \tilde{\rho}, \tilde{z}, \phi)$ we observe that at constant t and $\tilde{z}$

$$ds^2 \sim d\tilde{\rho}^2 + \frac{\tilde{\rho}^2}{256a^4m^4}d\phi^2.$$

An angle defect emerges, but a redefinition of the azimuthal angle $\phi = 16a^2m^2$ suffices to fix it. Hence the spacetime is free of conical singularities.

$\star$ Due to the fact that $|\partial_\phi|^2 \sim \tilde{\rho}^2$ the spacetime is free of Misner strings.

$\star$ In the original canonical coordinate system $\{t, \rho, z, \varphi\}$, the asymptotic form of the metric reads

$$ds^2 \sim ds^2_{\mathrm{LC}} - 64a^3m^3\frac{3\rho^4 + 12\rho^2z^2 + 8z^4}{\rho^2(\rho^2+z^2)^{3/2}}\,dtd\varphi,$$

where:

$$ds^2_{\rm LC} = \rho^4 \left(-dt^2 + d\rho^2 + dz^2\right) + \frac{256 a^4 m^4}{\rho^2} d\varphi^2,$$

which belongs to the Levi-Civita class. Hence, asymptotically the spacetime behaves as a rotating generalisation of the Levi-Civita spacetime.

⋆ ⋆ The Kerr–Levi-Civita spacetime is algebraically general, type I in the Petrov classification.

⋆ On the original solution two limits help to reveal the origin of the rotation of the Kerr-Levi-Civita spacetime. First, if $a = 0$ we recover the Schwarzschild-Levi-Civita spacetime. The simple static black hole embedded on a Levi-Civita cylindrical background.

$$ds^2_{SLC} = r^4 \sin^4\theta \left[-\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2 d\theta^2\right] + \frac{d\varphi^2}{r^2 \sin^2\theta}.$$

$\star$ On the other hand, for $m = 0$ the spacetime is nothing else than the aforementioned Levi-Civita background, hence there is no trace of the rotation at all

$$ds^2_{LC} = r^4 \sin^4\theta \left[-dt^2 + dr^2 + r^2 d\theta^2\right] + \frac{d\varphi^2}{r^2\sin^2\theta}.$$

$\star$ What is effectively happening is the following: we have a Levi-Civita cylindrical background on which a spinning mass has been placed. When this asymptotic is dragged by the spinning source it produces a rotational effect akin to swirling spacetimes. Reason why the spacetime rotates asymptotically. However, the background has no intrinsic rotation, as removed the source ($m = 0$) staticity is recovered.

$\star$ Recall that in a Kerr spacetime, the asymptotic geometry is static; the angular velocity falls off like $\sim 2am/r^3$ as $r \to \infty$. In the Kerr–Levi-Civita spacetime, we have that

$$\Omega \sim \frac{3 + 6x^2 - x^4}{8am} r,$$

at large distances and fixed latitude. The swirling acquired behaviour of the rotation is then manifest.

⋆ The horizon structure of the solution is that of Kerr, namely, we have outer and inner horizons of the form

$$r_+ = m + \sqrt{m^2 - a^2}, \quad r_- = m - \sqrt{m^2 - a^2}.$$

⋆ Nevertheless, curvature invariants are regular there. Surprisingly, they are **regular everywhere**; there is no ring singularity like in Kerr neither a pathological axis as in the static Schwarzschild-Levi-Civita metric. This is also valid in the extremal and over-spinning cases.

⋆ The outer horizon r_+ is a Killing horizon, as the norm of the combination $\partial_t + \Omega|_{\mathrm{H}}\partial_\varphi$, with

$$\Omega|_{\mathrm{H}} = \frac{3r_+ + r_-}{2\sqrt{r_+ r_-}(r_+ + r_-)},$$

vanishes on it. The latter for a ZAMO, assuming that ∂_t is the asymptotically timelike Killing vector.

$\star$ Notice that the concept of "rest" is ambiguous in spacetimes with an asymptotic behaviour of the swirling(rotating Levi-Civita) type.

$\star$ There is no Killing vector that is timelike everywhere at infinity, and $\boldsymbol{\partial}_{t^{(\alpha)}} = \boldsymbol{\partial}_t + \alpha \boldsymbol{\partial}_\varphi$ all define a time flow in some parts of the asymptotic region.

$\star$ One should, in theory, make that ambiguity manifest in the spacetime metric by moving to a rotating frame, i.e., switching over to coordinates

$$\left\{ t^{(\alpha)} = t, r, x, \varphi^{(\alpha)} = \varphi - \alpha t \right\},$$

adapted to $\boldsymbol{\partial}_{t^{(\alpha)}}$. For our purposes, it suffices to consider

$$|\partial_{t^{(\alpha)}}|^2 \geq 0, \quad \Omega^{(\alpha)} = -\frac{\partial_{t^{(\alpha)}} \cdot \partial_\varphi}{|\partial_\varphi|^2},$$

as the definitions for the ergoregions and the angular velocity of a ZAMO, respectively, having in mind that fixing α amounts to choosing a frame. This has consequences on the ergoregions:

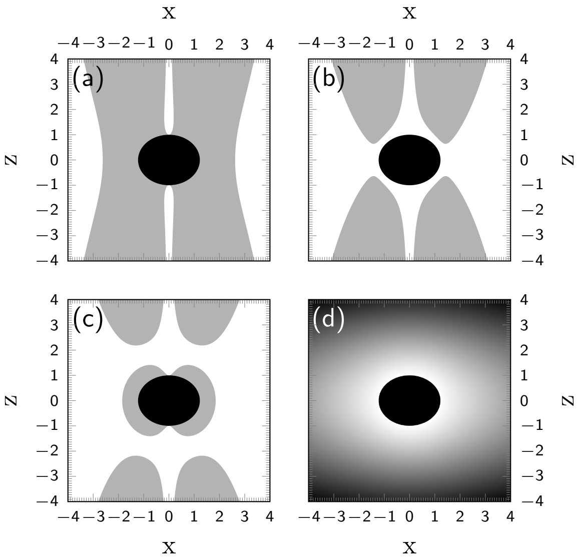


Figure: Cross section, taken at $y = 0$, of the ergoregion (gray fill) dressing the event-horizon of a Kerr–LC black hole with $r_+ = 1$ and $r_- = 1/2$ in rotating frames with angular velocity $\alpha = 0$ (a), $\alpha = 1.64$ (b), and $\alpha = 3$ (c). In panel (d), we provide a density map of the absolute difference $\Delta\Omega$ in the above spacetime. We use total white for $\Delta\Omega = 0$. The darker the color is, the bigger the difference becomes. The black ellipsoid represents the black hole.

Conclusions

⋆ Although Lie Point Symmetries in the electrovacuum have been explored in detail, recently several new developments in the physics of exact solutions have been reported.

⋆ The interplay between Ehlers and Harrison symmetries and accelerating spacetimes provided a full type I generalisation of the expanding Plebański-Demiański class of spacetimes.

⋆ Even more, the standard expanding Plebański-Demiański type D class has been completed via the characterisation of a type D accelerating-NUT geometry.

$\star$ A novel rotating vacuum solution of Einstein field equations has been constructed. This represents the exterior field of a spinning mass different to the other known examples, the Kerr and Tomimatsu-Sato spacetimes.

$\star$ It has been obtained via the discrete inversion symmetry of Ernst equations when acting on a Kerr metric written in magnetic WLP form.

$\star$ It has been named Kerr-Levi-Civita spacetime as it lands on the Schwarzschild-Levi-Civita spacetime in the static limit.

$\star$ It is proven to be **everywhere regular**!

$\star$ Although the spacetime acquires some features of swirling spacetimes there is a crucial difference: The Levi-Civita background does not posses any intrinsic rotation, it rotates due to the dragging produced by the spinning source, and the swirling characteristic emerges only due to this dragging over the non-trivial cylindrical background.

$\star$ In principle an entire Levi-Civita extension of the Plebański-Demiański spacetime can be formulated.

Most of what is next relates to the study of our new rotating geometry. To be done:

⋆ A deeper geometric analysis, geodesic motion, etc...

⋆ Devising a mechanism to compute the charges.

⋆ Thermodynamics if possible.

⋆ The Kerr–Newman–Levi-Civita geometry is on its way.

⋆ Extension with a minimally coupled scalar field. Due to asymptotic it could be free of naked singularities. Maybe a way to evade no hair theorems.

⋆ Construction of binaries, as superposition, and the resolution of their conical defects.

Merci